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On a Generalized Problem of M. Biernacki for Subordinate Functions

Uogólniony problem M. Biernackiego dla funkcji podporządkowanych

Обобщенная проблема М. Бернацкого для подчиненных функций

1. Introduction and notations. Let f and F be functions analytic in the unit disk $K = \{z : |z| < 1\}$. The function f is said to be subordinate to the function F in the disk K (we write $f \prec F$) if there exists an analytic function ω such that $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in K$ and $f(z) = F(\omega(z))$.

For classes of functions and for sets we assume the following notations:

S is the class of functions f analytic and univalent in the disk K and such that $f(0) = f'(0) - 1 = 0$;

$S_0 \subset S$ is a fixed family of functions such that for any function $f \in S_0$ and for any s, $|s| < 1$, we have $f(sz)/s \in S_0$;

S^c and S^*_α are the well-known subclasses of S of functions convex and starlike of order α, $0 \leq \alpha < 1$, respectively;

A_n $(n=1,2,\ldots)$ is the class of functions analytic in the disk

K of the form $f(z) = a_n z^n + a_{n+1} z^{n+1} + \dots$, $a_n \geqslant 0$;

B_n $(n=1,2,\dots)$ is the class of functions ω analytic in K and such that $\omega(z) = c_n z^n + c_{n+1} z^{n+1} + \dots$, $c_n \geqslant 0$ and $|\omega(z)| < 1$.

$H_n(z) = \left\{ \omega(z) : \omega \in B_n \right\}$, $z \in K$.

$H_n(z)$ is the generalized Rogosinski's set whose boundary is composed of three arcs [5]:

$$(1.1) \quad w = z_0(\theta) = |z|^{n+1} e^{i\theta}, \quad \arg z^n + \frac{\pi}{2} \leqslant \theta \leqslant \arg z^n + \frac{3}{2}\pi,$$

$$(1.2) \quad w = z_1(a) \text{ and } w = \overline{z_1}(a), \text{ where } z_1(a) = \frac{a + i|z|}{1 + ai|z|} z^n, \quad 0 \leqslant a \leqslant 1.$$

Properties of the set $H_n(z)$.

$$(1.3) \quad H_1(ze^{it}) = e^{it} H_1(z), \quad H_1(s_1 z) \subset H_1(s_2 z),$$

where t is an arbitrary real number and $0 < s_1 < s_2 \leqslant 1$.

$$Q_n(z, S_0) = \left\{ u : u = F(b)/F(z), \ F \in S_0 \right\},$$

where $z \in K$ and the point b runs over the set $H_n(z)$.

The following relations hold [2]:

$$(1.4) \quad Q_1(z, S_0) = Q_1(|z|, S_0), \quad Q_1(r, S_0) \subset Q_1(R, S_0), \quad 0 < r < R < 1,$$

$$D(a, S_0) = \bigcap_{f \in S_0} D_f(a), \quad D_f(a) = \left\{ z \in K : |f(z)| < |f(a)| \right\}.$$

For any real t and $0 < r < R < 1$ we have [3]:

$$(1.5) \quad D(ae^{it}, S_0) = e^{it} D(a, S_0) \text{ and } D(r, S_0) \subset D(R, S_0).$$

In paper [3] the authors determined, among other facts the set $D(R,S^{*})$, $0 < R < 1$, whose boundary has in polar coordinates the following equation

$$(1.6)\qquad \varrho(\theta) = \frac{1}{2R}\left[R^2+4R\sin\frac{\theta}{2}+1-\sqrt{\left(R^2+4R\sin\frac{\theta}{2}+1\right)^2-4R^2}\right], \quad 0 \leqslant \theta \leqslant 2\pi .$$

In 1935 M. Biernacki posed and partially solved the following problem: determine the number

$$(1.7)\qquad r_0 = r(A,S_0) = \inf r(f,F) ,$$

where $f \in A$, $F \in S_0$, $f \prec F$ and

$$r(f,F) = \sup\left\{r : \left[f \not\equiv F \wedge (0 < |z| < r)\right] \Rightarrow |f(z)| < |F(z)|\right\}.$$

From (1.7) it follows that if $z \in K$ and $|z| = R > r_0$, then the inequality $|f(z)| < |F(z)|$ ceases to hold for all functions f and F. The aim of the present paper is to solve the following problem:

Let R, $0 < R \leqslant 1$, $f \in A_n$, $F \in S_0$, $f \not\equiv F$ and $f \prec F$,

$$t(R,f,F) = \sup\left\{t : (0 < |z| < R) \Rightarrow (|f(zt)| < |F(z)|)\right\} .$$

Find the number

$$t_0 = t(R,A_n,S_0) = \inf_{f,F} t(R,f,F) .$$

If $0 < R \leqslant r_0$, then $t_0=1$, hence we may assume that $r_0 < R \leqslant 1$. In this paper we are going to determine the numbers $t(R,A_n,S_0)$ for $S_0 = S^{*}$, $S^{*}_{1/2}$, and S^{c}.

2. Main results.

Lemma. Let $p=r^2(1+R^2)$, $q=rR(1+r^2)$, $A=A(r,R,a)=$ $=pa^2-2qa+r^4R^2+1$, $B=B(r,R,a)=pa^2-2qa+r^4+R^2$, $G=G(r,R,a)=$ $=\frac{1+r}{1+R}\frac{\sqrt{A}+\sqrt{B}}{1-r^4}\sqrt{r^2+a^2}$, $G_1=G(r,r,a)$, $A_1=A(r,r,a)$, $B_1=B(r,r,a)$, $1/2<r\leqslant R\leqslant 1$, $0\leqslant a\leqslant 1$. Then $G\leqslant\frac{r}{1-r}$.

Proof. It is known [4] that $G_1\leqslant r/(1-r)$. Therefore it is sufficient to establish the inequality $G\leqslant G_1$, that is $(1+r)^2\sqrt{AB}\leqslant(1+R)^2\sqrt{A_1B_1}+(R-r)(1-rR)W$, where $W=2r^2a^2+2r(1+r^2)a+$ $+1+r^4$. Since $\sqrt{A_1B_1}=r(1+r^2)(1-a)\sqrt{H}$, $H=r^2(a^2-2a)+1-r^2+r^4$, on taking the squares of both sides of the above inequality and dividing through by $2r(1+r^2)(R-r)(1-Rr)(1+R)^2(1-a)$ we get

$$(2.1)\qquad 2r^2K\leqslant W\sqrt{H},$$

$$K=ra^3+(1-r+r^2)a^2+(r^{-1}-1-r-r^2+r^3)a+0,5(r^{-2}-1-2r-r^2+r^4).$$

Since $0\leqslant H\leqslant 1$, the inequality (2.1) is a fortiori satisfied when $2r^2K\leqslant WH$, and it takes, after some transformations, the form $P(a)\geqslant 0$, $P(a)=2r\big[a^4+(r-2)a^3+(0,5r^{-2}-r^{-1}-2-2r+1,5r^2)a^2$ $+(r^{-1}+1-r-r^2+r^3)a+r^{-1}+1,5-r^2+0,5r^4\big]$.

We easily see that $P''(a)<0$ and $\min\limits_{\langle 0,1\rangle}P(a)=\min\big[P(0),P(1)\big]>0$ which completes the proof the lemma.

Let $r_0<r\leqslant R=|z|$, r_0 being defined by (1.7). Let us now set

$$(2.2)\qquad L(R,r,n,S_0) = \sup\left\{|f(\tfrac{r}{R}z)|\,/\,|F(z)|\ :\ f\in A_n\ ,\ F\in S_0\ ,\ f\prec F\right\},$$

and $l(r,n,S_0) = L(r,r,n,S_0)$.

Theorem 1.

$$(2.3)\qquad L(R,r,n,S^{*}_{1/2}) = \begin{cases} \dfrac{r^2}{R}\,\dfrac{1+R}{1-r} & \text{if} \quad n=1\ , \\[2ex] \dfrac{r^n}{R}\,\dfrac{1+R}{1-r^n} & \text{if} \quad n\geq 2\ . \end{cases}$$

Proof. It is known [7] that for fixed $a,b\in K$ the range of the functional

$$(2.4)\qquad \left\{\left[\frac{a}{b}\,\frac{F(b)}{F(a)}\right]^{\frac{1}{2(1-\alpha)}}\ ;\ F\in S^{*}_{\alpha}\right\}$$

is a closed disk whose boundary is given by the equation

$$z(\theta) = (1-ae^{i\theta})/(1-be^{-i\theta})\ ,\quad 0\leq\theta\leq 2\pi\ .$$

Hence we obtain for $\alpha = 1/2$ that the boundary of the range of the functional $\left\{F(b)/F(a)\ ,\ F\in S^{*}_{1/2}\right\}$ is the circle

$$(2.5)\qquad w(\theta) = \frac{b}{a}(1-ae^{-i\theta})/(1-be^{-i\theta})\ ,\quad 0\leq\theta\leq 2\pi\ ,$$

with center s and radius ϱ , where

$$(2.6)\qquad s = \frac{b/a - |b|^2}{1-|b|^2}\ ,\qquad \varrho = \frac{|b|}{|a|}\,\frac{|b-a|}{1-|b|^2}\ .$$

From (2.2) and (2.5) it follows that (2.3) is equivalent to

$$\sup_{F\in S^{*}_{1/2}}\left|\frac{F(\omega(\frac{r}{R}z))}{F(z)}\right| = \sup_{\theta,\omega}\left|\frac{\omega(\frac{r}{R}z)}{z}\,\frac{1-ze^{-i\theta}}{1-\omega(\frac{r}{R}z)}\right|\ ,$$

where $0\leq\theta\leq 2\pi$, $\omega\in B_n$.

It is therefore sufficient to find the maximum of the function

$$h(b,\Theta) = \left|\frac{b}{z}\,\frac{1-ze^{-i\theta}}{1-be^{-i\theta}}\right| \quad , \quad 0 \leqslant \Theta \leqslant 2\pi \quad , \quad b \in \partial H_n(\tfrac{r}{R}z) \quad ,$$

The variable b is restricted to the boundary of $H_n(\frac{r}{R}z)$, since the right member of (2.5) is an analytic function of that variable.

Suppose first $n=1$. By (1.4) we may assume that $z = R$. The boundary $\partial H_1(r)$ takes on the form

$$w = r^2 e^{it} \quad , \quad \frac{\pi}{2} \leqslant t \leqslant \frac{3}{2}\pi \tag{2.7}$$

and

$$w = z_1(a) \;\text{ and }\; w = \overline{z_1}(a) \quad , \quad z_1(a) = \frac{r(a-ri)}{1-ari} \quad , \tag{2.8}$$

$$0 \leqslant a \leqslant 1 \quad .$$

Hence if b belongs to the arc (2.7), then

$$h(b,\Theta) = \frac{r^2}{R}\left|\frac{1 - Re^{-i\theta}}{1 - r^2e^{-i(\theta-t)}}\right| \leqslant \frac{r^2}{R}\,\frac{1+R}{1-r^2} \quad ,$$

On the other hand, if b belongs to the arc (2.8), then, on account of (2.5) and (2.6), we have

$$h(b,\Theta) \leqslant |s| + \varrho = J(R,b) \quad \text{and} \quad J(R,b) = J(R,\bar{b}) \quad .$$

Now

$$J(R,z_1(a)) = \frac{r}{R}\,\frac{1-R^2}{g(a)} \quad , \quad 0 \leqslant a \leqslant 1 \quad ,$$

$g(a)$ being defined in the lemma.

If $a=0$, the point $b=z_1(0)$ belongs to the arc (2.7) and we find, by virtue of the lemma, that

$$\max_{b \in H_1(r)} h(\Theta,b) = \frac{r^2}{R}\,\frac{1+R}{1-r^2} \quad ,$$

which proves (2.3) for $n=1$.

Let now $n \geqslant 2$. It is easy to show that if b belongs to the arc (1.1), then $h(b,\theta)$ $\frac{r^{n+1}}{R}\frac{1+R}{1-r^{n+1}}$.

If, on the other hand, b belongs to the arcs (1.2), that is, if $b=(\frac{r}{R}z)^n\frac{a+ir}{1+ari}$, then $|b|\leqslant r^n$, whence

$$h(b,\theta)\leqslant\frac{r^n}{R}\frac{1+R}{1-r^n}.$$

If now $a=1$, $\theta=\frac{n\pi}{1-n}$ and $z=Re^{i\pi/(1-n)}$, then $h(b,\theta)=$ $=\frac{r^n}{R}\frac{1+R}{1-r^n}$ and the proof of theorem 1 is thus completed.

<u>Theorem 2.</u> $L(R,r,n,S^c)=L(R,r,n,S^*_{1/2})$.

<u>Proof.</u> It is known [6] that $S^c\subset S^*_{1/2}$. From theorem 1 we get :

$$(2.9)\qquad |f(\tfrac{r}{R}z)|\,/\,|F(z)|\leqslant L(R,r,n,S^*_{1/2})$$

if $f\in A_n$, $F\in S^c$ and $f \prec F$.

Since for the pair of functions

$F(z)=\frac{z}{1+z}\in S^c$, $f(\frac{r}{R}z)=F(-\frac{r^2}{R^2}z^2)$ for $z=R$ and $n=1$,

and

$F_n(z)=\frac{z}{1-ze^{in\pi/(n-1)}}\in S^c$, $f_n(\frac{r}{R}z)=F_n(\frac{r^n}{R^n}z^n)$ for $z=Re^{i\pi/(1-n)}$

and $n\geqslant 2$, the formula (2.9) becomes an equality, the theorem 2 is thus established.

From theorems 1 and 2 we get for $r=R$ the results obtained by Z. Bogucki and J. Waniurski.

Corollary 1. [4].

$$l(r,n,S^c) = l(r,n,S^*_{1/2}) = \begin{cases} r/(1-r) & \text{if } n=1,\ r \geqslant 1/2, \\ r^{n-1}\cdot \dfrac{1+r}{1-r^n} & \text{if } n \geqslant 2,\ 0 < r < 1. \end{cases}$$

3. The generalized problem of M. Biernacki.

From Corollary 1 we obtain

Corollary 2.

$$r_o = r(A_n, S^c) = r(A_n, S^*_{1/2}) = \begin{cases} 1/2 & \text{if } n=1, \\ r_n & \text{if } n \geqslant 2, \end{cases}$$

where r_n is the unique root of the equation $2r^n + r^{n-1} - 1 = 0$.

Theorem 3. If $r_o \leqslant R \leqslant 1$, then

$$(3.1) \qquad t(R,A_n,S^c) = t(R,A_n,S^*_{1/2}) = \begin{cases} \dfrac{1}{\sqrt{R(1+2R)}} & \text{if } n=1, \\ \dfrac{1}{\sqrt[n]{R^{n-1}(1+2R)}} & \text{if } n \geqslant 2. \end{cases}$$

Proof. It is sufficient to use Theorem 2 and determine the desired numbers (3.1) from the equation $L(R,r,n,S^c) = 1$, Q.E.D.

In the case where $R=1$, Theorem 3 implies:

Corollary 3. *If* $f \in A_n$, $F \in S^c$ *or* $F \in S^*_{1/2}$ *and* $f \prec F$, *then*

$$|f(t_n z)| < |F(z)| \quad \text{for} \quad 0 < |z| < 1,$$

where

$$t_n = \begin{cases} \dfrac{1}{\sqrt{3}} & \text{if } n=1, \\[2mm] \dfrac{1}{\sqrt[n]{3}} & \text{if } n \geqslant 2. \end{cases}$$

The number t_n *is best possible.*

It is known [4] that

$$l(r,n,S^*) = \begin{cases} r/(1-r)^2 & \text{if } n=1, \\[2mm] r^{n-1} \cdot \left(\dfrac{1+r}{1-r^n} \right)^2 & \text{if } n \geqslant 2. \end{cases}$$

Hence

$$r_0^* = r(A_n, S^*) = \begin{cases} \dfrac{3-\sqrt{5}}{2} & \text{if } n=1, \\[2mm] r_n & \text{if } n \geqslant 2, \end{cases}$$

where r_n^* is the unique root of the equation $r^{n-1}\left(\dfrac{1+r}{1-r^n}\right)^2 = 1$.

Theorem 4. *If* $r_0 \leqslant R \leqslant 1$, *then*

$$(3.2) \qquad t(R,A_n,S^*) = \begin{cases} \dfrac{2}{\sqrt{R}\,(1+R+\sqrt{1+6R+R^2})} & \textit{if } n=1, \\[2mm] \dfrac{1}{R}\left[\dfrac{2\sqrt{R}}{1+R+\sqrt{1+6R+R^2}}\right]^{2/n} & \textit{if } n \geqslant 2. \end{cases}$$

Proof. Making use of (2.4) and (2.5) with $\alpha=0$ and proceedings as in the proof of Theorem 1 we show that

$$L(R,r,n,S^*) = \frac{r^n}{R}\left(\frac{1+R}{1-r^n}\right)^2 \quad \text{if} \quad n \geqslant 2,$$

whence we immediately obtain the number (3.2) for $n \geqslant 2$. Suppose now $n=1$. In view of (1.3) and (1.5) the inequality $|f(zt)| < |F(z)|$ for $0<|z|<R$, $t = t(R,A_1,S^*)$ takes on the equivalent form

$$(3.3) \qquad |F(\omega(r))| < |F(R)| \quad , \quad r = Rt \; ,$$

which is satisfied if

$$(3.4) \qquad H_1(r) \subset D(R,S^*) \; .$$

The boundary $\partial H_1(r)$ has in polar coordinates the equation

$$(3.5) \qquad R(\;\;) = \begin{cases} \frac{1}{2}\left[(r^2-1)\sin\theta + \sqrt{(1-r^2)^2\sin^2\theta + 4r^2}\right] , \\ \qquad\qquad \theta \in \langle 0,2\pi\rangle - (\pi/2 \,,\, 3\pi/2) \; , \\ r^2 \qquad\qquad\qquad , \quad \theta \in \langle \pi/2 \,,\, 3\pi/2\rangle \; . \end{cases}$$

Hence the relation (3.4) holds if

$R(\theta) < \varrho(\theta)$ for $\theta \in (0,\pi)$, $\varrho(\theta)$ being defined by (1.6).

Since the function $\varrho(\theta)$ is decreasing in the interval $\langle 0,\pi\rangle$ it is sufficient to show that

$$(3.6) \qquad R(\theta) < \varrho(\theta) \; , \quad 0 < \theta < \pi/2 \quad \text{and} \quad R(\pi) = \varrho(\pi) \; .$$

Taking account of (1.6) and (3.5) and using the notations

$$(3.7) \qquad A = \frac{1+R}{\sqrt{R}} \; , \quad H = A\sin\theta \; , \quad P = R + R^{-1} + 4\sin\theta/2$$

the inequality in (3.6) takes on the form

$$r\sqrt{H^2+4}+\sqrt{P^2-4} < P+rH \ , \quad 0<\theta<\pi/2 \ .$$

Hence, in view of (3.7) we have

$$(3.8)\qquad (A^2+4\sin\theta/2)^2-4(A^2+4\sin\theta/2) <$$
$$< A^2\sin\theta(A^2-2+4\sin\theta/2)+A^2(\sin^2\theta+1) \ , \quad 0<\theta<\pi/2.$$

Suppose first that $0<\theta\leqslant 2\operatorname{arc}\cos 4/5$. Then $\sin\theta\geqslant 1{,}6\sin\theta/2$. Setting $u=\sin\theta/2$, to prove (3.8) it suffices to show that the following inequality holds:

$$(A^2+4u)^2-4(A^2+4u) < 1{,}6uA^2(A^2+4u-2)+A^2(2{,}56u^2+1) \ ,$$
$$0<u\leqslant 0{,}6 \ ,$$

whence $16(0{,}56A^2-1)u^2+(5-A^2)\left[A^2-1{,}6(A^2-2)u\right]>0$.
Now, this latter inequality does hold, since $4\leqslant A^2\leqslant 5$.

Let now $0{,}6\leqslant u<\sqrt{2}/2$. Taking into account the fact that $\sqrt{2}\sin\theta/2<\sin\theta$ if $0<\theta<\pi/2$, we find in a similar way that in this case, too, the inequality (3.8) holds.

Finally, we find that the inequality in (3.6) takes on the form

$$r^2=\frac{1}{2R}\left[R^2+4R-1-\sqrt{(R^2+4R+1)^2-4R^2}\right]=\left[Rt(R,S^*)\right]^2 \ .$$

Since the sets $H_1(r)$ and $D(R,S^*)$ are extremal, the number (3.2) is best possible and the proof of Theorem 4 is thus completed.

<u>Corollary 4. If</u> $f\in A_n$, $F\in S^*$ <u>and</u> $f\prec F$, <u>then</u>

$$|f(t_n^* z)|<|F(z)| \quad \text{if} \quad 0<z<1 \ ,$$

<u>where</u>

$$t_n^* = \begin{cases} \sqrt{2}-1 & \underline{\text{if}}\ n=1, \\ (\sqrt{2}-1)^{2/n} & \underline{\text{if}}\ n \geqslant 2. \end{cases}$$

<u>The number</u> t_n^* <u>is best possible.</u>

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STRESZCZENIE

Niech S_o będzie ustaloną podklasą klasy S i niech funkcja $f(z) = a_n z^n + a_{n+1} z^{n+1} + \ldots$, $a_n \geq 0$, będzie podporządkowana funkcji F S_o, $f \not\equiv F$. Przy danym $R \in (0; 1]$ oznaczamy przez $t(R, f; F)$ kres górny tych $t > 0$, że dla każdego z, $0 < |z| < R$, zachodzi nierówność $|f(zt)| < |F(z)|$, zaś przez $t_o(R, f, F)$ kres dolny $t(R, f, F)$ przy $f \in A_n$, $F \in S_o$. W pracy wyznaczono $t(R, f, F)$ dla $S_o = S^*$, $S^*_{1/2}$, S^c.

РЕЗЮМЕ

Пусть S_0 фиксированный подкласс класса S , пусть $f(z) = a_n z^r + a_{nt} z^{n+1} + \ldots$, $a_n \geq 0$ будет подчинена функции $F \in S_0$, $f \not\equiv F$ для данного $R \in (0; 1)$ обозначим через t R,f,F точную верхнюю грань этих $t > 0$, что для всех z, $0 < |z| < R$ исполнено неравенство $|f(zt)| < |F(z)|$. Пусть t_0 это точная нижняя грань $t(R,f,F)$ для всех f, $F \in S_0$. В работе определено $t(R,f,F)$ для $S_0 = S^*$, $S^*_{1/2}$, S^0.