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Department of Mathematical Sciences
University of Delaware
Instytut Matematyki
Uniwersytet Marii Curie-Skłodowskiej

R. J. LIBERA, E. J. ZŁOTKIEWICZ

Bounded Univalent Functions with Montel Normalization

Funkcje jednolistne ograniczone z normalizacją Montela

Однолистные ограниченные функции нормированные по Монтелю

In the usual fashion we let S denote the class of functions $f(z) = z + a_2z^2 + \dots$ regular and univalent in the open unit disk Δ. The present authors have studied properties of families of functions with normalizations different from that of S. This report covers work included in the papers given below. All references to other work are given in those papers.

I. Notation. For fixed a, $0 < a < 1$, $M(a)$ denotes the family of all functions regular and univalent in Δ and normalized so that

$$F(0) = 0 \quad \text{and} \quad F(a) = a . \tag{1}$$

This is called Montels normalization. For $B > 1$, $M(a,B)$ denotes all members of $M(a)$ such that $|F(z)| < B$ for z in Δ. The related class $E(a,B)$ consists of functions $F(z)$ for which $F(0) = 0$, $F(a) = \frac{a}{B}$ and $|F(z)| < 1$, $z \in \Delta$.

II. Algebraic transformations. Here we review algebraically defined transformations among our classes and view some of their consequences.

(T_1) if $f(z) \in S$, then $F(z) = \frac{af(z)}{f(a)} \in M(a)$.

(T_2) if $F(z) \in M(a,B)$, then, for $0 \leqslant \varphi \leqslant 2\pi$,

$$G(z) = \frac{F(z)}{\left(1+e^{i\varphi}\cdot\frac{F(z)}{B}\right)^2}\left(1+e^{i\varphi}\cdot\frac{a}{B}\right)^2 \in M(a) .$$

(T_3) if $F(z) \in M(a,B)$, $H(z) = B^2 \dfrac{a-F\left(\frac{a-z}{1-az}\right)}{B^2-aF\left(\frac{a-z}{1-az}\right)} \in M(a,B)$.

The class $M(a)$ intherits some readily observable properties from S. For example, de Branges theorem and (T_1) yields

(2) $$|A_n| \leqslant n(1+a)^2 , \quad \text{all } n .$$

On the other hand, (T_2) gives

(3) $$\left(\frac{1-a}{B-a}B\right)^2 \leqslant |A_1| \leqslant \left(\frac{1+a}{1-a}B\right)^2$$

and

(4) $$\left|1-\frac{a}{B}e^{i\varphi}\right|^2 \left|A_2+\frac{2}{B}e^{2i\varphi}A_1^2\right| \leqslant 2(1+a)^2 .$$

The last result is particularly interesting because it suggests that more information can be obtained from it about A_1 or perhaps A_2. However technical difficulties in isolating, say A_1 in (4) seems to preclude this.

The transformation (T_3) may be considered the analog for $M(a,B)$ of the Koebe transformation for S; for $F(z)$ in $M(a,B)$ it gives

(5) $$\frac{1-a}{1+a}\,\frac{B-a}{B+a} \leqslant |F'(a)| \leqslant \frac{1+a}{1-a}\,\frac{B+a}{B-a}\,.$$

Using a powerful variational method, V. Singh obtained (5) in 1957.

If $f(z)$ is in S, then for each real φ, $e^{i\varphi} f(e^{-i\varphi} z)$ is in S also. This rotation is very useful in some investigations. However it is not generally available for functions in $M(a)$. This makes some problems in $M(a)$, or $M(a,B)$ such as, say the region of values of $F'(0)$, more challenging and, of course, more interesting. To address such questions we have developed appropriate variational methods for $M(a,B)$. In comparison with the transformations above, the transformations induced by variations may be considered transcendental.

<u>III. Transcendental transformations.</u> To simplify calculations, we derive variations for the class $M(a,B)$. As is typical of variational methods we construct two kinds of variations: the first for functions $F(z)$ in $M(a,B)$ for which $D_F = \Delta \setminus \overline{F(\Delta)}$ is an open, non-empty set and another for the case when $F(\Delta)$ consists of the disk Δ minus slits.

For the first case we let

(6) $$\varphi(w) = \sum_{k=1}^{2}\left[\lambda_k \frac{w+w_k}{w-w_k} + \overline{\lambda}_k \frac{\overline{w}_k w+1}{\overline{w}_k w-1} + \lambda_k - \overline{\lambda}_k\right]$$

which is meromorphic in C and purely imaginary for $|w| = 1$. Then, for small ε and δ,

$$g(w) = w \exp\left[\varepsilon\,\varphi(w)\right] = w + \varepsilon\, w\,\varphi(w) + o(\varepsilon)$$

is analytic and univalent in $\Delta \setminus \bigcup_{k=1}^{2} \{w : |w-w_k| \leqslant \delta\}$.

Now, for suitable choices of w_k in D_F and for appropriate δ,

$G(z,\varepsilon) = g(F(z))$ is univalent in Δ, $G(0,\varepsilon) = 0$ and $|G(z,\varepsilon)| < 1$. The parameters must be chosen so that $G(a,\varepsilon) = \frac{a}{b}$. This means we must choose λ_1, and λ_2 so that $\varphi(a) = 0$.

If we set $\psi(\chi) = \frac{a+\chi B}{a-\chi B} + 1$, then the last condition is equivalent to finding λ_2 for which

$$(7)\qquad 2\left[\left|\psi\left(\frac{1}{\bar{w}_2}\right)\right|^2 - \left|\psi(w_2)\right|^2\right] =$$

$$= \lambda_1\left[\psi(w_1)\psi(\bar{w}_2) - \psi\left(\frac{1}{w_1}\right)\psi\left(\frac{1}{\bar{w}_2}\right)\right] =$$

$$= \bar{\lambda}_1\left[\psi\left(\frac{1}{\bar{w}_1}\right)\psi(w_2) - \psi(\bar{w}_1)\psi\left(\frac{1}{w_2}\right)\right].$$

Hence, the coefficient of λ_2 is zero when $|\psi(w_2)| = \left|\psi\left(\frac{1}{\bar{w}_2}\right)\right|$ which is equivalent to $\left|\frac{a}{B} - w_2\right| = \left|\frac{a}{B} - \frac{1}{\bar{w}_2}\right|$ for all w_2 in Δ. But this cannot be so, consequently λ_1 and λ_2 can be chosen so that $G(z,\varepsilon)$ is in $E(a,B)$.

Suppose now that $F(z)$ is a slit mapping in $E(a,B)$. Then the method of Golusin modified by Slionskii gives the variation

$$(8)\qquad F^{*}(z) = F(z) + \varepsilon F(z)\varphi(F(z)) -$$

$$- \varepsilon zF'(z)\sum_{k=1}^{2}\left[\lambda_k\left(\frac{F(z_k)}{z_kF'(z_k)}\right)^2\frac{z+z_k}{z-z_k} - \bar{\lambda}_k\left(\frac{\overline{F(z_k)}}{\bar{z}_k\bar{F}'(z_k)}\right)^2\frac{1+\bar{z}_kz}{1-\bar{z}_kz}\right] + O(\varepsilon),$$

with $\varphi(w)$ as in (6).

The computations in this case are more difficult than those above, but are similarly structured. For example, the computation analogous to showing (7) has a solution is equivalent to showing $F(z)$ satisfies the differential equation

$$(9) \qquad \left(\frac{zF'(z)}{F(z)}\right)^2 \cdot \frac{W_1(F(z))}{(a-BF(z))(aF(z)-B)} = \frac{W_2(z)}{(a-z)(1-az)}$$

for W_k a polynomial of degree k, each k.

Assuming the analog of (7) has no solution yields the contradiction that the solution of (9) is not a slit mapping.

An application of these variational methods shows that region of values of A_1, $A_1 = F'(0)$, is given by

$$\left|\log A_1 + \log \frac{B^2 - a^2}{B(1-a^2)}\right| \leqslant \log \frac{(1+a)(B-a)}{(1-a)(B+a)} .$$

REFERENCES

[1] Libera, R.J., Złotkiewicz, E.J., Bounded Montel univalent functions (to appear, Colloquium Mathematicum).

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STRESZCZENIE

Niech $M(B,z_o)$ oznacza klasę funkcji analitycznych i jednolistnych w kole jednostkowym i spełniających tam warunki

$$f(0) = 0, \quad f(z_o) = z_o, \quad |f(z)| < B$$

gdzie $0 < |z_o| < 1$, $1 < B$.

Posługując się metodą wariacyjną G. M. Gołuzina autorzy wyprowadzają wzory wariacyjne w klasie $M(B, z_o)$ i stosują je do wyznaczenia obszaru zmienności funkcjonału $f'(0)$.

РЕЗЮМЕ

Пусть $M(B, z_0)$ класс аналитических и однолистных функций в единичном круге и таких что

$f(0)=0$, $f(z_0)=z_0$, $|f(z)|<B$

для фиксированных z_0, B, $0<|z_0|<1$, $1<B$. Пользуясь вариационной теоремой Г.М. Голузина, авторы установили вариационные формулы для каласса $M(B, z_0)$ и дали их применения для определения макорнтной области функционала $f'(0)$.