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On the Inequality of Type $A_1^{-1} + A_2^{-1} \geq 4(A_1 + A_2)^{-1}$

O nierówności typu $A_1^{-1} + A_2^{-1} \geq 4(A_1 + A_2)^{-1}$

О неравенстве типа $A_1^{-1} + A_2^{-1} \geq 4(A_1 + A_2)^{-1}$

In [1] the following questions has been posed by W. Anderson and G. Trapp: if A_1, A_2 are quadratic Hermitian positive matrices, does the inequality

$$A_1^{-1} + A_2^{-1} \geq 4(A_1 + A_2)^{-1}$$

hold? We shall consider this problem and analogous questions in the case when A_1, A_2 are operators.

Theorem 1. *Let H be an Hilbert space and let $A_1, A_2 : H \to H$ be linear bounded, symmetric and positive definite operators ($A_1 \geq \alpha$, $A_2 \geq \beta$; α , $\beta > 0$). Then*

$$A_1^{-1} + A_2^{-1} \geq 4(A_1 + A_2)^{-1} \ . \tag{1}$$

Proof. Let us denote by $A_1^{1/2}$, $A_2^{1/2}$, the positive definite square roots of A_1, A_2, respectively, and by $A_1^{-1/2}$, $A_2^{-1/2}$, the inverses of $A_1^{1/2}$, $A_2^{1/2}$. Because the operator $A_1^{-1/2}A_2A_1^{-1/2}$ is bounded and symmetric, we can use the spectral representation of it:

$$A_1^{-1/2}A_2A_1^{-1/2} = \int_m^M \lambda dE_\lambda \ ; \ 0 < m < M < \infty \ , \tag{2}$$

$$I = \int_m^M dE_\lambda \ , \tag{3}$$

where I denotes the identity operator and E_λ is the spectral family of $A_1^{-1/2}A_2A_1^{-1/2}$. Thus

$$A_2 = A_1^{1/2}\int_m^M \lambda dE_\lambda A_1^{1/2} \ ,$$

$$A_2^{-1} = A_1^{-1/2}\int_m^M \lambda^{-1} dE_\lambda A_1^{-1/2} \ ,$$

$$A_1^{-1} = A_1^{-1/2}\int_m^M \lambda dE_\lambda A_1^{-1/2} \ ,$$

$$A_1^{-1}+A_2^{-1} = A_1^{-1/2}\int_m^M (1+\lambda^{-1}) dE_\lambda A_1^{-1/2} \ ,$$

$$(A_1+A_2)^{-1} = A_1^{-1/2}\int_m^M (1+\lambda)^{-1} dE_\lambda A_1^{-1/2} \ ,$$

and the inequality (1) is equivalent with

$$A_1^{-1/2}\int_m^M (1+\lambda^{-1}-4(1+\lambda)^{-1}) dE_\lambda A_1^{-1/2} \geq 0 \ . \tag{4}$$

But we have: $1+\lambda^{-1}-4(1+\lambda)^{-1} \geq 0$ for all $\lambda > 0$, thus

$$C \overset{\text{def}}{=} \int_m^M (1+\lambda^{-1}-4(1+\lambda)^{-1}) dE_\lambda \geq 0$$

and therefore (4) holds, provided that $A_1^{-1/2}$ is symmetric and positively definite.

Theeorem 2. *The equality*

$$A^{-1}+A_2^{-1} = 4(\mathrm{A}_1+\mathrm{A}_2)^{-1}$$

holds if and only if $A_1 = A_2$.

Proof. Suppose

$$A^{-1}+A_2^{-1} = 4(A_1+A_2)^{-1} \ , \text{ i.e.}$$

$$A_1^{-1/2}\int_m^M (1+\lambda^{-1}-4(1+\lambda)^{-1}) dE_\lambda A_1^{-1/2} = 0 \ ,$$

which is equivalent to

$$\int_m^M (1+\lambda^{-1}-4(1+\lambda)^{-1}) dE_\lambda = 0 \ . \tag{5}$$

We have:

$$1+\lambda^{-1}-4(1+\lambda)^{-1} = \left(\left(\frac{\lambda}{\lambda+1}\right)^{1/2} - \left(\frac{\lambda}{\lambda(\lambda+1)}\right)\right)^2 > 0 \ \text{ for } \lambda \neq 1 \ .$$

Because the function $\lambda \to (E_\lambda x, x)$ is non-decreasing on $[m, M]$ for any $x \in H$, it follows that $(E_\lambda x, x)$ is a constant on the intervals $[m, 1-\varepsilon]$, $[1+\varepsilon, M]$ for every $\varepsilon > 0$.

Hence, by (2) and (3) we have:

$$(A_1^{-1/2} A_2 A_1^{-1/2} x, x) = \int_{1-\varepsilon}^{1+\varepsilon} \lambda d(E_\lambda x, x) \ ,$$

$$(x, x) = \int_{1-\varepsilon}^{1+\varepsilon} d(E_\lambda x, x)$$

and

$$|(A_1^{-1/2} A_2 A_1^{-1/2} x - x, x)| \leq \varepsilon \|x\|^2 \quad \text{for any } x \in H \text{ and } \varepsilon > 0$$

which means that

$$A_1^{-1/2} A_2 A_1^{-1/2} = I \ , \quad \text{i.e. } A_1 = A_2$$

Remark 1. Using the method of induction we are able to prove a more general result: if A_i, $i = 1, \ldots, n$, are as in Theorem 1 then

$$A_1^{-1} + \cdots + A_n^{-1} \geq n^2 (A_1 + \cdots + A_n)^{-1} \tag{6}$$

Proof. Assume that

$$A_1^{-1} + A_2^{-1} + \cdots + A_{n-1}^{-1} \geq (n-1)^2 (A_1 + \cdots + A_{n-1})^{-1} \ ,$$

and consider the following spectral representation:

$$(A_1 + \cdots + A_{n-1})^{-1/2} A_n (A_1 + \cdots + A_{n-1})^{-1/2} = \int_r^R \lambda dE_\lambda \ , \tag{7}$$

$$I = \int_r^R dE_\lambda \ , \tag{8}$$

We have:

$$A_n = (A_1 + \cdots + A_{n-1})^{1/2} \int_r^R \lambda dE_\lambda (A_1 + \cdots + A_{n-1})^{1/2} \ ,$$

$$A_n^{-1} = (A_1 + \cdots + A_{n-1})^{-1/2} \int_r^R \lambda^{-1} dE_\lambda (A_1 + \cdots + A_{n-1})^{-1/2} \ ,$$

$$A_1 + \cdots + A_{n-1} = (A_1 + \cdots + A_{n-1})^{1/2} \int_r^R dE_\lambda (A_1 + \cdots + A_{n-1})^{1/2} \ ,$$

$$(A_1 + \cdots + A_{n-1})^{-1} = (A_1 + \cdots + A_{n-1})^{-1/2} \int_r^R dE_\lambda (A_1 + \cdots + A_{n-1})^{-1/2} \ ,$$

$$A_1 + \cdots + A_{n-1} + A_n = (A_1 + \cdots + A_{n-1})^{1/2} \int_r^R (1+\lambda) dE_\lambda (A_1 + \cdots + A_{n-1})^{1/2} \ ,$$

$$(A_1+\dots+A_{n-1}+A_n)^{-1} = (A_1+\dots+A_{n-1})^{-1/2}\int_{r}^{R}(1+\lambda)^{-1}dE_\lambda(A_1+\dots+A_{n-1})^{-1/2},$$

and, under our assumptions

$$A_1^{-1}+\dots+A_n^{-1}-n^2(A_1+\dots+A_n)^{-1} \geq (n-1)^2(A_1+\dots+A_{n-1})^{-1}+ \\ +A_n^{-1}-n^2(A_1+\dots+A_n)^{-1} =$$

$$= (A_1+\dots+A_{n-1})^{-1/2}\int_{r}^{R}((n-1)^2+\lambda^{-1}-n^2(1+\lambda)^{-1})dE_\lambda(A_1+\dots+A_{n-1})^{-1/2}$$

But

$$(n-1)^2+\lambda^{-1}-n^2(1+\lambda)^{-1} = \left((n-1)\left(\frac{\lambda}{\lambda+1}\right)^{1/2}-\left(\frac{1}{\lambda(\lambda+1)}\right)^{1/2}\right)^2 \geq 0,$$

and thus Remark 1 is proved.

Remark 2. Similarly, by induction, one can prove the following:

$$A_1^{-1}+\dots+A_n^{-1} = n^2(A_1+\dots+A_n)^{-1} \text{ if and only if } A_1=\dots=A_n .$$

Proof. Suppose $A_1^{-1}+\dots+A_{n-1}^{-1} = (n-1)^2(A_1+\dots+A_{n-1})^{-1}$ implies $A_1=\dots=A_{n-1}$ and assume that $A_1^{-1}+\dots+A_n^{-1} = n^2(A_1+\dots+A_n)^{-1}$.

By (7) we have

$$A_1^{-1}+\dots+A_n^{-1}-n^2(A_1+\dots+A_n)^{-1} \geq \\ (n-1)^2(A_1+\dots+A_{n-1})^{-1}+A_n^{-1}+n^2(A_1+\dots+A_n)^{-1} \geq 0,$$

hence

$$A_1^{-1}+\dots+A_n^{-1} \geq (n-1)^2(A_1+\dots+A_{n-1})^{-1}+A_n^{-1} \geq \\ n^2(A_1+\dots+A_n)^{-1} = A_1^{-1}+\dots+A_n^{-1} .$$

Thus

$$A_1^{-1}+\dots+A_{n-1}^{-1} = (n-1)^2(A_1+\dots+A_{n-1})^{-1}$$

and by assumptions $A_1=\dots=A_{n-1}$.

Now we have

$$(n-1)A_1^{-1}+A_n^{-1} = n^2((n-1)A_1+A_n)^{-1}$$

and using once again the spectral family

$$A_1^{-1/2}A_nA_1^{-1/2} = \int_{q}^{Q}\lambda dE_\lambda , \quad I = \int_{q}^{Q}dE_\lambda ,$$

we obtain

$$A_1^{-1/2}\int_{q}^{Q}(n-1-\lambda^{-1}-n^2(n-1-\lambda)^{-1})\,dE_\lambda A_1^{-1/2} = 0 .$$

Here

$$n - 1 - \lambda^{-1} - n^2(n - 1 - \lambda)^{-1} = \frac{(n-1)(\lambda - 1)^2}{\lambda(n - 1 + \lambda)} > 0 \text{ for } \lambda \neq 1$$

and, similarly as in Theorem 2 we deduce that

$$A_1^{-1/2} A_n A_1^{-1/2} = I \ ,$$

e.g. $A_1 = A_n$.

RERERENCES

[1] SIAM Rev., Vol. 18, No 2, April 1976.

[2] Marcus, M., Minc, H., *A Survey of Matrix Theory and Matrix Inequalities*, Allyn Bacon Inc. Boston 1964.

STRESZCZENIE

Zakładając, że $A_1, \ldots, A_n$ są liniowo ciągłymi, symetrycznymi i dodatnio określonymi operacjami działającymi w przestrzeni Hilberta, dowodzi się nierówności

$$A_1^{-1} + \cdots + A_n^{-1} \geq n^2(A_1 + \cdots + A_n)^{-1}$$

oraz pokazuje, że równość jest możliwa tylko wtedy, jeśli $A_1 = \cdots = A_n$.

РЕЗЮМЕ

Предполагая, что $A_1, \ldots, A_n$ линейные ограниченные, симетрические и положитвльно определенны операторы в гильбертовом пространстве, доказывается неравенства

$$A_1^{-1} + \cdots + A_n^{-1} \geq n^2(A_1 + \cdots + A_n)^{-1} \ ,$$

а также, что равенство возможно только тогда, если $A_1 = \cdots = A_n$.