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On a Certain Class of Meromorphic and Univalent Functions

O pewnej klasie funkcji meromorficznych jednolistnych

О некотором классе мероморфных однолистных функция

1. Introduction. Let $U(p)$, $0 < p < 1$, denote the family of functions meromorphic and univalent in the unit disc $K = \{z : |z| < 1\}$ which have a simple pole at the point $z = p$ and the power series expansion:

$$f(z) = z + a_2 z^2 + \cdots \quad \text{for } |z| < p \, . \tag{1}$$

In this paper we will be concerned with the class $UR(p) \subset U(p)$ defined as follows:

$$UR(p) = \{f : f \in U(p) \wedge (f(z) \in \mathbf{R} \Leftrightarrow z \in \{(-1,1) - \{p\}\}\} \, ,$$

where $\mathbf{R}$ denotes the set of all real numbers.

The class $UR(p)$ is contained in the class TM of functions meromorphic and typically-real which was investigated by A. Goodman [2]. The bounds, obtained in [2] for some functionals in the class TM are also valid in the class $UR(p)$, sometimes they are even sharp, e.g. in the case of upper bounds for the coefficients a_n of the expansion (1) and for functional $\left|\dfrac{f'(a)}{\operatorname{Im}(a)}\right|$, $a \in K$, $a \notin \mathbf{R}$.

In Section 2 of this paper we will obtain the variational formulae for the class $UR(p)$. Using these formulae we will derive a differential equation for a function $f \in UR(p)$ corresponding to a regular boundary point of the set:

$$\mathbf{R}(z_0) = \{w : w = g(z_0), g \in UR(p)\} \, , \quad \text{for } z_0 \neq p \, .$$

This differential equation is solved effectively only when z_0 is real (Section 3). Finally, in Section 4 we give a characterization of the set of extreme points of the class $UR(p)$. We prove that each function such that the set $\overline{\mathbb{C}} - f(|z| < 1)$ has zero area is an extreme point of $UR(p)$. This shows that the set of extreme points of $UR(p)$ is too large to be of value in studying linear extremal problems over this class.

I wish to express my sincere gratutide to Professor E.Zlotkiewicz for his interest in this work and for his remarks, and also to Professor Z.Lewandowski for helpful discussions and criticism.

2. Variational formulae for the class $UR(p)$. Let Σ denote the class of functions F:

$$F(\varsigma) = \varsigma + A_0 + \frac{A_1}{\varsigma} + \cdots, \qquad |\varsigma| > 1 \tag{2}$$

analytic and univalent in $K^* = \{\varsigma : |\varsigma| > 1\}$, and let $\Sigma\left(\frac{1}{p}\right)$, $0 < p < 1$, denote the subclass of all $F \in \Sigma$ vanishing at $\frac{1}{p}$.

In 1963 Z.Lewandowski and E.Zlotkiewicz [4] obtained variational formulae for the class $U(p)$ using a simple relation between the classes $U(p)$ and $\Sigma\left(\frac{1}{p}\right)$. The way we will derive the variational formulae for $UR(p)$ is quite similar to that of [4].

We shall prove the following:

Theorem 1. *Suppose $f \in UR(p)$, $z_0 \neq p$ is an arbitrary fixed point of the unit disc K, A is an arbitrary complex number and $\alpha = -\dfrac{p}{\operatorname{res}(p, f(z))}$. Then a positive λ_0 can be chosen so that for $\lambda \in \langle 0, \lambda_0)$ the functions:*

$$f^*(z) = f(z) - \lambda\left(\frac{Aw_0^2 f^2(z)}{w_0 - f(z)} + \frac{\bar{A}\bar{w}_0^2 f^2(z)}{\bar{w}_0 - f(z)}\right) + o(\lambda), \tag{3}$$

where w_0 is an arbitrary point of the complement of the set $\overline{f(K)}$,

$$\begin{aligned} f^{**}(z) &= f(z) - \lambda\left(\frac{Af^2(z)f^2(z_0)}{f(z_0) - f(z)} + \frac{\bar{A}f^2(z)f^2(\bar{z}_0}{f(\bar{z}_0 - f(z)}\right) - \\ &-\frac{1}{2}\lambda A\left(\frac{f^2(z_0)}{z_0 f'(z_0}\right)^2\left[2f(z) - zf'(z)\left(\frac{z+z_0}{z_0 - z} + \frac{1+zz_0}{1-zz_0}\right) + \right.\\ &\left.+\alpha f^2(z)\left(\frac{p+z_0}{z_0-p} + \frac{1+pz_0}{1-pz_0}\right)\right] - \frac{1}{2}\lambda\bar{A}\left(\overline{\frac{f^2(z_0)}{z_0 f'(z_0}}\right)^2 [2f(z) - \\ &-zf'(z)\left(\frac{z+\bar{z}_0}{\bar{z}_0 - z} + \frac{1+z\bar{z}_0}{1-z\bar{z}_0}\right) + \alpha f^2(z)\left(\frac{p+\bar{z}_0}{\bar{z}_0 - p} + \frac{1+p\bar{z}_0}{1-p\bar{z}_0}\right)\Bigg] + o(\lambda), \end{aligned} \tag{4}$$

$$\begin{aligned} f^{***}(z) &= f(z) + \frac{\lambda}{2}\left[2f(z) - zf'(z)\left(\frac{z+e^{i\theta}}{e^{i\theta}-z} + \frac{z+e^{-i\theta}}{e^{-i\theta}-z}\right) + \right. \\ &\left. +\alpha f^2(z)\left(\frac{p+e^{i\theta}}{e^{i\theta}-p} + \frac{p+e^{-i\theta}}{e^{-i\theta}-p}\right)\right] + o(\lambda)\,, \end{aligned} \tag{5}$$

where θ is an arbitrary real number, belong to the class $UR(p)$.

Remark 1. In the case $p \to 1$ the formulae (3)–(5) turn out to be the known variational formulae for the class of functions analytic and univalent in K with real coefficients [8].

Proof. Let Σ_R $\left(\Sigma_R\left(\frac{1}{p}\right)\right)$ denote the subclass of all $F \in \Sigma$ $(F \in \left(\Sigma\left(\frac{1}{p}\right)\right)$ for which the coefficients A_n, $n = 1, 2, \ldots$, in the power series expansion (2) are real. According to the Goluzin–Shlionsky variational method [6] the following variational formulae for the class Σ_R can be obtained:

$$F^{\Delta}(\varsigma) = F(\varsigma) + \lambda\left(\frac{A}{F(\varsigma) - U_0} + \frac{\bar{A}}{F(\varsigma) - \bar{U}_0}\right) + o(\lambda)\,, \tag{3'}$$

$$\begin{aligned} F^{\Delta\Delta}(\varsigma) &= F(\varsigma) + \lambda\left(\frac{A}{F(\varsigma) - F(\varsigma_0)} + \frac{\bar{A}}{F(\varsigma) - \bar{F}(\varsigma_0)}\right) + \\ &+\frac{1}{2}\lambda A\left(\frac{1}{\varsigma_0 F'(\varsigma_0)}\right)^2\left[2F(\varsigma) - \varsigma F'(\varsigma)\left(\frac{\varsigma+\varsigma_0}{\varsigma-\varsigma_0} + \frac{\varsigma\varsigma_0+1}{\varsigma\varsigma_0-1}\right)\right] + \\ &+\frac{1}{2}\lambda\bar{A}\left(\frac{1}{\varsigma_0 F'(\varsigma_0)}\right)^2\left[2F(\varsigma) - \varsigma F'(\varsigma)\left(\frac{\varsigma+\bar{\varsigma}_0}{\varsigma-\bar{\varsigma}_0} + \frac{\varsigma\bar{\varsigma}_0+1}{\varsigma\bar{\varsigma}_0-1}\right)\right] + o(\lambda)\,, \end{aligned} \tag{4'}$$

where A is an arbitrary, fixed complex number and $\lambda \in < 0$, $\lambda_0, U_0 \in \mathbf{C} - \overline{F(K^*)}$, $|\varsigma_0| > 1$.

From (3') and (4') we derive the following variational formulae for the class $\Sigma_R\left(\frac{1}{p}\right)$:

$$F^{*}(\varsigma) = F(\varsigma) + \lambda\left[\frac{AF(\varsigma)}{W_0(F(\varsigma) - W_0)} + \frac{\bar{A}F(\varsigma)}{\bar{W}_0(F(\varsigma) - \bar{W}_0}\right] + o(\lambda)\,, \tag{3''}$$

$W_0 \in \mathbf{C} - \overline{F(K^*)}$,

$$\begin{aligned} F^{**}(\varsigma) &= F(\varsigma) + \lambda\left[\frac{AF(\varsigma)}{F(\varsigma_0)(F(\varsigma) - F(\varsigma_0))} + \frac{\bar{A}F(\varsigma)}{\overline{F(\varsigma_0)}(F(\varsigma) - \overline{F(\varsigma_0)})}\right] + \\ &+\frac{1}{2}\lambda A\left(\frac{1}{\varsigma_0 F'(\varsigma_0)}\right)^2\left[2F(\varsigma) - \varsigma F'(\varsigma)\left(\frac{\varsigma+\varsigma_0}{\varsigma-\varsigma_0} + \frac{\varsigma\varsigma_0+1}{\varsigma\varsigma_0-1}\right) + \right. \\ &\left. +\alpha\left(\frac{1+\varsigma_0 p}{1-\varsigma_0 p} + \frac{\varsigma_0+p}{\varsigma_0-p}\right)\right] + \frac{1}{2}\lambda\bar{A}\overline{\left(\frac{1}{\varsigma_0 F'(\varsigma_0)}\right)}^2 [2F(\varsigma) - \\ &-\varsigma F'(\varsigma)\left(\frac{\varsigma+\bar{\varsigma}_0}{\varsigma-\bar{\varsigma}_0} + \frac{\varsigma\bar{\varsigma}_0+1}{\varsigma\bar{\varsigma}_0-1}\right) + \alpha\left(\frac{1+\bar{\varsigma}_0 p}{1-\bar{\varsigma}_0 p} + \frac{\bar{\varsigma}_0+p}{\bar{\varsigma}_0-p}\right)\right] + o(\lambda), \end{aligned} \tag{4''}$$

where $\alpha = \frac{1}{p}F\left(\frac{1}{p}\right)$. Next using the fact $f(z) \in UR(p)$ if and only if $F(\varsigma) = \frac{1}{f\left(\frac{1}{\varsigma}\right)} \in \Sigma_R\left(\frac{1}{p}\right)$ we obtain the formulae (3) and (4).

In order to obtain (5) let us first consider the function:

$$F_\theta(\varsigma) = \frac{(\varsigma-1)^{1-\cos\theta}(\varsigma+1)^{1+\cos\theta}}{\varsigma}, \qquad \theta \in \mathbf{R}$$

defined in the exterior of the unit disc (K^*). This function is univalent in K^* and maps K^* onto the plane $\mathbf{C}$ with two slits emerging from the origin and symmetric with respect the real axis. Hence the function:

$$\varsigma^*(\varsigma,\lambda) = F_\theta^{-1}(e^\lambda F_\theta(\varsigma)), \qquad \lambda > 0,$$

maps K^* into itself and:

$$\varsigma^*(\varsigma,\lambda) = \varsigma - \frac{1}{2}\lambda\varsigma\left(\frac{1+\varsigma e^{i\theta}}{1-\varsigma e^{i\theta}} + \frac{1+\varsigma e^{-i\theta}}{1-\varsigma e^{-i\theta}}\right) + o(\lambda).$$

If $G \in \Sigma_R$ then the function $\psi(\varsigma) = G(\varsigma^*(\varsigma,\lambda))$ is univalent in K^* and has real coefficients in the power series expansion. Moreover:

$$\psi(\varsigma) = G(\varsigma) - \frac{\lambda}{2}\varsigma G'(\varsigma)\left(\frac{e^{i\theta}+\varsigma}{e^{i\theta}-\varsigma} + \frac{e^{-i\theta}+\varsigma}{e^{-i\theta}-\varsigma}\right) + O(\lambda^2).$$

Taking into account the normalization of the class Σ_R we have:

$$G^0(\varsigma) = \frac{\psi(\varsigma)}{\psi'(\infty)} = \frac{\psi(\varsigma)}{1+\lambda} \in \Sigma_R.$$

Now putting $F^{***}(\varsigma) = G^0(\varsigma) - G^0\left(\frac{1}{p}\right)$, $F(\varsigma) = G(\varsigma) - G\left(\frac{1}{p}\right)$ we have the following variational formula for the class $\Sigma_R\left(\frac{1}{p}\right)$:

$$\begin{aligned} F^{***}(\varsigma) &= F(\varsigma) - \frac{\lambda}{2}\left[2F(\varsigma) + \varsigma F'(\varsigma)\left(\frac{e^{i\theta}+\varsigma}{e^{i\theta}-\varsigma} + \frac{e^{-i\theta}+\varsigma}{e^{-i\theta}-\varsigma}\right) + \right. \\ &\quad \left. +\alpha\left(\frac{pe^{i\theta}+1}{pe^{i\theta}-1} + \frac{pe^{-i\theta}+1}{pe^{-i\theta}-1}\right)\right] + o(\lambda). \end{aligned}$$

Hence we can obtain (5) in the same way as above.

3. Values assumed by functions of the class $UR(p)$. In this section we shall use the variational formulae (3)–(5) to find the variability region of $f(z)$, z being fixed and f ranging over the whole class $UR(p)$. Let

$$R(z) = \{w : w = g(z),\ g \in UR(p)\}, \qquad z \neq p.$$

Assume that b is a regular boundary point of $R(z)$ i.e. there exists a point a in $\mathbf{C} - \overline{R(z)}$ and $r > 0$ such that $R(z) \cap \overline{K(a,r)} = \{b\}$ ($K(a,r)$ denotes a disc of radius r centered at a). Let $f \in UR(p)$ be the function corresponding to the regular boundary point b i.e. $f(z) = b$. Then:

$$|f(z) - a| \leq |g(z) - a|$$

if g is any function of $UR(p)$. In particular:

$$|f(z) - a| \leq |f^{**}(z) - a| \tag{i}$$

where f^{**} is given by (4). If we write (4) in the form:

$$f^{**}(z) = f(z) - \lambda W(z, z_0, f) + o(\lambda)\,, \qquad \lambda \in < 0, \lambda_0)\,,$$

where:

$$\begin{aligned} W(z, z_0, f) = {} & \frac{A f^2(z) f^2(z_0)}{f(z_0) - f(z)} + \frac{\bar{A} f^2(z) f^2(\bar{z}_0)}{f(z_0) - f(z)} + \\ & + \frac{1}{2} A \left(\frac{f^2(z_0)}{z_0 f'(z_0)} \right)^2 \left[2f(z) + z f'(z) \left(\frac{z + z_0}{z_0 - z} + \frac{1 + z z_0}{1 - z z_0} \right) + \right. \\ & \qquad \left. + \alpha f^2(z) \left(\frac{p + z_0}{z_0 - p} + \frac{1 + p z_0}{1 - p z_0} \right) \right] + \\ & + \frac{1}{2} \bar{A} \left(\frac{\overline{f^2(z_0)}}{z_0 f'(z_0)} \right)^2 \left[2f(z) + z f'(z) \left(\frac{z + \bar{z}_0}{\bar{z}_0 - z} + \frac{1 + z \bar{z}_0}{1 - z \bar{z}_0} \right) + \right. \\ & \qquad \left. \alpha f^2(z) \left(\frac{p + \bar{z}_0}{\bar{z}_0 - p} + \frac{1 + p \bar{z}_0}{1 - p \bar{z}_0} \right) \right], \end{aligned} \tag{ii}$$

then (i) is equivalent to this inequality:

$$|f(z) - a|^2 \leq |f(z) - a - \lambda W(z, z_0, f) + o(\lambda)|^2 =$$

$$= |f(z) - a|^2 - 2\lambda \operatorname{Re} \left[W(z, z_0, f) \overline{(f(z) - a)} \right] + o(\lambda)\,.$$

Hence:

$$\operatorname{Re} \left[W(z, z_0, f) \overline{(f(z) - a)} \right] \leq 0\,. \tag{iii}$$

Because the complex number A in (ii) is arbitrary we conclude from (iii) that the extremal function satisfies the following equation:

$$\left(e^{-i\phi}\frac{f^2(z)f^2(z_0)}{f(z_0)-f(z)}+e^{i\phi}\frac{\overline{f^2(z)}f^2(z_0)}{f(z_0)-\overline{f(z)}}\right)\left(\frac{z_0f'(z_0)}{f^2(z_0)}\right)^2=$$

$$=-\left\{\frac{e^{-i\phi}}{2}\left[2f(z)-zf'(z)\left(\frac{z+z_0}{z-z_0}+\frac{1+zz_0}{1-zz_0}\right)+\right.\right.$$

$$\left.+\alpha f^2(z)\left(\frac{z_0+p}{z_0-p}+\frac{1+pz_0}{1-pz_0}\right)\right]+ \tag{6}$$

$$+\frac{e^{i\phi}}{2}\left[2\overline{f(z)}-\overline{zf'(z)}\left(\frac{\bar z+z_0}{\bar z-z_0}+\frac{1+\bar zz_0}{1-\bar zz_0}\right)+\right.$$

$$\left.\left.+\alpha \overline{f^2(z)}\left(\frac{z_0+p}{z_0-p}+\frac{1+pz_0}{1-pz_0}\right)\right]\right\}$$

where $\phi=\arg(f(z)-a)$. The right side of equation (6) is real for $|z_0|=1$. Applying the variation (5) we conclude that it is a non-positive number. Using the formula (3) we can prove that the complement of the set $f(K)$ has no interior points. In the case $z=r$, $r\in(-1,1)$, $r\neq p$, in view of the compactness of the class $UR(p)$, the set $R(r)$ is a segment of the real axis, so our problem reduces to that of finding $\max\limits_{g\in UR(p)} g(r)=R_1(r)$ and $\min\limits_{g\in UR(p)} g(r)=R_2(r)$.

a)

$$R_1(r)\,,\qquad 0<r<p\,.$$

In this case $R_1(r)$ can be determined by using the following Goluzin's result ([1] p.127-140). If $F\in\Sigma$ then:

$$\mathrm{Re}\left(\sum_{v,v'=1}^{n}\gamma_v\gamma_{v'}\log\frac{F(\zeta_v)-F(\zeta_{v'})}{\zeta_v-\zeta_{v'}}\right)\leq-\sum_{v,v'=1}^{n}\gamma_v\bar\gamma_{v'}\log\left(1-\frac{1}{\zeta_v\bar\zeta_{v'}}\right) \tag{$*$}$$

for arbitrary, fixed $\zeta_1,\zeta_2,\ldots,\zeta_n\in K^*$ and equality holds only for the function F defined by the equation:

$$\sum_{v=1}^{n}\gamma_v\log\frac{F(\zeta)-F(\zeta_v)}{\zeta-\zeta_v}=-\sum_{v=1}^{n}\bar\gamma_v\log\left(1-\frac{1}{\zeta\bar\zeta_v}\right)$$

If we take $n=2$, $\gamma_1=\gamma_2=e^{i\alpha}$ then we obtain from $(*)$:

$$\mathrm{Re}\left[e^{2i\alpha}\log\frac{F(\zeta_1)-F(\zeta_2)}{\zeta_1-\zeta_2}\right]\leq-\log\left|1-\frac{1}{\zeta_1\bar\zeta_2}\right|.$$

Hence, if $\alpha=\frac{\pi}{2}$ then we have:

$$\left|\frac{F(\zeta_1)-F(\zeta_2)}{\zeta_1-\zeta_2}\right|\geq\left|1-\frac{1}{\zeta_1\bar\zeta_2}\right| \tag{$**$}$$

and the equality will ocurr only for the function F satisfying the following equations:

$$F(\varsigma) = F(\varsigma_v) + (\varsigma - \varsigma_v)\left(1 - \frac{1}{\varsigma_1\bar{\varsigma}_v}\right), \qquad v = 1,2 .$$

If $f \in U(p)$ then there exists the function $F \in \Sigma$ such that $f(z) = \dfrac{1}{F(\frac{1}{z}) - F(\frac{1}{p})}$.

Hence by substituting $\varsigma_1 = \dfrac{1}{r}$, $\varsigma_2 = \dfrac{1}{p}$ in (**) we obtain:

$$|f(r)| \leq \frac{rp}{|r-p||1-rp|} \qquad \text{for } f \in U(p) .$$

Moreover:

$$R_1(r) \leq \sup_{f \in U(p)} |f(r)| \leq \frac{r}{(r-p)(r-\frac{1}{p})} .$$

Equality occurs precisely for the function $f(z) = \dfrac{z}{(z-p)(z-\frac{1}{p})}$ which belongs to $UR(p)$. This result can also be obtained using the equation (6).

$$\text{b)} \qquad R_2(r) , \qquad 0 < r < p .$$

Let $f \in UR(p)$ be the function for which $\min\limits_{g \in UR(p)} g(r)$ occurs.
Then $z = r$, $\phi = 0$ and the equation (6) takes the form:

$$\left(\frac{zf'(z)}{f(z)}\right)^2 \frac{f^2(r)}{f(z) - f(r)} = Q(z) , \tag{7}$$

where we have replaced z_0 by z and:

$$Q(z) = -\left\{ f(r) - rf'(r)\frac{z(1-r^2)}{(z-r)(1-zr)} + \alpha f^2(r)\frac{z(1-p^2)}{(z-p)(1-zp)} \right\} ,$$

$Q(z) \leq 0$ on $|z| = 1$. Making the substitutions: $f(z) = w$, $f(r) = a > 0$ we obtain the equivalent form of (7):

$$\frac{a^2}{w^2(w-a)} dw^2 = Q(z)\frac{dz^2}{z^2} . \tag{7'}$$

It follows from the elementary properties of quadratic differentials [3], [5] that the trajectories of the differential:

$$\frac{a^2}{w^2(w-a)} dw^2 , \qquad a > 0 ,$$

in some neighbourhoods of its critical points, are: near the origin–closed Jordan curves each of which is contained in the annulus

$$\eta(1-\varepsilon)^{-1} \le |z| \le \eta(1+\varepsilon)\,, \qquad \varepsilon,\eta > 0\,,$$

and the half line $w > a$ emanating from the point a. The boundary slit $f(|z|=1)$ is contained in a trajectory of (8). Points $f(1)$, $f(-1)$ belong to the negative real axis, but neither the slit $f(|z|=1)$ nor any its non–degenerated subset can be contained in the negative axis (the half line $w < 0$ is contained in an orthogonal trajectory of
(8): $w < a$).

Hence $f(|z|=1)$ is an analytic arc, symmetric with respect to the real axis. This arc intersects the real axis at the point $f(1) = f(-1) < 0$. In this case $Q(z)$ can be written in the form:

$$Q(z) = A\frac{(z-e^{i\beta})^2(z-e^{-i\beta})^2}{(z-r)(1-zr)(z-p)(1-zp)}\,, \qquad \beta \ne 0,\pi\,, \tag{9}$$

$$A = -rpf(r)$$

In order to determine the parameter β we use the equation ([7],p.112-114):

$$\int_{\beta}^{2\pi-\beta} |Q(e^{i\phi})|^{\frac{1}{2}}\,d\theta = \int_{-\beta}^{\beta} |Q(e^{i\phi})|^{\frac{1}{2}}\,d\theta$$

From this, due to the periodicity of the function $Q(e^{i\phi})$, we obtain:

$$\cos\beta = \frac{\displaystyle\int_0^{2\pi} \frac{\cos\theta}{|e^{i\theta}-r||e^{i\theta}-p|}d\theta}{\displaystyle\int_0^{2\pi} \frac{1}{|e^{i\theta}-r||e^{i\theta}-p|}d\theta}\,. \tag{10}$$

Next, if we multiply the both sides of (7) by $z-r$ and take $z \to r$ then we obtain:

$$\frac{f'(r)}{f(r)} = -\frac{p}{r}\frac{|r-e^{i\beta}|^4}{(1-r^2)(r-p)(1-rp)}\,.$$

It follows from (7) that:

$$\mathrm{Re}\left\{\ln\frac{\sqrt{f(z)-f(r)}-i\sqrt{f(r)}}{\sqrt{f(z)-f(r)}+i\sqrt{f(r)}}\right\} = \text{const} \qquad \text{for } |z|=1\,.$$

Let $h(z) = \dfrac{f(z)}{f'(r)}$ and $h(r) = \dfrac{-r(1-r^2)(r-p)(r-p)(1-rp)}{p\,|r-e^{i\beta}|^4} = R^*$.

The function h maps the unit circumference onto the analytic arc defined by the equation:

$$\left|\frac{\sqrt{w-R^*}-i\sqrt{R^*}}{\sqrt{w-R^*}+i\sqrt{R^*}}\right| = \text{const}\,.$$

Moreover the function h satisfies the equation:

$$\left(\frac{zh'(z)}{h(z)}\right)^2 \frac{R^*}{h(z)-R^*} = -rp\frac{(z-e^{i\beta})^2(z-e^{-i\beta})^2}{(z-r)(1-zr)(z-p)(1-zp)}$$

Hence

$$2\,\mathrm{arctg}\sqrt{\frac{h(z)}{R^*}-1} = i\sqrt{rp}\int_r^z \frac{(\varsigma-e^{i\beta})(\varsigma-e^{-i\beta})d\varsigma}{\varsigma\sqrt{(\varsigma-r)(1-\varsigma r)(\varsigma-p)(1-\varsigma p)}}$$

and

$$h(z) = R^*\,\mathrm{tg}^2\left[\frac{i\sqrt{rp}}{2}\int_r^z \frac{(\varsigma-e^{i\beta})(\varsigma-e^{-i\beta})d\varsigma}{\varsigma\sqrt{(\varsigma-r)(1-\varsigma r)(\varsigma-p)(1-\varsigma p)}}\right] + R^*$$

The extremal function f has the form $f(z) = \dfrac{h(z)}{h'(0)}$, so we can calculate:

$$\begin{aligned} R_2(r) &= f(r) = \frac{h(r)}{h'(0)} = \\ &= \frac{1}{4} r \exp - \int_0^r \frac{\sqrt{rp}(\varsigma-e^{i\beta})(\varsigma-e^{-i\beta}) - \sqrt{(\varsigma-r)(1-\varsigma r)(\varsigma-p)(1-\varsigma p)}}{\varsigma\sqrt{(\varsigma-r)(1-\varsigma r)(\varsigma-p)(1-\varsigma p)}} d\varsigma \end{aligned}$$

where $\cos\beta$ is given by (10).

Determining $R_1(r)$, $R_2(r)$ for $r \in (p,1) \cup (-1,0)$ reduces for the analogous consideration, e.g. we can prove that:

$$R_2(r) = \frac{r}{(r-p)(r-\frac{1}{p})} < 0 \quad \text{for } r \in (p,1)\,.$$

c) $R(z_0)$, $z_0 \in K$, $z_0 \notin \mathbf{R}$ The function f corresponding to a regular boundary point of the set $R(z_0)$ satisfies the differential equation:

$$d\frac{f(z)-c}{(f(z)-b)(f(z)-\bar{b})}\left(\frac{zf'(z)}{f(z)}\right)^2 = Q(z)\,, \qquad (11)$$

where

$$\begin{aligned}
b &= f(z_0)\,,\\
c &= \frac{|b|^2(e^{i\phi}\bar{b}+e^{-i\phi}b)}{e^{i\phi}\bar{b}^2+e^{-i\phi}b^2}\in\mathbf{R}\,,\\
d &= e^{i\phi}\bar{b}^2+e^{-i\phi}b^2\in\mathbf{R}\,,\\
Q(z) &= -\left\{\frac{e^{-i\phi}}{2}\left[2f(z_0)-z_0f'(z_0)\left(\frac{z+z_0}{z-z_0}+\frac{1+zz_0}{1-zz_0}\right)+\right.\right.\\
&\quad \left.+\alpha f^2(z_0)\left(\frac{z+p}{z-p}+\frac{1+pz}{1-pz}\right)\right]+\\
&\quad +\frac{e^{i\phi}}{2}\left[2f(\bar{z}_0)-\bar{z}_0f'(\bar{z}_0)\left(\frac{z+\bar{z}_0}{z-\bar{z}_0}+\frac{1+z\bar{z}_0}{1-z\bar{z}_0}\right)+\right.\\
&\quad \left.\left.+\alpha f^2(z_0)\left(\frac{z+p}{z-p}+\frac{1+pz}{1-pz}\right)\right]\right\}
\end{aligned}$$

and $Q(z)\le 0$ on $|z|=1$.

$Q(z)$ is a rational function whose denominator is a polynomial of 6 degree and it follows from the form of the equation (11) that the numerator is also a polynomial of 6 degree. If $c\in f(K)$ then $Q(z)$ has the form:

$$Q(z)=A\frac{(z-e^{i\beta})^2(z-e^{-i\beta})^2(z-x)(z-\frac{1}{x})}{(z-z_0)(z-\bar{z}_0)(1-zz_0)(1-z\bar{z}_0)(z-p)(1-zp)}\,,\quad \beta\neq 0,\pi\,,\qquad (12)$$

$$Q(z)=A\frac{(z-1)^2(z+1)^2(z-x)(z-\frac{1}{x})}{(z-z_0)(z-\bar{z}_0)(1-zz_0)(1-z\bar{z}_0)(z-p)(1-zp)}\,,\qquad (12')$$

$$\text{where}\quad x\in\{(-1,1)-\{p\}\}\,,\quad c=f(x)\,,\quad A=\frac{cp|z_0|^2}{d|b|^2}$$

The differential $d\dfrac{(w-c)dw^2}{(w-b)(w-\bar{b})w^2}$ has three simple poles at the points b, $\bar{b}$, ∞ and a zero of first order at the point $w=c$. At zero is a pole of second order. So, if $c\in f(K)$ then by analysing the trajectories of both sides of (12) and (12') we conclude that the function f reduces to the mapping of the unit disc K onto the complement of a segment of the real axis (if $d<0$) or onto the complement of an analytic arc symmetric with respect the real axis which meets it at the point $f(1)=f(-1)$ (if $d>0$).

If $c\notin f(K)$ then the function f satisfies the equation of the form:

$$\begin{aligned}
&d\frac{f(z)-c}{(f(z)-b)(f(z)-\bar{b})}\left(\frac{zf'(z)}{f(z)}\right)^2=\\
&=A\frac{(z-e^{i\beta})^2(z-e^{-i\beta})^2(z^2-1)}{(z-z_0)(z-\bar{z}_0)(1-zz_0)(1-z\bar{z}_0)(z-p)(1-zp)}\,. \qquad (13)
\end{aligned}$$

In this case the set $f(|z|=1)$ consists of two analytic arcs symmetric with respect the real axis. The point $c = f(1) = f(-1) < 0$ is their common end and the points $f(e^{i\beta})$, $f(e^{-i\beta})$ are their symmetric endpoints.The parameter β can be determined as in the case b).

4. A remark on extreme points. Let $\Sigma_R^+\left(\frac{1}{p}\right)$ denotes the class of functions univalent in the exterior of the unit disc which have the power series expansion:

$$F(\zeta) = a_1\zeta + a_0 + \frac{a_{-1}}{\zeta} + \cdots, \qquad |\zeta| > 1,$$

with $a_i \in \mathbf{R}$, $i = 1, 0, -1, \ldots$, $a_1 > 0$, and satisfy the conditions: $F\left(\frac{1}{p}\right) = 0$, $F'\left(\frac{1}{p}\right) = 1$.

There is the following relation between the classes $U\mathbf{R}(p)$ and $\Sigma_R^+\left(\frac{1}{p}\right)$:

$$f(z) \in U\mathbf{R}(p) \Leftrightarrow \frac{1-p^2}{p^2} f\left(\frac{p - \frac{1}{\zeta}}{1 - \frac{p}{\zeta}}\right) = F(\zeta) \in \Sigma_R^+\left(\frac{1}{p}\right),$$

$$F(\zeta) = \frac{-\gamma}{p^2}\zeta + a_0 + \frac{a_{-1}}{\zeta} + \cdots, \qquad \gamma = \operatorname{res}\,(p, f(z)) < 0\,.$$

Let $\mathbf{A}$ denotes the linear space of functions

$$F(\zeta) = a_1\zeta + a_0 + \frac{a_{-1}}{\zeta} + \cdots$$

that are analytic in K^* and satisfy the condition: $\sum_{n=1}^{\infty} n|a_{-n}|^2 < \infty$. For each $F(\zeta) = a_1\zeta + a_0 + \frac{a_{-1}}{\zeta} + \cdots$ and $G(\zeta) = b_1\zeta + b_0 + \frac{b_{-1}}{\zeta} + \cdots$ in $\mathbf{A}$ the hermitian product:

$$< F, G >= a_1\bar{b}_1 - \sum_{n=1}^{\infty} n a_{-n}\bar{b}_{-n}$$

is well defined.

Let

$$\mathbf{A}\left(\frac{1}{p}\right) = \Big\{F \in \mathbf{A} : F(\zeta) = a_1\zeta + a_0 + \frac{a_{-1}}{\zeta} + \cdots, \ |\zeta| > 1,$$

$$a_i \in \mathbf{R}\,,\ i = 1, 0, -1, \ldots,\ a_1 > 0 \text{ and } F\left(\frac{1}{p}\right) = F'\left(\frac{1}{p}\right) - 1 = 0\,,$$

$$\text{and } < F, F > \geq 0\Big\}\,.$$

We have

Lemma. $\mathbf{A}\left(\frac{1}{p}\right)$ is convex and closed in the topology of locally uniform convergence and

$$\left\{F : F \in \mathbf{A}\left(\frac{1}{p}\right) \text{ and } <F,F>=0\right\} \subset E\mathbf{A}\left(\frac{1}{p}\right),$$

where $E\mathbf{A}\left(\frac{1}{p}\right)$ denotes the set of extreme points of $\mathbf{A}\left(\frac{1}{p}\right)$.

Proof. Let $F_k(\varsigma) = a_1^{(k)}\varsigma + a_0^{(k)} + \frac{a_{-1}^{(k)}}{\varsigma} + \cdots \in \mathbf{A}A\left(\frac{1}{p}\right)$ and suppose that $F_k \to F$ locally uniformly in $|\varsigma| > 1$ where $F(\varsigma) = a_1\varsigma + a_0 + \frac{a_{-1}}{\varsigma} + \cdots$. Then:

$$\lim_{k\to\infty} a_i^{(k)} = a_i \qquad \text{for } i = 1,0,-1,\ldots .$$

Moreover:

$$\sum_{n=1}^{N} n(a_{-n}^{(k)})^2 \le \sum_{n=1}^{\infty} n(a_{-n}^{(k)})^2 \le a_1^{(k)}$$

and by letting $k \to \infty$:

$$\sum_{n=1}^{N} n(a_{-n})^2 \le a_1^2 \quad \text{for each } N .$$

Hence:

$$\sum_{n=1}^{\infty} n(a_{-n})^2 \le a_1^2 .$$

Therefore $<F,F> \ge 0$ and $F \in \mathbf{A}\left(\frac{1}{p}\right)$ so $\mathbf{A}\left(\frac{1}{p}\right)$ is closed. Next let us take $F = tF_1 + (1-t)F_2$, $F_1, F_2 \in \mathbf{A}A\left(\frac{1}{p}\right)$, $t \in (0,1)$. Then:

$$<F,F>= t^2 <F_1,F_1> +2t(1-t) <F_1,F_2> +(1-t^2) <F_2,F_2> . \qquad (*)$$

We first shall prove that if $F_1, F_2 \in bfA\left(\frac{1}{p}\right)$ then $<F_1,F_2> \ge 0$.

Let $F_1(\varsigma) = a_1\varsigma + a_0 + \frac{a_{-1}}{\varsigma} + \cdots$, $F_2(\varsigma) = b_1\varsigma + b_0 + \frac{b_{-1}}{\varsigma} + \cdots$. By the Cauchy-

Schwarz inequality:

$$\begin{aligned}< F_1,F_2 > &= a_1b_1 - \sum_{n=1}^{\infty} na_{-n}b_{-n} = a_1b_1\left(1-\sum_{n=1}^{\infty}\frac{na_{-n}b_{-n}}{a_1b_1}\right) \geq \\ &\geq a_1b_1\left(1-\sum_{n=1}^{\infty}\frac{n|a_{-n}||b_{-n}|}{a_1b_1}\right) \geq \\ &\geq a_1b_1\left[1-\left(\sum_{n=1}^{\infty}\frac{n(a_{-n})^2}{(a_1)^2}\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\frac{n(b_{-n})^2}{(b_1)^2}\right)^{\frac{1}{2}}\right] \geq 0 .\end{aligned}$$

The last inequality follows from the fact

$$1-\sum_{n=1}^{\infty}\frac{n(a_{-n})^2}{(a_1)^2} \geq 0 \text{ and } 1-\sum_{n=1}^{\infty}\frac{n(b_{-n})^2}{(b_1)^2} \geq 0$$

for $F_1,F_2 \in \mathbf{A}\left(\frac{1}{p}\right)$.

It follows from above and from (*) that $< F,F > \geq 0$. Therefore $F \in \mathbf{A}\left(\frac{1}{p}\right)$ and $\mathbf{A}\left(\frac{1}{p}\right)$ is convex. Moreover if $< F_1,F_2 >= 0$ then $F_1 = F_2$. Suppose now that $F \in \mathbf{A}\left(\frac{1}{p}\right)$, $< F,F >= 0$ and $F \notin E\mathbf{A}\left(\frac{1}{p}\right)$. So we have

$$0 =< F,F >= t^2 < F_1,F_1 > +2t(1-t) < F_1,F_2 > +(1-t^2) < F_2,F_2 > .$$

Hence $F_1 = F_2$ which is a contradiction.

If $F \in \Sigma_R^+\left(\frac{1}{p}\right)$ and $B = \overline{\mathbf{C}} - f(|\varsigma| > 1)$ then $0 \leq < F,F >= \frac{1}{\pi}$ area B. So $\Sigma_R^+\left(\frac{1}{p}\right) \subset \mathbf{A}\left(\frac{1}{p}\right)$. Hence if $F \in E\mathbf{A}\left(\frac{1}{p}\right)$ and $F \in \Sigma_R^+\left(\frac{1}{p}\right)$ then $F \in E\Sigma_R^+\left(\frac{1}{p}\right)$. Because the formula (14) defines a linear homeomorphism of $UR(p)$ onto $\Sigma_R^+\left(\frac{1}{p}\right)$ we have:

Theorem 2. *If $f \in UR(p)$ and f maps the unit disc onto a domain whose complement has the zero area then f is an extreme point of the class $UR(p)$.*

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STRESZCZENIE

W pracy rozpatruję klasę $U\mathbb{R}(p)$ funkcji określonych w kole jednostkowym, meromorficznych i jednolistnych oraz przyjmujących wartości rzeczywiste dla argumentu rzeczywistego. Zostały podane wzory wariacyjne dla klasy $U\mathbb{R}(p)$, za pomocą których otrzymujemy równanie różniczkowo-funkcyjne dla funkcji odpowiadającej regularnemu punktowi brzegowemu zbioru $R(z_0) = \{w : w = f(z_0), f \in U\mathbb{R}(p)\}$. Podany jest również warunek konieczny na to, by funkcja f była punktem extremalnym klasy $U\mathbb{R}(p)$.

РЕЗЮМЕ

В данной работе рассмотрен класс $U\mathbb{R}(p)$ функций определенных в единичном круге, мероморфных и однолистных и принимающих вещественные значения для вещественного аргумента. Полученные вариационные формулы для класса $U\mathbb{R}(p)$ с помощью которых получаем дифференциально-функциональное уравнение для функции соответствующей регулярной граничной точке множества $R(z_0) = \{w : w = f(z_0), f \in U\mathbb{R}(p)\}$. Также получено достаточное условие на то, чтобы функция f была экстремальной точкой класса $U\mathbb{R}(p)$.