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Instytut Matematyki
Uniwersytet Marii Curie-Skłodowskiej

M. Maksym, A. Żmurek

On the Generalized 3-structures Induced on the Hypersurface in Riemannian Manifold

Uogólnione 3-struktury indukowane na hiperpowierzchni rozmaitości riemannowskiej

Обобщённые 3-структуры индуцированные на гиперповерхности в римановом многообразии

Introduction. In the present paper we will consider algebraic properties of 3-structures induced on the hypersurfaces in the Riemannian manifold by generalized 3-structures given on the Riemannian manifold.

By M^K, TM^K we will denote a k-dimensional C^∞- manifold and tangent space to M^K, respectively. The indices α, β, γ will run over the set $\{1,2,3\}$.

Let M^{4n} be a $4n$-dimensional differentiable manifold of class C^∞ which admits a set of three tensor fields $\{\underset{\alpha}{\tilde{F}}\}$ of type $(1,1)$ satisfying the conditions:

$$\underset{\alpha}{\tilde{F}} \circ \underset{\alpha}{\tilde{F}} = \underset{\alpha}{\tilde{F}}^2 = \underset{\alpha}{\varepsilon} \tilde{I}, \qquad \underset{\alpha}{\varepsilon} = \pm 1 \tag{1}$$

$$\underset{\alpha}{\tilde{F}} \circ \underset{\beta}{\tilde{F}} = \underset{\alpha\beta\gamma}{\varepsilon} \underset{\gamma}{\tilde{F}}, \qquad \underset{\alpha\beta}{\varepsilon} = \pm 1, \qquad \alpha \neq \beta \neq \gamma \neq \alpha, \tag{2}$$

$\tilde{I}$ denotes here the identity tensor field.

The set of these tensors fields $\{\underset{\alpha}{\tilde{F}}\}$, $(\alpha = 1,2,3)$, will be called a generalized 3-structure, or shortly 3-structure.

The formulas (1), (2) imply the following conditions:

$$(\underset{\alpha}{\tilde{F}} \circ \underset{\beta}{\tilde{F}}) \circ \underset{\gamma}{\tilde{F}} = \underset{\alpha\beta}{\varepsilon}(\underset{\gamma}{\tilde{F}} \circ \underset{\gamma}{\tilde{F}}),$$

$$\underset{\alpha}{\tilde{F}} \circ (\underset{\beta}{\tilde{F}} \circ \underset{\gamma}{\tilde{F}}) = \underset{\alpha\beta}{\varepsilon}\underset{\gamma}{\varepsilon} I,$$

$$\underset{\alpha}{\tilde{F}} \circ (\underset{\beta\gamma}{\varepsilon}\underset{\alpha}{\tilde{F}}) = \underset{\alpha\beta}{\varepsilon}\underset{\gamma}{\varepsilon} I$$

$$\underset{\alpha}{\varepsilon}\underset{\beta\gamma}{\varepsilon} I = \underset{\alpha\beta}{\varepsilon}\underset{\gamma}{\varepsilon} I,$$

hence we have

$$\underset{\alpha}{\varepsilon}\underset{\beta\gamma}{\varepsilon} = \underset{\alpha\beta}{\varepsilon}\underset{\gamma}{\varepsilon}. \tag{3}$$

Analogously

$$\underset{\alpha}{\tilde{F}} \circ (\underset{\alpha}{\tilde{F}} \circ \underset{\beta}{\tilde{F}}) = \underset{\alpha\beta}{\varepsilon}(\underset{\alpha}{\tilde{F}} \circ \underset{\gamma}{\tilde{F}}),$$

$$(\underset{\alpha}{\tilde{F}} \circ \underset{\alpha}{\tilde{F}}) \circ \underset{\beta}{\tilde{F}} = \underset{\alpha\beta}{\varepsilon}\underset{\alpha\gamma}{\varepsilon}\underset{\beta}{\tilde{F}},$$

$$\underset{\alpha}{\varepsilon}\underset{\beta}{\tilde{F}} = \underset{\alpha\beta}{\varepsilon}\underset{\alpha\gamma}{\varepsilon}\underset{\beta}{\tilde{F}},$$

and

$$\underset{\alpha}{\varepsilon} = \underset{\alpha\beta}{\varepsilon}\underset{\alpha\gamma}{\varepsilon}. \tag{4}$$

From (3) and (4) we obtain

$$\underset{\alpha}{\varepsilon} = \underset{\beta\alpha}{\varepsilon}\underset{\gamma\alpha}{\varepsilon}. \tag{5}$$

There exist four types of 3–structures $\{\underset{\alpha}{\tilde{F}}\}$ on M^{4n} which satisfy the conditions (1), (2) :

I.

$$\underset{1}{\tilde{F}}{}^2 = \underset{2}{\tilde{F}}{}^2 = \underset{3}{\tilde{F}}{}^2 = -I, \qquad (\underset{1}{\varepsilon} = \underset{2}{\varepsilon} = \underset{3}{\varepsilon} = -1).$$

Taking into account (3), (4) we have

$$\underset{12}{\varepsilon} = -\underset{21}{\varepsilon} = \varepsilon = \pm 1, \qquad \underset{13}{\varepsilon} = -\underset{31}{\varepsilon} = -\varepsilon, \qquad \underset{23}{\varepsilon} = -\underset{32}{\varepsilon} = \varepsilon.$$

This structure is called an almost quaternion structure ([1], [2]).

II.

$$\underset{1}{\tilde{F}}{}^2 = I, \qquad \underset{2}{\tilde{F}}{}^2 = \underset{3}{\tilde{F}}{}^2 = -I, \qquad (\underset{1}{\varepsilon} = 1, \qquad \underset{2}{\varepsilon} = \underset{3}{\varepsilon} = -1).$$

Making use of (3) and (4) we obtain

$$\underset{12}{\varepsilon} = \underset{21}{\varepsilon} = \varepsilon = \pm 1, \qquad \underset{13}{\varepsilon} = \underset{31}{\varepsilon} = \varepsilon, \qquad \underset{23}{\varepsilon} = \underset{32}{\varepsilon} = -\varepsilon.$$

This structure is called an almost quaternion structure of the first kind ([2]).

III.

$$\underset{1}{\bar F}^2 = \underset{2}{\bar F}^2 = I\,, \qquad \underset{3}{\bar F}^2 = -I\,, \qquad (\underset{1}{\varepsilon} = \underset{2}{\varepsilon} = 1\,, \qquad \underset{3}{\varepsilon} = -1)\,.$$

Taking into account (3), (4) we have

$$\underset{12}{\varepsilon} = -\underset{21}{\varepsilon} = \varepsilon = \pm 1\,, \qquad \underset{13}{\varepsilon} = -\underset{31}{\varepsilon} = \varepsilon\,, \qquad \underset{23}{\varepsilon} = -\underset{32}{\varepsilon} = -\varepsilon\,.$$

This structure is called an almost quaternion structure of the second kind ([2]).

IV.

$$\underset{1}{\bar F}^2 = \underset{2}{\bar F}^2 = \underset{3}{\bar F}^2 = I\,, \qquad (\underset{1}{\varepsilon} = \underset{2}{\varepsilon} = \underset{3}{\varepsilon} = 1)\,.$$

(3) and (4) imply

$$\underset{12}{\varepsilon} = \underset{21}{\varepsilon} = \varepsilon = \pm 1\,, \qquad \underset{13}{\varepsilon} = \underset{31}{\varepsilon} = \varepsilon\,, \qquad \underset{23}{\varepsilon} = \underset{32}{\varepsilon} = \varepsilon\,.$$

This structure is called the 3-product structure.

The values $\underset{\alpha}{\varepsilon}$, $\underset{\alpha\beta}{\varepsilon}$ for these 3-structures are illustrated by the following tables

I.

	1	2	3
1	-1	ε	$-\varepsilon$
2	$-\varepsilon$	-1	ε
3	ε	$-\varepsilon$	-1

II.

	1	2	3
1	1	ε	ε
2	ε	-1	$-\varepsilon$
3	ε	$-\varepsilon$	-1

III.

	1	2	3
1	1	ε	ε
2	$-\varepsilon$	1	$-\varepsilon$
3	$-\varepsilon$	ε	-1

IV.

	1	2	3
1	1	ε	ε
2	ε	1	ε
3	ε	ε	1

(we denote $\underset{\alpha}{\varepsilon} = \underset{\alpha\alpha}{\varepsilon}$).

Theorem 1. *Let M^{4n} be a Riemannian manifold with a 3-structure $\{\underset{\alpha}{\bar F}\}$. There exists a metric $\bar g$ which satisfies the condition*

$$\bar g(\underset{\alpha}{\bar F}(\bar X)\,, \qquad \underset{\alpha}{\bar F}(\bar Y)) = \bar g(\bar X, \bar Y) \tag{6}$$

for all $\alpha = 1,2,3$ and for arbitrary vector fields $\bar X, \bar Y \in TM^{4n}$.

Proof. Taking an arbitrary Riemannian metric $\tilde{a}$ in M^{4n} we put

$$\begin{aligned}\tilde{g}(\tilde{X},\tilde{Y}) = \ & \tilde{a}(\tilde{X},\tilde{Y}) + \tilde{a}(\underset{1}{\tilde{F}}(\tilde{X}),\underset{1}{\tilde{F}}(\tilde{Y}))+ \\ & +\tilde{a}(\underset{2}{\tilde{F}}(\tilde{X}),\underset{2}{\tilde{F}}(\tilde{Y})) + \tilde{a}(\underset{3}{\tilde{F}}(\tilde{X}),\underset{3}{\tilde{F}}(\tilde{Y}))\end{aligned}$$

(see [1]).

1. The 3-structures induced on the hypersurface in a Riemannian manifold. Let M^{4n} be a Riemannian manifold with a 3-structure $\{\underset{a}{\tilde{F}}\}$ and let M^{4n-1} be an orientable manifold such that there exists a differentiable immersion

$$i : M^{4n-1} \to M^{4n}$$

A submanifold M^{4n-1} will be identified with a hypersurface $i(M^{4n-1})$ in the Riemannian manifold M^{4n}.

We denote by N the unit vector field normal to $i(M^{4n-1})$ with respect to the Riemannian metric $\tilde{g}$ satisfying (6):

$$\tilde{g}(N,N) = 1\,. \tag{7}$$

For an arbitrary vector field $\tilde{X} \in TM^n$ we put:

$$\underset{a}{\tilde{F}}(\tilde{X}) = \underset{a}{X_1} + \underset{a}{X_2} \tag{8}$$

where $\underset{a}{X_1} \in TM^{4n-1}$, $\underset{a}{X_2} \in TM_{\perp}^{4n-1}$. Let us denote:

$$\underset{a}{X_1} = \underset{a}{\acute{F}}(\tilde{X})\,, \qquad \underset{a}{X_2} = \underset{a}{\varepsilon}\underset{a}{\breve{\omega}}(\tilde{X})N\,,$$

where $\underset{a}{\acute{F}} : TM^{4n} \to TM^{4n-1}$, $\underset{a}{\breve{\omega}} : TM^{4n} \to R$ are given by the decomposition (8). We have

$$\underset{a}{\tilde{F}}(\tilde{X}) = \underset{a}{\acute{F}}(\tilde{X}) + \underset{a}{\varepsilon}\underset{a}{\breve{\omega}}(\tilde{X})N\,. \tag{8'}$$

The condition

$$\underset{a}{\acute{F}}(\tilde{X}) \in TM^{4n-1}$$

implies

$$\tilde{g}(\underset{a}{\acute{F}}(\tilde{X}),N) = 0\,. \tag{9}$$

Putting $\tilde{X} = N$ into (8) we can find

$$\underset{a}{\tilde{F}}(N) = \underset{a}{\acute{F}}(N) + \underset{a}{\varepsilon}\underset{a}{\breve{\omega}}(N)N\,. \tag{10}$$

Let

$$\underset{a}{\eta} = \underset{a}{F}(N) \in TM^{4n-1}, \qquad \underset{a}{\lambda} = \underset{a}{\breve{\omega}}(N) \in R. \tag{11}$$

Then we have

$$\underset{a}{\tilde{F}}(N) = \underset{a}{\eta} + \underset{a}{\varepsilon}\underset{a}{\lambda} N. \tag{10'}$$

The restrictions $\underset{a}{\tilde{F}}|_{TM^{4n-1}}$, $\underset{a}{\breve{\omega}}|_{TM^{4n-1}}$ of $\underset{a}{\tilde{F}}$ and $\underset{a}{\breve{\omega}}$ will be denoted by $\underset{a}{F}$ and $\underset{a}{\omega}$, respectively. $\underset{a}{F}$ is a tensor field of type (1,1) on TM^{4n-1} and $\underset{a}{\omega}$ is a 1-form on TM^{4n-1}.

In this way on the submanifold M^{4n-1} there are given three tensor fields $\underset{a}{F}$ of type (1,1), three vector fields $\underset{a}{\eta}$ and three 1-form fields $\underset{a}{\omega}$ induced by $\underset{a}{\tilde{F}}$.

Thus the 3-structure $\{\underset{a}{\tilde{F}}\}$ induces the 3-structure $\{\underset{a}{F}, \underset{a}{\omega}, \underset{a}{\eta}\}$ on the submanifold M^{4n-1}.

We will consider the kind of the 3-structure $\{\underset{a}{F}, \underset{a}{\omega}, \underset{a}{\eta}\}$. From (1), (8) and (11) we have

$$\begin{aligned}
\underset{a}{\varepsilon}\tilde{X} &= \underset{a}{\tilde{F}}(\underset{a}{F}(\tilde{X}) + \underset{a}{\varepsilon}\underset{a}{\omega}(\tilde{X})N) = \\
&= \underset{a}{F}(\underset{a}{F}(\tilde{X})) + \underset{a}{\varepsilon}\underset{a}{\omega}(\tilde{X})\underset{a}{\eta} + \underset{a}{\varepsilon}(\underset{a}{\omega}\underset{a}{F}(\tilde{X}) + \underset{a}{\varepsilon}\underset{a}{\omega}(\tilde{X})\underset{a}{\lambda})N = \\
&= \underset{a}{F}^2(\tilde{X}) + \underset{a}{\varepsilon}\underset{a}{\omega}(\tilde{X})\underset{a}{\eta} + \underset{a}{\varepsilon}(\underset{a}{\omega} \circ \underset{a}{F})(\tilde{X})N + \underset{a}{\lambda}\underset{a}{\omega}(\tilde{X})N
\end{aligned}$$

for an arbitrary vector field $\tilde{X} \in TM^{4n}$.

From here, for $\tilde{X} = N$ we have

$$\underset{a}{\varepsilon}N = \underset{a}{F}(\underset{a}{\eta}) + \underset{a}{\varepsilon}\underset{a}{\lambda}\underset{a}{\eta} + \underset{a}{\varepsilon}\underset{a}{\omega}(\underset{a}{\eta})N + (\underset{a}{\lambda})^2 N$$

and

$$\underset{a}{F}(\underset{a}{\eta}) = -\underset{a}{\varepsilon}\underset{a}{\lambda}\underset{a}{\eta}, \qquad \underset{a}{\omega}(\underset{a}{\eta}) = 1 - \underset{a}{\varepsilon}(\underset{a}{\lambda})^2.$$

However, for each vector field $X \in TM^{4n-1}$ we have

$$\underset{a}{F}^2(X) = \underset{a}{\varepsilon}X - \underset{a}{\varepsilon}\underset{a}{\omega}(X)\underset{a}{\eta}, \qquad \text{or} \qquad \underset{a}{F}^2 = \underset{a}{\varepsilon}(I - \underset{a}{\omega} \otimes \underset{a}{\eta})$$

and therefore,

$$\underset{a}{\omega} \circ \underset{a}{F} = -\underset{a}{\varepsilon}\underset{a}{\lambda}\underset{a}{\omega}.$$

In this way we obtained the 3-structure, $\{\underset{\alpha}{F},\underset{\alpha}{\omega},\underset{\alpha}{\eta}\}$ which satisfies the conditions:

$$
\begin{aligned}
\underset{\alpha}{F}^2 &= \underset{\alpha}{\varepsilon}(I - \underset{\alpha}{\omega}\otimes\underset{\alpha}{\eta})\\
\underset{\alpha}{\omega}\circ\underset{\alpha}{F} &= -\underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda}\,\underset{\alpha}{\omega}\\
\underset{\alpha}{F}(\underset{\alpha}{\eta}) &= -\underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda}\,\underset{\alpha}{\eta}\\
\underset{\alpha}{\omega}(\underset{\alpha}{\eta}) &= 1-\underset{\alpha}{\varepsilon}(\underset{\alpha}{\lambda})^2 .
\end{aligned}
\tag{12}
$$

It is a generalized contact or an almost contact 3-structure (with respect to the values $\underset{\alpha}{\lambda} = \underset{\alpha}{\omega}(N)$).

We will consider dependences of this 3-structure derived from the condition (2). For an arbitrary $\bar{X}\in TM^{4n}$ we have

$$\underset{\alpha}{\bar{F}}(\underset{\beta}{\bar{F}}(\bar{X})) = \underset{\alpha\beta}{\varepsilon}\,\underset{\gamma}{\bar{F}}, \qquad \alpha\neq\beta\neq\gamma\neq\alpha,$$

$$\underset{\alpha}{\bar{F}}(\underset{\beta}{F}(\bar{X}) + \underset{\beta\beta}{\varepsilon\omega}(\bar{X})N) = \underset{\alpha\beta\gamma}{\varepsilon}\,\bar{F}(\bar{X}),$$

$$\underset{\alpha}{F}(\underset{\beta}{F}(\bar{X}) + \underset{\beta\beta}{\varepsilon\omega}(\bar{X})N + \underset{\alpha}{\varepsilon}(\underset{\alpha}{\omega}(\underset{\beta}{F}(\bar{X}) + \underset{\beta\beta}{\varepsilon\omega}(\bar{X})N)N = \underset{\alpha\beta}{\varepsilon}(\underset{\gamma}{F}(\bar{X}) + \underset{\gamma\gamma}{\varepsilon\omega}(\bar{X})N) .$$

Now making use of (11) we get

$$(\underset{\alpha}{F}\circ\underset{\beta}{F})(\bar{X}) + \underset{\beta\beta}{\varepsilon\omega}(\bar{X})\underset{\alpha}{\eta} + \underset{\alpha}{\varepsilon}(\underset{\alpha}{\omega}\circ\underset{\beta}{F})(\bar{X})N + \underset{\alpha\beta\alpha\beta}{\varepsilon\varepsilon\lambda\omega}(\bar{X})N = \underset{\alpha\beta\gamma}{\varepsilon\,F}(\bar{X}) + \underset{\alpha\beta\gamma\gamma}{\varepsilon\;\varepsilon\omega}(\bar{X})N . \tag{13}$$

Thus we have

$$\underset{\alpha}{F}\circ\underset{\beta}{F} = \underset{\alpha\beta\gamma}{\varepsilon\,F} - \underset{\beta\beta}{\varepsilon\omega}\otimes\underset{\alpha}{\eta},$$

$$\underset{\alpha}{\omega}\circ\underset{\beta}{F} = \underset{\alpha\alpha\beta\gamma\gamma}{\varepsilon\,\varepsilon\,\varepsilon\omega} - \underset{\beta\alpha\beta}{\varepsilon\lambda\omega} = \underset{\beta\gamma\gamma}{\varepsilon\,\omega} - \underset{\beta\alpha\beta}{\varepsilon\lambda\omega} .$$

(taking into account the equality (3)).

Putting $\bar{X} = N$ into (13) and using (11) we obtain

$$\underset{\alpha}{F}(\underset{\beta}{\eta}) + \underset{\beta\beta\alpha}{\varepsilon\lambda\eta} + \underset{\alpha\alpha}{\varepsilon\omega}(\underset{\beta}{\eta})N + \underset{\alpha\beta\alpha\beta}{\varepsilon\varepsilon\lambda\lambda}N = \underset{\alpha\beta\gamma}{\varepsilon\;\eta} + \underset{\alpha\beta\gamma\gamma}{\varepsilon\;\eta\lambda}N .$$

It implies

$$
\begin{aligned}
\underset{\alpha}{F}(\underset{\beta}{\eta}) &= \underset{\alpha\beta\gamma}{\varepsilon\,\eta} - \underset{\beta\beta\alpha}{\varepsilon\lambda\eta},\\
\underset{\alpha}{\omega}(\underset{\beta}{\eta}) &= \underset{\beta\gamma\gamma}{\varepsilon\;\eta} - \underset{\beta\alpha\beta}{\varepsilon\lambda\lambda}.
\end{aligned}
$$

In this way we find: for $\alpha \neq \beta \neq \gamma \neq \alpha$

$$\left\{\begin{aligned} \underset{\alpha}{F} \circ \underset{\beta}{F} &= \underset{\alpha\beta}{\varepsilon}\underset{\gamma}{F} - \underset{\beta}{\varepsilon}\underset{\beta}{\omega} \otimes \underset{\alpha}{\eta} \\ \underset{\alpha}{\omega} \circ \underset{\beta}{F} &= \underset{\beta\gamma}{\varepsilon}\underset{\gamma}{\omega} - \underset{\beta}{\varepsilon}\underset{\alpha}{\lambda}\underset{\beta}{\omega} \\ \underset{\alpha}{F}(\underset{\beta}{\eta}) &= \underset{\alpha\beta}{\varepsilon}\underset{\gamma}{\eta} - \underset{\beta}{\varepsilon}\underset{\beta}{\lambda}\underset{\alpha}{\eta} \\ \underset{\alpha}{\omega}(\underset{\beta}{\eta}) &= \underset{\beta\gamma}{\varepsilon}\underset{\gamma}{\lambda} - \underset{\beta}{\varepsilon}\underset{\alpha}{\lambda}\underset{\beta}{\lambda} \end{aligned}\right. \tag{14}$$

Thus we have proved

Theorem 2. *The 3-structure $\{\underset{\alpha}{\tilde{F}}\}$ given on the $4n$-dimensional Riemannian manifold induces the 3-structure $\{\underset{\alpha}{F},\underset{\alpha}{\omega},\underset{\alpha}{\eta}\}$ on an orientable hypersurface which satisfies the conditions* (12) *and* (14).

Collolary. *A linear subspace spanned by the vectors $\underset{1}{\eta}$, $\underset{2}{\eta}$, $\underset{3}{\eta}$ is an invariant subspace with respect to linear mappings $\underset{\alpha}{F}$.*

Four types of 3–structures $\{\underset{\alpha}{\tilde{F}}\}$ on the $4n$–dimensional Riemannian manifold given on the pages 3 - -5 induce four types of 3–structures $\{\underset{\alpha}{F},\underset{\alpha}{\omega},\underset{\alpha}{\eta}\}$ on an orientable hypersurface, which will be called: I - almost contact 3–structure [3],
II - generalized almost contact 3–structure of the first kind,
III - generalized almost contact 3–structure of the second kind,
IV - generalized almost paracontact 3–structure,
respectively.

2. A metric induced on a hypersurface. Suppose that on a manifold M^{4n} with a 3–structure $\{\underset{\alpha}{\tilde{F}}\}$ there is a metric $\tilde{g}$ which satisfies condition (6):

$$\tilde{g}(\underset{\alpha}{\tilde{F}}(\tilde{X}),\underset{\alpha}{\tilde{F}}(\tilde{Y})) = \tilde{g}(\tilde{X},\tilde{Y}), \quad \tilde{X},\tilde{Y} \in TM^{4n}.$$

With respect to (7), (9), (10') we obtain

$$\tilde{g}(N,\underset{\alpha}{\tilde{F}}(N)) = \tilde{g}(N,\underset{\alpha}{\eta} + \underset{\alpha\alpha}{\varepsilon\lambda} N) = \underset{\alpha\alpha}{\varepsilon\lambda}.$$

On the other hand, using (6), (1) and the above equality we get

$$\tilde{g}(N,\underset{\alpha}{\tilde{F}}(N)) = \tilde{g}(\underset{\alpha}{\tilde{F}}(N),\underset{\alpha}{\varepsilon}N) = \underset{\alpha}{\varepsilon}\tilde{g}(N,\underset{\alpha}{\tilde{F}}(N)) = \underset{\alpha}{\lambda}.$$

Thus we have two cases:

1) $\underset{\alpha}{\varepsilon} = -1$. Then $\underset{\alpha}{\lambda} = 0$ and $\tilde{g}(N,\underset{\alpha}{\tilde{F}}(N)) = 0$.

2) $\underset{\alpha}{\varepsilon} = 1$. Then $\tilde{g}(N,\underset{\alpha}{\tilde{F}}(N)) = \underset{\alpha}{\lambda}$ For each type of the 3-structures we obtain

I. $\underset{1}{\lambda} = \underset{2}{\lambda} = \underset{3}{\lambda} = 0$,

II. $\underset{1}{\lambda} \neq 0$, $\underset{2}{\lambda} = \underset{3}{\lambda} = 0$,

III. $\underset{1}{\lambda} \neq 0\,,\ \underset{2}{\lambda} \neq 0\,,\ \underset{3}{\lambda} = 0\,,$

IV. $\underset{1}{\lambda} \neq 0\,,\ \underset{2}{\lambda} \neq 0\,,\ \underset{3}{\lambda} \neq 0\,.$

The submanifold M^{4n-1} will be considered with the metric g induced by $\tilde{g}$:

$$g(X,Y) = \tilde{g}(X,Y) \qquad \text{for } X,Y \in TM^{4n-1}.$$

Theorem 8. *The induced metric g satisfies the following conditions:*

$$\begin{aligned}
g(\underset{a}{F}(X),\underset{a}{F}(Y)) &= g(X,Y) - \underset{a}{\omega}(X)\underset{a}{\omega}(Y)\,, \qquad g(X,\underset{a}{\eta}) = \underset{a}{\omega}(X)\,,\\
g(\underset{a}{F}(X),\underset{\beta}{F}(Y)) &= \underset{a}{\varepsilon}\underset{a\beta}{\varepsilon} g(X,\underset{\gamma}{F}(Y)) - \underset{a\beta a}{\varepsilon\varepsilon\omega}(X)\underset{\beta}{\omega}(Y)\,, \quad \alpha \neq \beta \neq \gamma \neq \alpha\,, \qquad (16)\\
g(\underset{\beta}{\eta},\underset{a}{\eta}) &= \underset{a}{\omega}(\underset{\beta}{\eta})\,.
\end{aligned}$$

Proof. For $X,Y \in TM^{4n-1}$ with respect to (6), (7), (8), (9) we obtain

$$\begin{aligned}
g(\underset{a}{F}(X),\underset{a}{F}(Y)) = \tilde{g}\underset{a}{F}(X),\underset{a}{F}(Y)) &= \tilde{g}(\underset{a}{\tilde{F}}(X) - \underset{aa}{\varepsilon\omega}(X)N, \underset{a}{\tilde{F}}(Y) - \underset{aa}{\varepsilon\omega}(Y)N) =\\
&= \tilde{g}\underset{a}{\tilde{F}}(X),\underset{a}{\tilde{F}}(Y)) - \underset{aa}{\varepsilon\omega}(X)\tilde{g}(N,\underset{a}{\tilde{F}}(Y)) -\\
&\quad - \underset{aa}{\varepsilon\omega}(Y)\tilde{g}(\underset{a}{\tilde{F}}(X),N) + \underset{a}{\omega}(X)\underset{a}{\omega}(Y)\tilde{g}(N,N) =\\
&= \tilde{g}(X,Y) - \underset{aa}{\varepsilon\omega}(X)\tilde{g}(N,\underset{a}{F}(Y) + \underset{aa}{\varepsilon\omega}(Y)N)) -\\
&\quad - \underset{aa}{\varepsilon\omega}(Y)\tilde{g}(\underset{a}{F}(X) + \underset{aa}{\varepsilon\omega}(X)N,N) + \underset{a}{\omega}(X)\underset{a}{\omega}(Y) =\\
&= g(X,Y) - \underset{a}{\omega}(X)\underset{a}{\omega}(Y) - \underset{a}{\omega}(X)\underset{a}{\omega}(Y) + \underset{a}{\omega}(X)\underset{a}{\omega}(Y) =\\
&= g(X,Y) - \underset{a}{\omega}(X)\underset{a}{\omega}(Y)\,.
\end{aligned}$$

In the similar way we can prove the equality:

$$\begin{aligned}
g(X,\underset{a}{\eta}) &= \tilde{g}(X,\underset{a}{\tilde{F}}(N) - \underset{aa}{\varepsilon\lambda}N) = \tilde{g}(X,\underset{a}{\tilde{F}}(N)) = \tilde{g}(\underset{a}{\tilde{F}}(X),\underset{a}{\tilde{F}}^2(N)) =\\
&= \tilde{g}(\underset{a}{F}(X) + \underset{aa}{\varepsilon\omega}(X)N,\underset{a}{\varepsilon}N) = \underset{a}{\omega}(X)\tilde{g}(N,N) = \underset{a}{\omega}(X)\,.
\end{aligned}$$

The third equality is obtained analogously. The fourth equality (16) directly results from the second one.

Similary, taking into account (16_4), (12_4) we obtain

$$g(\underset{a}{\eta},\underset{a}{\eta}) = \underset{a}{\omega}(\underset{a}{\eta}) = 1 - \underset{a}{\varepsilon}(\underset{a}{\lambda})^2\,.$$

The above equality implies the following conditions for $\underset{a}{\varepsilon} = 1$ and Riemannian metric g:

$$-1 < \underset{a}{\lambda} < 1\,, \qquad \underset{a}{\lambda} \neq 0\,, \qquad \underset{a}{\omega}(\underset{a}{\eta}) > 0\,.$$

The second equality (16) and (14) give us

$$g(\underset{\beta}{\eta},\underset{\alpha}{\eta}) = \underset{\alpha}{\omega}(\underset{\beta}{\eta}) = \underset{\beta\gamma}{\varepsilon}\underset{\gamma}{\lambda} - \underset{\beta}{\varepsilon}\underset{\alpha}{\lambda}\underset{\beta}{\lambda}\,.$$

Thus for each 3-structures we obtain

$$\begin{aligned} \text{I.}\quad g(\underset{\beta}{\eta},\underset{\alpha}{\eta}) &= g(\underset{\alpha}{\eta},\underset{\beta}{\eta}) = 0\,,\\ g(\underset{\alpha}{\eta},\underset{\alpha}{\eta}) &= 1, \quad \alpha \neq \beta\,. \end{aligned}$$

$$\begin{aligned} \text{II.}\quad g(\underset{1}{\eta},\underset{2}{\eta}) &= g(\underset{2}{\eta},\underset{1}{\eta}) = 0\,,\\ g(\underset{1}{\eta},\underset{3}{\eta}) &= g(\underset{3}{\eta},\underset{1}{\eta}) = 0\,,\\ g(\underset{2}{\eta},\underset{3}{\eta}) &= g(\underset{3}{\eta},\underset{2}{\eta}) = \varepsilon\underset{1}{\lambda}\,,\\ g(\underset{1}{\eta},\underset{1}{\eta}) &= 1 - (\underset{1}{\lambda})^2\,,\\ g(\underset{2}{\eta},\underset{2}{\eta}) &= g(\underset{3}{\eta},\underset{3}{\eta}) = \text{[illegible]} \end{aligned}$$

$$\begin{aligned} \text{III.}\quad g(\underset{1}{\eta},\underset{2}{\eta}) &= g(\underset{2}{\eta},\underset{1}{\eta}) = \text{[illegible]}\,,\\ g(\underset{1}{\eta},\underset{3}{\eta}) &= g(\underset{3}{\eta},\underset{1}{\eta}) = \varepsilon\underset{2}{\lambda}\,,\\ g(\underset{2}{\eta},\underset{3}{\eta}) &= g(\underset{3}{\eta},\underset{2}{\eta}) = -\varepsilon\underset{1}{\lambda}\,,\\ g(\underset{1}{\eta},\underset{1}{\eta}) &= 1 - (\underset{1}{\lambda})^2\,,\\ g(\underset{2}{\eta},\underset{2}{\eta}) &= 1 - (\underset{2}{\lambda})^2\,,\\ g(\underset{3}{\eta},\underset{3}{\eta}) &= 1\,. \end{aligned}$$

$$\begin{aligned} \text{IV.}\quad g(\underset{\beta}{\eta},\underset{\alpha}{\eta}) &= g(\underset{\alpha}{\eta},\underset{\beta}{\eta}) = \varepsilon\underset{\gamma}{\lambda} - \underset{\alpha}{\lambda}\underset{\beta}{\lambda}\,,\\ g(\underset{\alpha}{\eta},\underset{\alpha}{\eta}) &= 1 - (\underset{\alpha}{\lambda})^2\,, \quad \alpha \neq \beta \neq \gamma \neq \alpha\,. \end{aligned}$$

We will investigate the linear independence of the vector fields $\underset{1}{\eta},\underset{2}{\eta},\underset{3}{\eta}$. Let

$$\overset{1}{a}\underset{1}{\eta} + \overset{2}{a}\underset{2}{\eta} + \overset{3}{a}\underset{3}{\eta} = 0\,, \qquad \overset{1}{a},\overset{2}{a},\overset{3}{a} \in R\,.$$

Making use of (12_4) and (14_4) we obtain

$$\begin{aligned} &\overset{1}{a}(1 - \underset{1}{\varepsilon}(\underset{1}{\lambda})^2) + \overset{2}{a}(\underset{23}{\varepsilon}\underset{3}{\lambda} - \underset{2}{\varepsilon}\underset{1}{\lambda}\underset{2}{\lambda}) + \overset{3}{a}(\underset{32}{\varepsilon}\underset{2}{\lambda} - \underset{3}{\varepsilon}\underset{1}{\lambda}\underset{3}{\lambda}) = 0\\ &\overset{1}{a}(\underset{13}{\varepsilon}\underset{3}{\lambda} - \underset{1}{\varepsilon}\underset{1}{\lambda}\underset{2}{\lambda}) + \overset{2}{a}(1 - \underset{2}{\varepsilon}(\underset{2}{\lambda})^2) + \overset{3}{a}(\underset{31}{\varepsilon}\underset{1}{\lambda} - \underset{3}{\varepsilon}\underset{2}{\lambda}\underset{3}{\lambda}) = 0\\ &\overset{1}{a}(\underset{12}{\varepsilon}\underset{2}{\lambda} - \underset{1}{\varepsilon}\underset{1}{\lambda}\underset{3}{\lambda}) + \overset{2}{a}(\underset{21}{\varepsilon}\underset{1}{\lambda} - \underset{2}{\varepsilon}\underset{2}{\lambda}\underset{3}{\lambda}) + \overset{3}{a}(1 - \underset{3}{\varepsilon}(\underset{3}{\lambda})^2) = 0. \end{aligned}$$

We must compute the determinant of the matrix

$$A=\begin{bmatrix} 1-\underset{1}{\varepsilon}(\underset{1}{\lambda})^2 & \underset{23}{\varepsilon}\underset{3}{\lambda}-\underset{2}{\varepsilon}\underset{1}{\lambda}\underset{2}{\lambda} & \underset{32}{\varepsilon}\underset{2}{\lambda}-\underset{3}{\varepsilon}\underset{1}{\lambda}\underset{3}{\lambda}) \\ \underset{13}{\varepsilon}\underset{3}{\lambda}-\underset{1}{\varepsilon}\underset{1}{\lambda}\underset{2}{\lambda} & 1-\underset{2}{\varepsilon}(\underset{2}{\lambda})^2 & \underset{31}{\varepsilon}\underset{1}{\lambda}-\underset{3}{\varepsilon}\underset{2}{\lambda}\underset{3}{\lambda}) \\ \underset{12}{\varepsilon}\underset{2}{\lambda}-\underset{1}{\varepsilon}\underset{1}{\lambda}\underset{3}{\lambda} & \underset{21}{\varepsilon}\underset{1}{\lambda}-\underset{2}{\varepsilon}\underset{2}{\lambda}\underset{3}{\lambda} & 1-\underset{3}{\varepsilon}(\underset{3}{\lambda})^2 \end{bmatrix}$$

$$\det A = 1+(\underset{1}{\lambda})^4+(\underset{2}{\lambda})^4+(\underset{3}{\lambda})^4+$$

$$\underset{1}{\lambda}\underset{2}{\lambda}\underset{3}{\lambda}(\underset{1}{\varepsilon}\underset{23}{\varepsilon}+\underset{1}{\varepsilon}\underset{32}{\varepsilon}+\underset{2}{\varepsilon}\underset{13}{\varepsilon}+\underset{2}{\varepsilon}\underset{31}{\varepsilon}+\underset{3}{\varepsilon}\underset{12}{\varepsilon}+\underset{3}{\varepsilon}\underset{21}{\varepsilon}+\underset{12}{\varepsilon}\underset{23}{\varepsilon}\underset{31}{\varepsilon}+\underset{13}{\varepsilon}\underset{32}{\varepsilon}\underset{21}{\varepsilon})-$$

$$-2(\underset{1}{\varepsilon}(\underset{1}{\lambda})^2+\underset{2}{\varepsilon}(\underset{2}{\lambda})^2+\underset{3}{\varepsilon}(\underset{3}{\lambda})^2-2(\underset{1}{\varepsilon}(\underset{2}{\lambda}\underset{3}{\lambda})^2+\underset{2}{\varepsilon}(\underset{1}{\lambda}\underset{3}{\lambda})^2+\underset{3}{\varepsilon}(\underset{1}{\lambda}\underset{2}{\lambda})^2\ .$$

For the I type of the 3-structures we have:

$$\det A = 1$$

Thus $\overset{1}{a}=\overset{2}{a}=\overset{3}{a}$ and the vector fields $\underset{1}{\eta},\underset{2}{\eta},\underset{3}{\eta}$ are linearly independent.

For the II type we obtain:

$$\det A = 1+(\underset{1}{\lambda})^4-2(\underset{1}{\lambda})^2=[1-(\underset{1}{\lambda})^2]^2.$$

The vector fields $\underset{1}{\eta},\underset{2}{\eta},\underset{3}{\eta}$ are linearly independent iff

$$1-(\underset{1}{\lambda})^2\neq 0\ .$$

For the III type we find:

$$\det A = 1+(\underset{1}{\lambda})^4+(\underset{2}{\lambda})^4-2((\underset{1}{\lambda})^2+(\underset{2}{\lambda})^2)+2(\underset{1}{\lambda}\underset{2}{\lambda})^2=[1-(\underset{1}{\lambda})^2-(\underset{2}{\lambda})^2]^2.$$

The vector fields $\underset{1}{\eta},\underset{2}{\eta},\underset{3}{\eta}$ are linearly independent iff

$$1-(\underset{1}{\lambda})^2-(\underset{2}{\lambda})^2\neq 0\ .$$

For the IV type we have:

$$\begin{aligned}\det A\ &= 1+(\underset{1}{\lambda})^4+(\underset{2}{\lambda})^4+(\underset{3}{\lambda})^4+8\varepsilon\underset{1}{\lambda}\underset{2}{\lambda}\underset{3}{\lambda}-2((\underset{1}{\lambda})^2+(\underset{2}{\lambda})^2+(\underset{3}{\lambda})^2)-\\ &\quad -2((\underset{1}{\lambda}\underset{2}{\lambda})^2+(\underset{1}{\lambda}\underset{3}{\lambda})^2+(\underset{2}{\lambda}\underset{3}{\lambda})^2)=\\ &= [1-(\underset{1}{\lambda})^2-(\underset{2}{\lambda})^2-(\underset{3}{\lambda})^2]^2+8\varepsilon\underset{1}{\lambda}\underset{2}{\lambda}\underset{3}{\lambda}-4[(\underset{1}{\lambda}\underset{2}{\lambda})^2+(\underset{1}{\lambda}\underset{3}{\lambda})^2+(\underset{2}{\lambda}\underset{3}{\lambda})^2]\ .\end{aligned}$$

The vector fields $\underset{1}{\eta},\underset{2}{\eta},\underset{3}{\eta}$ are linearly independent iff

$$[1-(\underset{1}{\lambda})^2-(\underset{2}{\lambda})^2-(\underset{3}{\lambda})^2]^2+8\varepsilon\underset{1}{\lambda}\underset{2}{\lambda}\underset{3}{\lambda}-4[(\underset{1}{\lambda}\underset{2}{\lambda})^2+(\underset{1}{\lambda}\underset{3}{\lambda})^2+(\underset{2}{\lambda}\underset{3}{\lambda})^2]\neq 0\,.$$

We have proved

Theorem 4. *The vector fields $\underset{1}{\eta},\underset{2}{\eta},\underset{3}{\eta}$ are linearly independent for the I type. For the types II, III, IV, we have: the vector fields $\underset{1}{\eta},\underset{2}{\eta},\underset{3}{\eta}$ are linearly independent iff*

$$1-(\underset{1}{\lambda})^2\neq 0 \qquad \text{(II type)}$$

$$1-(\underset{1}{\lambda})^2-(\underset{2}{\lambda})^2\neq 0 \qquad \text{(III type)}$$

$$[1-(\underset{1}{\lambda})^2-(\underset{2}{\lambda})^2-(\underset{3}{\lambda})^2]^2+8\varepsilon\underset{1}{\lambda}\underset{2}{\lambda}\underset{3}{\lambda}-4[(\underset{1}{\lambda}\underset{2}{\lambda})^2+(\underset{1}{\lambda}\underset{3}{\lambda})^2+(\underset{2}{\lambda}\underset{3}{\lambda})^2]\neq 0 \quad \text{(IV type)}$$

3. A metric on the submanifold invariant with respect to $\underset{a}{F}$. On a Riemannian submanifold M^{4n-1} with the 3–structure $\{\underset{a}{F},\underset{a}{\omega},\underset{a}{\eta}\}$ we can define such metric $\hat{g}$ that

$$\hat{g}(\underset{a}{F}(X),\underset{a}{F}(Y))=\hat{g}(X,Y) \tag{17}$$

for arbitrary $X,Y\in TM^{4n-1}$.

Considering an arbitrary metric g on M^{4n-1} which satisfies (16) we shall look for a metric $\hat{g}$ on M^{4n-1} of the following form

$$\begin{aligned}\hat{g}(X,Y) &= g(X,Y)+A[\underset{1}{\omega}(X)\underset{1}{\omega}(Y)+\underset{2}{\omega}(X)\underset{2}{\omega}(Y)+\underset{3}{\omega}(X)\underset{3}{\omega}(Y)]+B[\underset{2}{\omega}(X)\underset{3}{\omega}(Y)]+\\ &+\underset{3}{\omega}(X)\underset{2}{\omega}(Y)]+C[\underset{1}{\omega}(X)\underset{3}{\omega}(Y)+\underset{3}{\omega}(X)\underset{1}{\omega}(Y)]+D[\underset{1}{\omega}(X)\underset{2}{\omega}(Y)+\underset{2}{\omega}(X)\underset{1}{\omega}(Y)]\,.\end{aligned}$$

We will choose the coefficients A,B,C,D in such a way that they satisfy (17). We must consider the system of linear equations

$$\begin{cases} A=-1+A[(\underset{1}{\lambda})^2+(\underset{2}{\lambda})^2+(\underset{3}{\lambda})^2]+2B\underset{2}{\lambda}\underset{3}{\lambda}+2C\underset{1}{\lambda}\underset{3}{\lambda}+2D\underset{1}{\lambda}\underset{2}{\lambda}\\ B=-\underset{2}{\varepsilon}\underset{23}{\varepsilon}[A\underset{1}{\lambda}+D\underset{2}{\lambda}+C\underset{3}{\lambda}]\\ C=-\underset{1}{\varepsilon}\underset{13}{\varepsilon}[D\underset{1}{\lambda}+A\underset{2}{\lambda}+B\underset{3}{\lambda}]\\ D=-\underset{1}{\varepsilon}\underset{12}{\varepsilon}[C\underset{1}{\lambda}+B\underset{2}{\lambda}+A\underset{3}{\lambda}]\\ B=\underset{1}{\varepsilon}B\\ C=\underset{2}{\varepsilon}C\\ D=\underset{3}{\varepsilon}D\,. \end{cases} \tag{19}$$

We will find a solution of the system (19) for each of four types of the 3-structures:

I. $A=-1, B=0, C=0, D=0$ is a solution of (19). Thus we have

$$\hat{g}(X,Y)=g(X,Y)-[\underset{1}{\omega}(X)\underset{1}{\omega}(Y)+\underset{2}{\omega}(X)\underset{2}{\omega}(Y)+\underset{3}{\omega}(X)\underset{3}{\omega}(Y)]$$

II. We can rewrite (19) in the form

$$\begin{cases} A=-1+A(\underset{1}{\lambda})^2 \\ B=-\varepsilon\underset{1}{\lambda}A \\ C=0 \\ D=0\,. \end{cases}$$

The solution of this system exists, iff

$$(\underset{1}{\lambda})^2\neq 1 \tag{21}$$

and it has the form

$$A=\frac{1}{(\underset{1}{\lambda})^2-1}\,,$$

$$B=\frac{-\varepsilon\underset{1}{\lambda}}{(\underset{1}{\lambda})^2-1}\,,$$

$$C=0\,,$$

$$D=0\,.$$

Thus

$$\begin{aligned}\hat{g}(X,Y) &= g(X,Y)+\frac{1}{(\underset{1}{\lambda})^2-1}[\underset{1}{\omega}(X)\underset{1}{\omega}(Y)+\underset{2}{\omega}(X)\underset{2}{\omega}(Y)+\underset{3}{\omega}(X)\underset{3}{\omega}(Y)]- \\ &-\frac{\varepsilon\underset{1}{\lambda}}{(\underset{1}{\lambda})^2-1}[\underset{2}{\omega}(X)\underset{3}{\omega}(Y)+\underset{3}{\omega}(X)\underset{2}{\omega}(Y)]\,.\end{aligned} \tag{22}$$

III. We can rewrite (19) in the form

$$\begin{cases} A=-1+A[(\underset{1}{\lambda})^2+(\underset{2}{\lambda})^2] \\ B=\varepsilon\underset{1}{\lambda}A \\ C=-\varepsilon\underset{2}{\lambda}A \\ D=0\,. \end{cases}$$

The solution of this system exists, iff

$$(\underset{1}{\lambda})^2 + (\underset{2}{\lambda})^2 \neq 1 \tag{23}$$

and it has the form

$$\begin{aligned}
A &= \frac{1}{(\underset{1}{\lambda})^2 + (\underset{2}{\lambda})^2 - 1}, \\
B &= \frac{\varepsilon \underset{1}{\lambda}}{(\underset{1}{\lambda})^2 + (\underset{2}{\lambda})^2 - 1}, \\
C &= \frac{-\varepsilon \underset{2}{\lambda}}{(\underset{1}{\lambda})^2 + (\underset{2}{\lambda})^2 - 1}, \\
D &= 0 .
\end{aligned}$$

Thus

$$\begin{aligned}
\hat{g}(X,Y) &= g(X,Y) + \\
&+ \frac{1}{(\underset{1}{\lambda})^2 + (\underset{2}{\lambda})^2 - 1}\left[\underset{1}{\omega}(X)\underset{1}{\omega}(Y) + \underset{2}{\omega}(X)\underset{2}{\omega}(Y) + \underset{3}{\omega}(X)\underset{3}{\omega}(Y)\right] + \\
&+ \frac{\varepsilon \underset{1}{\lambda}}{(\underset{1}{\lambda})^2 + (\underset{2}{\lambda})^2 - 1}\left[\underset{2}{\omega}(X)\underset{3}{\omega}(Y) + \underset{3}{\omega}(X)\underset{2}{\omega}(Y)\right] - \\
&- \frac{\varepsilon \underset{2}{\lambda}}{(\underset{1}{\lambda})^2 + (\underset{2}{\lambda})^2 - 1}\left[\underset{1}{\omega}(X)\underset{3}{\omega}(Y) + \underset{3}{\omega}(X)\underset{1}{\omega}(Y)\right] .
\end{aligned} \tag{24}$$

IV. We can rewrite (19) in the form

$$\left\{\begin{aligned}
A &= -1 + A[(\underset{1}{\lambda})^2 + (\underset{2}{\lambda})^2 + (\underset{3}{\lambda})^2] + 2B\underset{2}{\lambda}\underset{3}{\lambda} + 2C\underset{1}{\lambda}\underset{3}{\lambda} + 2D\underset{1}{\lambda}\underset{2}{\lambda} \\
B &= -\varepsilon[A\underset{1}{\lambda} + D\underset{2}{\lambda} + C\underset{3}{\lambda}] \\
C &= -\varepsilon[D\underset{1}{\lambda} + A\underset{2}{\lambda} + B\underset{3}{\lambda}] \\
D &= -\varepsilon[C\underset{1}{\lambda} + B\underset{2}{\lambda} + A\underset{3}{\lambda}]
\end{aligned}\right.$$

The solution of this system exists, iff

$$[(\underset{1}{\lambda})^2 + (\underset{2}{\lambda})^2 + (\underset{3}{\lambda})^2 - 1]^2 + 8\varepsilon\underset{1}{\lambda}\underset{2}{\lambda}\underset{3}{\lambda} - 4[(\underset{1}{\lambda}\underset{2}{\lambda})^2 + (\underset{1}{\lambda}\underset{3}{\lambda})^2 + (\underset{2}{\lambda}\underset{3}{\lambda})^2] \neq 0 \tag{25}$$

and it has the form

$$
\begin{aligned}
A &= \frac{(\underset{1}{\lambda})^2 + (\underset{2}{\lambda})^2 + (\underset{3}{\lambda})^2 - 1 - 2\varepsilon\underset{1}{\lambda}\underset{2}{\lambda}\underset{3}{\lambda}}{[(\underset{1}{\lambda})^2 + (\underset{2}{\lambda})^2 + (\underset{3}{\lambda})^2 - 1]^2 + 8\varepsilon\underset{1}{\lambda}\underset{2}{\lambda}\underset{3}{\lambda} - 4[(\underset{1}{\lambda}\underset{2}{\lambda})^2 + (\underset{1}{\lambda}\underset{3}{\lambda})^2 + (\underset{2}{\lambda}\underset{3}{\lambda})^2]}\\
B &= \frac{\varepsilon\underset{1}{\lambda}[1 - (\underset{1}{\lambda})^2 + (\underset{2}{\lambda})^2 + (\underset{3}{\lambda})^2] - 2\underset{2}{\lambda}\underset{3}{\lambda}}{[(\underset{1}{\lambda})^2 + (\underset{2}{\lambda})^2 + (\underset{3}{\lambda})^2 - 1]^2 + 8\varepsilon\underset{1}{\lambda}\underset{2}{\lambda}\underset{3}{\lambda} - 4[(\underset{1}{\lambda}\underset{2}{\lambda})^2 + (\underset{1}{\lambda}\underset{3}{\lambda})^2 + (\underset{2}{\lambda}\underset{3}{\lambda})^2]}\\
C &= \frac{\varepsilon\underset{2}{\lambda}[1 + (\underset{1}{\lambda})^2 - (\underset{2}{\lambda})^2 + (\underset{3}{\lambda})^2] - 2\underset{1}{\lambda}\underset{3}{\lambda}}{[(\underset{1}{\lambda})^2 + (\underset{2}{\lambda})^2 + (\underset{3}{\lambda})^2 - 1]^2 + 8\varepsilon\underset{1}{\lambda}\underset{2}{\lambda}\underset{3}{\lambda} - 4[(\underset{1}{\lambda}\underset{2}{\lambda})^2 + (\underset{1}{\lambda}\underset{3}{\lambda})^2 + (\underset{2}{\lambda}\underset{3}{\lambda})^2]}\\
D &= \frac{\varepsilon\underset{3}{\lambda}[1 + (\underset{1}{\lambda})^2 + (\underset{2}{\lambda})^2 - (\underset{3}{\lambda})^2] - 2\underset{1}{\lambda}\underset{2}{\lambda}}{[(\underset{1}{\lambda})^2 + (\underset{2}{\lambda})^2 + (\underset{3}{\lambda})^2 - 1]^2 + 8\varepsilon\underset{1}{\lambda}\underset{2}{\lambda}\underset{3}{\lambda} - 4[(\underset{1}{\lambda}\underset{2}{\lambda})^2 + (\underset{1}{\lambda}\underset{3}{\lambda})^2 + (\underset{2}{\lambda}\underset{3}{\lambda})^2]}
\end{aligned}
\tag{26}
$$

The metric $\hat{g}$ has the form (18) with coefficients A, B, C, D given by (26). Theorem 4 and the conditions (21), (23), (25) imply the linear independence of the vector fields $\underset{1}{\eta}, \underset{2}{\eta}, \underset{3}{\eta}$.

Theorem 5. *Let $\{\underset{a}{F}, \underset{a}{\omega}, \underset{a}{\eta}\}$ be a 3-structure on the submanifold M^{4n-1}. Then there exists the metric $\hat{g}$ which satisfies the condition* (17) *iff the vector fields $\underset{1}{\eta}, \underset{2}{\eta}, \underset{3}{\eta}$ are linearly independent. The metric $\hat{g}$ is given by* (20) *- I type,* (22) *- II type,* (24) *- III type,* (18) *with coefficients A, B, C, D given by* (26) *- IV type.*

4. Eigenvectors of the tensors $\underset{a}{\tilde{F}}$ and $\underset{a}{F}$. The real eigenvalues do not exists for $\underset{a}{\tilde{F}}, \underset{a}{\tilde{F}}^2 = -\mathrm{I}$ Let us assume that

$$\underset{a}{\tilde{F}}^2 = \mathrm{I} \tag{27}$$

By $\kappa, \tilde{X}$ we denote the real eigenvalue and eigenvector of the tensor $\underset{a}{\tilde{F}}$, respectively. Thus $\underset{a}{\tilde{F}}(X) = \kappa\tilde{X}$. Hence using (27) we obtain

$$\tilde{X} = \underset{a}{\tilde{F}}^2(\tilde{X}) = \kappa\underset{a}{\tilde{F}}(\tilde{X}) = \kappa^2\tilde{X}$$

and

$$\kappa^2 = 1\ . \tag{28}$$

I. Let us assume $\tilde{X} = N$. Making use of (8) and (11) we get

$$\underset{a}{F}(N) = \underset{a}{\eta} = 0\ ,$$

$$\underset{a}{\omega}(N) = \underset{a}{\lambda} = \kappa\ .$$

Then the relations (12) can be rewrite as follows

$$\underset{a}{F}^2 = \mathrm{I}\ ,$$

$$\underset{a}{F}(\underset{a}{\eta}) = 0 ,$$

$$\underset{a}{\omega}(\underset{a}{\eta}) = 0 ,$$

$$\underset{a}{\omega} \circ \underset{a}{F} = -\kappa \underset{a}{\omega} .$$

Thus the 3–structure $\{\underset{a}{F}, \underset{a}{\omega}, \underset{a}{\eta}\}$ on M^{4n-1} is not complete.

II. Let us assume

$$\underset{a}{\tilde{F}}(\tilde{X}) = \kappa \tilde{X} , \qquad \tilde{X} \neq N . \tag{29}$$

The equality (8) implies

$$\underset{a}{F}(\tilde{X}) + \underset{a}{\omega}(\tilde{X}) N = \kappa \tilde{X} .$$

Hence using (28), (29) and (12) we obtain

$$\underset{a}{F}(\kappa \underset{a}{\tilde{F}}(\tilde{X})) + \underset{a}{\omega}(\tilde{X}) N = \kappa \tilde{X} ,$$

$$\kappa \underset{a}{F}(\underset{a}{F}(\tilde{X}) + \underset{a}{\omega}(\tilde{X}) N) + \underset{a}{\omega}(\tilde{X}) N = \kappa \tilde{X} ,$$

$$\kappa \tilde{X} - \kappa \underset{a}{\omega}(\tilde{X}) \underset{a}{\eta} + \kappa \underset{a}{\omega}(\tilde{X}) \underset{a}{\eta} + \underset{a}{\omega}(\tilde{X}) N + \underset{a}{\omega}(\tilde{X}) N = \kappa \tilde{X} .$$

So $\underset{a}{\omega}(\tilde{X}) = 0$ and $\underset{a}{F}(\tilde{X}) = \kappa \tilde{X}$. Therefore $\tilde{X} \in TM^{4n-1}$.

Thus we proved that the eigenvector $\tilde{X}$ of $\underset{a}{\tilde{F}}$ is the eigenvector of $\underset{a}{F}$.

It implies that $\tilde{X} \in TM^{4n-1}$.

We will find all eigenvectors of $\underset{a}{F}$. Let $\tilde{X} \in TM^{4n-1}$ and

$$\underset{a}{F}(X) = \rho X \tag{30}$$

for some real ρ. Making use of the first equality (12) and (30) we get

$$\begin{aligned} &\underset{a}{F}^2(X) = \underset{a}{\varepsilon}(X - \underset{a}{\omega}(X)\underset{a}{\eta}) , && \underset{a}{F}^2(X) = \rho \underset{a}{F}(X) = \rho^2 X , \\ &\underset{a}{\varepsilon}(X - \underset{a}{\omega}(X)\underset{a}{\eta}) = \rho^2 X , && (1 - \underset{a}{\varepsilon}\rho^2) X = \underset{a}{\omega}(X)\underset{a}{\eta} . \end{aligned} \tag{31}$$

The last equality of (12) and (30) imply

$$(\underset{a}{\omega} \circ \underset{a}{F})(X) = \rho \underset{a}{\omega}(X) , \quad -\underset{a}{\varepsilon}\underset{a}{\lambda}\underset{a}{\omega}(X) = \rho \underset{a}{\omega}(X) , \quad (\rho + \underset{a}{\varepsilon}\underset{a}{\lambda})\underset{a}{\omega}(X) = 0 . \tag{32}$$

Thus we have two possibilities:

$$1. \quad \rho = -\underset{a}{\varepsilon}\underset{a}{\lambda} .$$

Then the equality (31) can be rewritten in the form

$$(1 - \underset{a}{\varepsilon}(\underset{a}{\lambda})^2) X = \underset{a}{\omega}(X)\underset{a}{\eta} , \qquad \underset{a}{\omega}(\underset{a}{\eta}) X = \underset{a}{\omega}(X)\underset{a}{\eta} .$$

(see (12)). The above relation implies that an arbitrary eigenvector of $\underset{a}{F}$ and the vector $\underset{a}{\eta} = \underset{a}{F}(N)$ are linearly dependent (In the case $\underset{a}{\varepsilon} = 1$ we must assume that $(\underset{a}{\lambda})^2 \neq 1$).

$$2. \quad \underset{a}{\omega}(X) = 0 .$$

Then $1 - \underset{a}{\varepsilon}\rho^2 = 0$. The real enginenvalues do not exist for $\underset{a}{\varepsilon} = -1$.
For $\underset{a}{\varepsilon} = 1$ we have $\rho = 1$ or $\rho = -1$ and we compute an eigenvector X from the condition $\underset{a}{\omega}(X) = 0$.

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STRESZCZENIE

Niech M^{4n} będzie 4n-wymiarową różniczkowalną rozmaitością Riemanna z zadaną na niej 3-strukturą $\{\underset{a}{\bar F}\}$ będącą uogólnieniem struktur rozważanych w pracach [1] i [2]. 3-struktura ta indukuje na hiperpowierzchni M^{4n-1} zanurzonej w M^{4n} pewną 3-strukturę $\{\underset{a}{F}, \underset{a}{\omega}, \underset{a}{\eta}\}$. W pracy tej badane są algebraiczne własności tej 3-struktury na hiperpowierzchni M^{4n-1}. W szczególności rozważania dotyczą metryki na M^{4n-1} indukowanej przez metrykę na M^{4n} niezmienniczą względem 3-struktury $\{\underset{a}{\bar F}\}$. Następnie wyprowadzone zostały warunki istnienia i postać metryki na M^{4n-1} niezmienniczej względem tensorów $\underset{a}{F}$. Rozważania w końcowej części pracy dotyczą wektorów własnych tensorów $\underset{a}{\bar F}$ i $\underset{a}{F}$.

РЕЗЮМЕ

Пусть M^{4n} есть $4n$-мерное дифференциальное римановое многообразие с заданной на нем обобщённой 3-структурой $\{\underset{a}{\bar F}\}$. Эта 3-структура индуцирует на гиперповерхности M^{4n-1} погруженной в M^{4n} некоторую 3-структуру $\{\underset{a}{F}, \underset{a}{\omega}, \underset{a}{\eta}\}$. В данной работе изучены алгебраические свойства 3-структуры на M^{4n-1}. В частном случае рассматриваем метрику на M^{4n-1} индуцированную метрикой на M^{4n} инвариантной относительно 3-структуры $\{\underset{a}{\bar F}\}$. Далее представлены условия существования и форма метрики $\{\underset{a}{F}, \underset{a}{\omega}, \underset{a}{\eta}\}$ на M^{4n-1} инвариантной относительно тензоров $\{\underset{a}{F}\}$. Дальнейшие рассуждения касаются собственных векторов для тензоров $\{\underset{a}{\bar F}\}$ и $\{\underset{a}{F}\}$.