ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN-POLONIA
VOL. XXXVIII, 8 SECTIO A 1984

Instytut Matematyki
Uniwersytet Wrocławski im. Bolesława Bieruta

H. MARCINKOWSKA

A Remark about the Surfaces of Lapunov Type

Uwaga o powierzchniach typu Lapunova

Замечание о поверхностях типа Лапунова

In the study of Dirichlet and Neumann problem for the Laplace operator bounded domains with boundary being a Lapunov surface play an essential role. In this class of domains namely the above boundary value problems may be reduced to solving integral equations of Fredholm type on the boundary. Usually the conditions of Lapunov for an $(n-1)$-dimensional surface $\Sigma \subset R^n$ of class C^1 are precised in the following way (see e.g. [2]):

i) there are positive constants c, κ $(\kappa \leqslant 1)$ such that

$$|\sphericalangle (n(x), n(y))| \leqslant c\,|x-y|^{\kappa}$$

where $n(x)$ denotes the continuous field of the unit normal vectors on Σ;

ii) there is a $\delta > 0$ such that for $x \in \Sigma$ each straight line parallel to $n(x)$ has at most one point in common with $\Sigma \cap K(x, \delta)$; here $K(x, \delta)$ denotes the ball with center x and radius δ.

The purpose of this note is to show that condition ii) is satisfied by each compact surface of class C^1, so it may be omitted in the definition of the Lapunov surface. We prove namely.

Theorem. *Let $\Sigma \subset R^n$ be a compact surface of class C^1 and $x \in \Sigma$ an arbitrarily fixed point. Let us choose the rectangular system of coordinate axes $(y_1, \ldots, y_n)$ in such a way that x would be the origin and $n(x)$ the unit vector of the y_n-axis. Then there are positive constants d, M (not depending on x) such that the part $\Sigma \cap K(x, d)$ may be described by the equation*

$$y_n = f(y') \tag{1}$$

where f is of class C^1 and $|D_j f(y')| \leqslant M$ for $j = 1, \ldots, n-1$ and $|y'| \leqslant d$.

The proof is based on the following slight modification of the well known implicit functions theorem.

Lemma. *Let us consider the function* $F(t, p, q)$ $(t \in R^k,\ p \in R^m,\ q \in R)$ *defined in a product* $A \times 0$, *where* A *is an arbitrary subset of* R^k *and* 0 *an* $(m+1)$*-dimensional neighbourhood of the point* $(\mathring{p}, \mathring{q})$. *We make the following assumptions:*

a) $F(t, \ldots) \in C^1(0)$ *for each fixed* $t \in A$,

b) $F(t, \ldots)$ *and* $D_q F(t, \ldots)$ *are continuous in* 0, *uniformly with respect to* $(t, p, q) \in \in A \times 0$,

c) *for each fixed* $t \in A$

$$F(t, \mathring{p}, \mathring{q}) = 0$$

and

$$D_q F(t, \mathring{p}, \mathring{q}) \geqslant a > 0$$

with a *not depending on* t.

Then there is a positive constant d *(not depending on* t*) such that*

1) *for every fixed* $t \in A$ *the equation* $F(t, p, q) = 0$ *has exactly one solution* $q = f_{(t)}(p)$ *defined and of class* C^1 *in the ball* $|p - \mathring{p}| \leqslant d$.

2) *the inequality*

$$D_q F(t, p, q) \geqslant a/2$$

holds in some neighbourhood of the graph of the function $f_{(t)}$.

The proof goes in exactly this same way as the usually given one in the case of function F not depending on t (see e.g. [1]), so it may be left to the reader.

Proof of the theorem. It is sufficient to consider the case, where Σ is described by the equation

$$z_n - g(z') = 0 \qquad (z' \in \Delta) \tag{2}$$

with $\bar{\Delta} \subset \Delta_0$ (Δ, Δ_0 two bounded domains of R^{n-1}) and $g \in C^1(\bar{\Delta}_0)$. In the general case Σ is a finite union of surfaces having the above property.

The unit normal vector to Σ in the point $z = (z', z_n)$ has the form

$$n(z') = (1 + |\operatorname{grad} g|^2)^{-1/2}(-D_1 g, \ldots, -D_n g, 1) \tag{3}$$

We introduce the new system of y-coordinates by means of the formulas

$$z_k = \sum_{j=1}^{n} a_{kj}(x') y_j + x_k \qquad (k = 1, \ldots, n) \tag{4}$$

where the columnas of the matrix $[a_{kj}]$ are formed by the vectors e_j ($j = 1, \dots, n$) of the y-axes, defined in the z-system of coordinates as follows:

$$e_1 = \sqrt{w_1 w_2^{-1}}\ (1, 0, \dots, 0, -\frac{\nu_n \nu_1}{w_1})$$

$$e_j = \sqrt{w_j w_{j+1}^{-1}}\ (-\frac{\nu_1 \nu_j}{w_j}, \dots, -\frac{\nu_{j-1} \nu_j}{w_j}, 1, 0, \dots, 0, -\frac{\nu_n \nu_j}{w_j})\ (j = 2, \dots, n-1)$$

$$e_n = (\nu_1, \dots, \nu_n).$$

Here ν_s ($s = 1, \dots\ n$) are coordinates of the vector $n(x')$ defined by formula (3) and $w_1 = \nu_n^2$, $w_j = \nu_1^2 + \dots + \nu_{j-1}^2 + \nu_n^2$ for $j = 2, \dots, n$. By means of elementary calculations it may be verified that e_j ($j = 1, \dots, n$) are unit vectors and form an orthogonal system.

It follows from our assumptions concerning the function g that a_{jk} are continuous functions in $\bar{\Delta}_0$. Introducing the new variables in equation (2) let us put

$$F(x', y', y_n) = z_n - g(z') \tag{5}$$

with z_k ($k = 1, \dots, n$) expressed by (4). We claim that F satisfies the assumptions of Lemma. In fact, let us put $p = y'$, $q = y_n$, $t = x' \in \bar{\Delta}$ and let $(\overset{0}{p}, \overset{0}{q})$ be the origin. Let us denote further $\mu = \max\limits_{k,j} \sup\limits_{\Delta_0} | a_{jk} |$ and let 0 be the cube in R^n defined by the inequalities

$$| y_j | \leqslant (n\mu)^{-1} \eta \qquad (j = 1, \dots, n) \tag{6}$$

with $\eta = (1/2)\ \mathrm{dist}\,(\partial \Delta, \partial \Delta_0)$. It follows from (6) that $z' \in \Delta_0$ for $y \in 0$, so assumptions a), b) are satisfied according to our suppositions about g. Calculating the derivative $D_{y_n} F$ we get from (4), (5)

$$D_{y_n} F(x', y', y_n) = a_{nn}(x') - \sum_{s=1}^{n-1} D_s g(z')\, a_{sn}(x')$$

and this yields in view of (3)

$$D_{y_n} F(x', 0, 0) = (1 + | \mathrm{grad}\, g(x')|^2)^{1/2}$$

Thus assumption c) is satisfied with $a = 1$.

Now we can solve the equation $F(x', y', y_n) = 0$ with respect to y_n and according to the Lemma the solution is defined and of class C^1 for $| y' | \leqslant d$ with some d not depending on x'. This means that the part Σ_d of the surface contained in the cylinder $| y' | \leqslant d$ of R^n may be described by equation (1). As obviously $\Sigma \cap K(x, d) \subset \Sigma_d$, the assertions of our theorem follow directly from the Lemma.

REFERENCES

[1] Leja, F., *Rachunek różniczkowy i całkowy*, PWN, Warszawa 1976.
[2] Pogorzelski, W., *Równania całkowe i ich zastosowania*, t. II, PWN Warszawa 1958.

STRESZCZENIE

W pracy wykazano, że warunki określające powierzchnię Lapunowa mogą być uproszczone w przypadku powierzchni zwartej.

РЕЗЮМЕ

В работе показано, что условия определяющие поверхность типа Лапунова даются упростить в случае компактной поверхности.