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On Maximizing Certain Fourth-Order Functionals of Bounded Univalent Functions

O maksymalizacji pewnych funkcjonałów czwartego rzędu
w klasie funkcji ograniczonych i jednolistnych

Об отыскании таксимима некоторых функционалов четвертого порядка
в классе ограниченных однолистных функций

1. Introduction. The class $S(b)$ consistsof bounded univalent functions f defined in the unit disc U: $|z| < 1$ and normalized so that

$$f(z) = b(z + a_2 z^2 + \dots), \quad |f(z)| < 1, \quad 0 < b < 1 .$$

The information concerning the coefficient body $(a_2, \dots, a_n)$ applies also for functionals of the coefficients involved. Thus, for sufficiently simple functionals extremal problems can be expected to be solvable.

Incomplete information is provided by Grunsky type inequalities, one form of which is the Power inequality (cf. e.g. [7]). By aid of these some of the lower coefficients and functionals determined by them are maximized for certain values of b. Actually, only the first nontrivial coefficient body (a_2, a_3) of $S(b)$ is completely governed for each value of b [7]. This allows maximizing $\mathrm{Re}\,(a_3 + \lambda a_2)$ [2], [8] and $\mathrm{Re}\,(a_3 + \lambda a_2^2)$ [4] in $S(b)$ for all values of the complex parameter λ. In the real subclass $S_R(b)$ of $S(b)$ the algebraic part of the second coefficient body (a_2, a_3, a_4) can be determined by aid of an extended inequality proved by Jokinen [1]. This recent development opens up possibilities in studying fourth order functionals in $S_R(b)$. Until now all results for them have concerned homogeneous functionals and the information available has been based on the Power inequality [3].

In this paper some homogeneous and some linear functionals of fourth order will be considered in $S_R(b)$. The homogeneous combinations of the a_ν-coefficients can be traced back to a classic question concerning the b_ν-coefficients of the logarithmic derivative of f, introducing the expansion

$$z\,\frac{f'(z)}{f(z)} = 1 + b_1 z + b_2 z^2 + \dots .$$

The a_ν- and b_ν-coefficient are connected:

$$n a_{n+1} = \sum_{1}^{n} a_{n-\nu+1}\, b_\nu \qquad (a_1 = 1;\ n = 1, 2, \dots).$$

By using Löwner's functions $f(z, u)$ obtained from

$$u\,\frac{df}{du} = f\,\frac{1+\kappa f}{1-\kappa f}, \quad f(z, 1) = z, \quad f(z, u) \in S(u),$$

generated by a step-function $\kappa(u) = e^{-i\vartheta(u)}$, $b \leq u \leq 1$, one can construct examples of the a_ν- and b_ν-coefficients. This allows estimating $\max |b_n|$ from below. In [5] the estimation is performed for the first indexes mainly for the purpose of showing that the b_ν-coefficients exceed the Koebe-function limit 2.

For the first b_ν-coefficients we have

$$\begin{cases} b_1 = a_2, \\ \frac{1}{2} b_2 = a_3 - \frac{1}{2} a_2^2, \\ \frac{1}{3} b_3 = a_4 - a_2 a_3 + \frac{1}{3} a_2^3 . \end{cases}$$

The coefficient b_1 is maximized with a_2. Similarly, the relatively simple technique of maximizing a_3 in $S(b)$ can be applied to b_2 too [6]. The problem for higher indexes is open. For b_3 in $S_R(b)$ the maximum will be determined in this paper.

In [9] Zyskowska introduces a linear functional $a_{2m} + \mu a_{2n+1}$ and proves that in $S_R(b)$, for $\mu > 0$ and fixed, there exists an interval $(0, b_\mu]$ where the functional is maximized by the left radial-slit-mapping. In [8] a complete solution in the case $a_3 + \lambda a_2$ is presented (if $\mu = \lambda^{-1}$ the result applies to the Zyskowska-functional). Let $\alpha : \beta$ be the name of a slit-domain where α is the amount of starting points and β the amount of end-points of the slits. Then the list of extremal domains is

$$0 < b \leq e^{-1}: \begin{cases} 1:2 \text{ for } |\lambda| < 4b, \\ 1:1 \text{ for } |\lambda| \geq 4b; \end{cases}$$

$$e^{-1} \leqslant b < 1: \begin{cases} 2:2 \text{ for } |\lambda| \leqslant 4b(1+\log b), \\ 1:2 \text{ for } 4b(1+\log b) < |\lambda| < 4b, \\ 1:1 \text{ for } 4b \leqslant |\lambda|. \end{cases}$$

Here 1 : 1 means the left radial-slit-mapping.

In this paper we introduce the functional $a_4 + \mu a_2$ and maximize it in $S_R(b)$ for an extensive set of values of the μ-parameter. If appears that the Zyskowska-type extremal occurs even for some negative μ-parameters in the case where both coefficients are even.

2. Preliminaries. Let us collect here results concerning the two inequalities which determine the algebraic part of the coefficient body (a_2, a_3, a_4) in $S_R(b)$. The first one follows from the Power inequality, mentioned above [7]:

$$\begin{cases} a_4 - 2a_2a_3 + \dfrac{13}{12}a_2^3 + \dfrac{b}{2}a_2^2 - \dfrac{2}{3}(1-b^3) + 2\lambda\left(a_3 - \dfrac{3}{4}a_2^2 + ba_2\right) + \\ + \lambda^2\,[a_2 - 2(1-b)] \leqslant 0, \\ \lambda \in R. \end{cases} \tag{1}$$

The equality function of this is defined by the generating function $\cos\vartheta$ for which [1], [8]

$$\cos\vartheta = \begin{cases} -1, & b \leqslant u < \sigma, \\ \dfrac{1}{3} + \dfrac{1-3\lambda}{6}u^{-3/2}, & \sigma \leqslant u \leqslant 1; \end{cases} \tag{2}$$

$$\frac{1}{3} - \frac{4}{3}\sigma^{3/2} \leqslant \lambda \leqslant \frac{1}{3} + \frac{8}{3}\sigma^{3/2}. \tag{3}$$

The corresponding extremal function f has the first coefficients:

$$\begin{cases} a_2 = 2(\sigma - b) - \dfrac{2}{3}(1-\sigma) + \dfrac{2}{3}(1-3\lambda)(1-\sigma^{-1/2}), \\ a_3 = a_2^2 + \dfrac{7}{9} + b^2 - \dfrac{16}{9}\sigma^2 - \dfrac{8}{9}(1-3\lambda)(1-\sigma^{1/2}) + \dfrac{1}{9}(1-3\lambda)^2(1-\sigma^{-1}). \end{cases} \tag{4}$$

For f there hold the conditions obtained by integrating Löwner's equation for $S_R(b)$ in two steps:

$$\begin{cases} \sigma^{3/2}(f_\sigma^{3/2} - f_\sigma^{-3/2}) + (3\lambda - 1 + \sigma^{3/2})(f_\sigma^{1/2} - f_\sigma^{-1/2}) = z^{3/2} - z^{-3/2} + 3\lambda(z^{1/2} - z^{-1/2}), \\ b^{1/2}(f^{1/2} - f^{-1/2}) = \sigma^{1/2}(f_\sigma^{1/2} - f_\sigma^{-1/2}). \end{cases} \tag{5}$$

The corresponding extremal domains are of the type 1:3 and 3:3.

The inequality (1) is sharp on a defined part of the boundary of the coefficient body when optimized by choosing λ so that the left side is maximized. This yields the estimate

$$a_4 \leq \frac{2}{3}(1-b^3) - \frac{1}{2} b a_2^2 + 2a_2 a_3 - \frac{13}{12} a_2^3 - \frac{(a_3 - \frac{3}{4} a_2^2 + b a_2)^2}{2(1-b) - a_2} \tag{6}$$

which is obtained for

$$\lambda = \frac{a_3 - \frac{3}{4} a_2^2 + b a_2}{2(1-b) - a_2} . \tag{7}$$

The right side of (6) can further be maximized in a_3. This yields for a_4 an estimate in a_2 and b:

$$a_4 \leq -\frac{7}{12} a_2^3 + \frac{1}{2}(4-9b) a_2^2 + \frac{2}{3}(1-b^3) = G_1, \quad |a_2| \leq 2(1-b). \tag{8}$$

This inequality is sharp on the parabola

$$1^\circ : a_3 = -\frac{1}{4} a_2^2 + (2-3b) a_2 . \tag{9}$$

By substituting (9) in (7) we see that on 1° $\lambda = a_2$. The maximum of a_4 thus gained is sharp so far as 1° remains in a defined subdomain I of the coefficient region (a_2, a_3) (cf. [8]). The extremal domains defined by (2)–(3) are of the type 1:3 or 3:3.

The second inequality is the one proved by Jokinen in [1]. It extends the Power inequality and reads

$$\begin{cases} a_4 - 2a_2a_3 + a_2^3 - b^2 a_2 + 2\lambda(a_3 - a_2^2 + 1 - b^2) \leq \frac{2}{3}(1+\lambda)^3, \\ -1 \leq \lambda \leq 0 . \end{cases} \tag{10}$$

For the extremal generating function there holds

$$\cos\vartheta = \begin{cases} -1, & b \leq u \leq \sigma_1 , \\ 1, & \sigma_1 < u \leq \sigma_2 , \\ \frac{1}{3} + \frac{1+3\lambda}{6} u^{-3/2}, & \sigma_2 < u \leq 1; \end{cases} \tag{11}$$

$$\begin{cases} \sigma_2 = \left(\frac{1-3\lambda}{4}\right)^{2/3} \in [b, 1] , \\ b \leq \sigma_1 \leq \sigma_2 \leq 1 . \end{cases} \tag{12}$$

The initial coefficients of the corresponding function f are in this case

$$\begin{cases} a_2 = -\frac{2}{3} - 2b + 4\sigma_1 - 4\sigma_2 + \frac{8}{3}\sigma_2^{3/2}, \\ a_3 = a_2^2 + \frac{7}{9} + b^2 - \frac{32}{9}\sigma_2^{3/2} + \frac{16}{9}\sigma_2^3 . \end{cases} \tag{13}$$

Löwner's equation, when integrated in three steps for (11), yields for the extremal f:

$$\begin{cases} \sigma_2^{3/2}(f_{\sigma_2}^{3/2} - f_{\sigma_2}^{-3/2}) - 3\sigma_2^{3/2}(f_{\sigma_2}^{1/2} - f_{\sigma_2}^{-1/2}) = \\ = z^{3/2} - z^{-3/2} + (1 - 4\sigma_2^{3/2})(z^{1/2} - z^{-1/2}), \\ \sigma_1^{1/2}(f_{\sigma_1}^{1/2} + f_{\sigma_1}^{-1/2}) = \sigma_2^{1/2}(f_{\sigma_2}^{1/2} + f_{\sigma_2}^{-1/2}), \\ b^{1/2}(f^{1/2} - f^{-1/2}) = \sigma_1^{1/2}(f_{\sigma_1}^{1/2} - f_{\sigma_1}^{-1/2}) . \end{cases} \tag{14}$$

The extremal domains are of the type 2:3.

The optimized form of (10) reads

$$a_4 \leqslant a_2^3 + (3b^2 - 2)a_2 + 2(a_2 + 1)x_0^2 - \frac{4}{3}x_0^3 \tag{15}$$

obtained by choosing

$$\begin{cases} 0 \leqslant x_0 = \lambda + 1 = \sqrt{a_3 - a_2^2 + 1 - b^2} \leqslant 1; \\ a_2^2 + b^2 - 1 \leqslant a_3 \leqslant a_2^2 + b^2. \end{cases} \tag{16}$$

Again, when maximized in a_3 this gives the maximum of the right side in a_2 and b:

$$a_4 = \begin{cases} a_2^3 + (3b^2 - 2)a_2 + \frac{2}{3}(a_2 + 1)^3 = G_2 & \text{for} \quad a_2 + 1 \geqslant 0, \\ a_2^3 + (3b^2 - 2)a_2 = G_3 & \text{for} \quad a_2 + 1 \leqslant 0 . \end{cases} \tag{17}$$

The maximizing choice of a_3 is such that $x_0 = a_2 + 1$ or $x_0 = 0$ which, in view of (16) implies $\lambda = a_2$ or $\lambda = -1$. From (10) we see that we have to restrict the use of (17) for the values $a_2 \leqslant 0$. This guarantees the validity of (16).

The upper limit G_2 is sharp on

$$2^\circ : a_3 = 2a_2^2 + 2a_2 + b^2 \tag{18}$$

and G_3 gives the sharp upper bound on

$$3^\circ : a_3 = a_2^2 - 1 + b^2 . \tag{19}$$

So far as the parabolic arc 2° lies in the subdomain II (cf. [8]) of (a_2, a_3) the estimation (17) remains to be sharp (3° lies on the lower boundary arc of II). The extremal domain connected with 2° is defined by (11) and is of the type 2:3. The extremal domain 2:2 having two horizontal slits is connected with 3°.

3. **The maximizing of b_3 in S_R (b).** Rewrite (6) for estimating the combination b_3:

$$b_3 - 2(1-b^3) \leq -\frac{3}{2} ba_2^2 + 3a_2 a_3 - \frac{9}{4} a_2^3 - 3 \frac{(a_3 - \frac{3}{4} a_2^2 + ba_2)^2}{2(1-b) - a_2} =$$

$$= \frac{3}{4}(2 - 8b - a_2) a_2^2 - \frac{3}{2(1-b) - a_2} \left[a_3 + (2b-1) a_2 - \frac{a_2^2}{4} \right]^2 \leq$$

$$\leq \frac{3}{4}(2 - 8b - a_2) a_2^2 .$$

Thus

$$\frac{b_3}{3} \leq \frac{2}{3}(1-b^3) + \frac{1-4b}{2} a_2^2 - \frac{1}{4} a_2^3 = M_1(a_2) \tag{20}$$

where the equality is reached for

$$a_3 = (1-2b) a_2 + \frac{a_2^2}{4} . \tag{21}$$

The value of λ in (7) for (21) is

$$\lambda = \frac{a_2}{2} . \tag{22}$$

Observe that we arrive at this choice also by starting from the unoptinuized inequality (1) which for b_3 implies

$$\frac{b_3}{3} - a_2 a_3 + 2\lambda \left(a_3 - \frac{3}{4} a_2^2 + ba_2\right) + \frac{3}{4} a_2^3 + \frac{b}{2} a_2^2 - \frac{2}{3}(1-b^3) +$$

$$+ \lambda^2 [a_2 - 2(1-b)] \leq 0 .$$

The choice (22) eliminates a_3, yielding (20).

The sharpness of the estimate can be interpreted in terms of (21); the inequality (20) is sharp as far as the parabola (21) lies in the subdomain I of (a_2, a_3) [8]. The equality conditions can also be expressed by aid of (3), (4) and (22): The existence of the equality function (2) is guaranteed by the existence of σ and a_2, such that

$$\begin{cases} 8\sigma \pm (3a_2 - 2)\sigma^{-1/2} - 6(a_2 + b) = 0, \\ \frac{2}{3} - \frac{8}{3}\sigma^{3/2} \leq a_2 \leq \frac{2}{3} + \frac{16}{3}\sigma^{3/2}, \\ b \leq \sigma \leq 1. \end{cases} \tag{23}$$

Next, rewrite (15) for b_3:

$$\frac{1}{3}b_3 \leq \frac{1}{3}a_2^3 + (2b^2 - 1)a_2 + (a_2 + 2)x_0^2 - \frac{4}{3}x_0^3 \tag{24}$$

where x_0 includes a_3 according to (16). When maximizing the right side in x_0 we obtain

$$\frac{b_3}{3} \leq \frac{2}{3} + 2b^2 a_2 + \frac{1}{2}a_2^2 + \frac{5}{12}a_2^3 = M_2(a_2). \tag{25}$$

The equality is reached for

$$\lambda + 1 = x_0 = \sqrt{a_3 - a_2^2 + 1 - b^2} = \frac{a_2}{2} + 1 \tag{26}$$

i.e. the choice (22) remains to hold for λ. As before, we arrive at the same result by starting from the unoptimized inequality (10), which for b_3 yields

$$\frac{b_3}{3} - a_2 a_3 + 2\lambda(a_3 - a_2^2 + 1 - b^2) + \frac{2}{3}a_2^3 - b^2 a_2 \leq \frac{2}{3}(1 + \lambda)^3,$$

and which by (22) reduces to the form (25).

The sharpness of (25), taken from (26), implies that the parabola

$$a_3 = b^2 + a_2 + \frac{5}{4}a_2^2 \tag{27}$$

lies in the subdomain II of (a_2, a_3) [8]. Similarly, from (12) and (13) we deduce that the equality function (11) exists provided that the numbers σ_1 and σ_2 can be determined to satisfy

$$\begin{cases} \sigma_2 = \left(\dfrac{1 - \frac{3}{2} a_2}{4}\right)^{2/3}, \\ \sigma_1 = \sigma_2 + \dfrac{a_2 + b}{2}, \\ b \leqslant \sigma_1 \leqslant \sigma_2 \leqslant 1 . \end{cases} \tag{28}$$

We will apply (25) for $-2(1-b) \leqslant a_2 \leqslant -b$ where $a_2/2 = \lambda$, $\lambda \in [-(1-b), -b/2] \subset [-1, 0]$. (20) will be applied for $-b \leqslant a_2 \leqslant 2(1-b)$.

$$\frac{b_3}{3} \leqslant F(a_2) = \begin{cases} M_1(a_2) = \dfrac{2}{3}(1 - b_3^3) + \dfrac{1-4b}{2} a_2^2 - \dfrac{1}{4} a_2^3, \ -b \leqslant a_2 \leqslant 2(1-b), \\ \\ M_2(a_2) = \dfrac{2}{3} + 2b^2 a_2 + \dfrac{1}{2} a_2^2 + \dfrac{5}{12} a_2^3, \ -2(1-b) \leqslant a_2 \leqslant -b. \end{cases} \tag{29}$$

This upper bound is differentiable even at the point $a_2 = -b$. Observe that the order of M_1 and M_2 is changed at this point, because

$$M_2(a_2) - M_1(a_2) = \frac{2}{3}(a_2 + b)^3 .$$

The roots of $M_2'(a_2) = 0$ are denoted by α and β. Denote $\gamma = -b$ and let δ be the non-vanishing root of $M_1'(a_2) = 0$:

$$\begin{cases} \alpha = -\dfrac{2}{5} - \dfrac{2}{5}\sqrt{1 - 10b^2}, \\ \beta = -\dfrac{2}{5} + \dfrac{2}{5}\sqrt{1 - 10b^2} \quad (0 < b \leqslant 10^{-1/2}), \\ \gamma = -b, \\ \delta = \dfrac{4}{3}(1 - 4b). \end{cases} \tag{30}$$

The upper bound F of (29) always has the local maximum

$$M_2(\alpha) = \frac{18}{25} - \frac{4}{5}b^2 + \frac{4}{75}(1 - 10b^2)^{3/2} .$$

The local nature of

$$M_1(\delta) = \frac{2}{3}(1-b_3) + \frac{8}{27}(1-4b)^3$$

depends of the sign of $1-4b$ as well as on the reality and order of the numbers (29). We omit the comparisons needed to check the following list of orders:

$0 < b < ¼: \ -2(1-b) < \alpha < \gamma < \beta < 0 < \delta < 2(1-b);$

$b = ¼: \ -2(1-b) < \alpha < \gamma < \beta < \delta = 0;$

$¼ < b < 4/13: -2(1-b) < \alpha < \gamma < \beta < \delta < 0;$

$b = 4/13: -2(1-b) < \alpha < \beta = \gamma = \delta < 0;$

$4/13 < b < 10^{-1/2}: -2(1-b) < \alpha < \beta < \delta < \gamma < 0;$

$b = 10^{-1/2}: -2(1-b) < \alpha = \beta < \delta < \gamma < 0;$

$10^{-1/2} < b: \delta < \gamma.$

From this list we read out the alternatives for the local maxima:

$0 < b \leq ¼$: local maxima are $M_2(\alpha)$, $M_1(\delta)$;

$¼ < b \leq 10^{-1/2}$: local maxima are $M_2(\alpha)$, $M_1(0)$;

$10^{-1/2} < b < 1$: the global maximum is $M_1(0)$.

In order to distinguish between the two competing candidates we have to solve the inequalities $M_1(0) \geq M_2(\alpha)$ and $M_2(\alpha) \geq M_1(\delta)$. This leads to the following:

Result.

1°. $0 < b \leq \tilde{b} = 0.077\,428\,918$

$$\max \frac{b_3}{3} = M_1(\delta) = \frac{2}{3}(1-b^3) + \frac{8}{27}(1-4b)^3.$$

The extremal domain is of the type 1:3 and $\tilde{b} \in (0, ¼)$ is the root of the equation $M_2(\alpha) = M_1(\delta)$.

2°. $\tilde{b} \leq b \leq \tilde{\tilde{b}} = 0.302\,279\,250$

$$\max \frac{b_3}{3} = M_2(\alpha) = \frac{18}{25} - \frac{4}{5}b^2 + \frac{4}{75}(1-10b^2)^{3/2}.$$

The type of the extremal domain is 2:3 and $\tilde{\tilde{b}} \in (10^{-1/2}, ¼)$ is the root of $M_1(0) = M_2(\alpha)$.

3°. $\tilde{\tilde{b}} \leq b < 1$

$$\max \frac{b_3}{3} = M_1(0) = \frac{2}{3}(1-b^3).$$

The extremal domain is 3:3 with three straight radial slits.

Observe, that at the points b and $\tilde{b}$ there exist two different extremal functions, – a phenomenon which holds in similar form also for a_4 in $S_R(b)$ [1].

Especially in the real unbounded case $S_R = S_R(0)$ we obtain

$$\max b_3(0) = \frac{26}{9}.$$

4. **The functional** $a_4 - a_2 a_3 + \frac{a_2^3}{4}$. Clearly, the above technique is applicable to the two-parametric functional

$$B_3(p,q) = a_4 + p a_2 a_3 + q a_2^3; \quad p, q \in R.$$

The results in p and q would remain rather implicit. As a curious example we mention here only the result which concerns the case $p = -1$, $q = 1/4$.

Result.

1°. $1/3 \leqslant b \leqslant 1$

$$\max B_3\left(-1, \frac{1}{4}\right) = \frac{2}{3}(1-b^3).$$

The extremal domain is 3:3.

2°. $0 \leqslant b \leqslant 1/3$

$$\max B_3\left(-1, \frac{1}{4}\right) = \frac{3}{4} - b^2 + \frac{1}{12}(1-8b^2)^{3/2}.$$

The extremal domain is 2:3.

3°. $b = 0$.

There exists also the extremal domain 1:3 for which

$$\max B_3\left(-1, \frac{1}{4}\right) = \left(\frac{2}{3}(1-b^3) + \frac{(1-4b)^3}{6}\right)_{b=0} = \frac{5}{6}.$$

5. **The linear combination** $a_4 + \mu a_2$. The inequalities (8) and (17) yield the corresponding estimates for $a_4 + \mu a_2$:

$$a_4 + \mu a_2 \leqslant \begin{cases} -\frac{7}{12}a_2^3 + \frac{1}{2}(4-9b)a_2^2 + \frac{2}{3}(1-b^3) + \mu a_2 = F_1, & -\frac{2}{3}b \leqslant a_2 \leqslant \\ & \leqslant 2(1-b), \\ a_2^3 + (3b^2-2)a_2 + \frac{2}{3}(a_2+1)^3 + \mu a_2 = F_2, & -1 \leqslant a_2 \leqslant -\frac{2}{3}b, \\ a_2^3 + (3b^2-2)a_2 + \mu a_2 = F_3, & -2(1-b) \leqslant a_2 \leqslant -1;\ b \leqslant \frac{1}{2}. \end{cases} \tag{31}$$

Observe that for $-2(1-b) \leq a_2 \leq -2/3\, b$ F_3 and F_2 are below F_1. Therefore, F_1 will be limited to the interval $-2/3\, b \leq a_2 \leq 2(1-b)$. Consider the derivatives.

1) $-\frac{2}{3} b \leq a_2 \leq 2(1-b)$; $F_1'(a_2) = -\frac{7}{4} a_2^2 + (4-9b)\, a_2 + \mu$

Denote the roots of $F_1'(a_2) = 0$ by

$$\alpha_1, \alpha_2 = \frac{2}{7}(4-9b) \pm \sqrt{\frac{4}{49}(4-9b)^2 + \frac{4\mu}{7}}. \tag{32}$$

At α_1 F_1 has a local maximum

$$F_1(\alpha_1) = \frac{2}{3}(1-b^3) + \frac{4}{147}(4-9b)^3 + \frac{2}{7}(4-9b)\mu + \frac{4}{147}\left[(4-9b)^2 + 7\mu\right]^{3/2}. \tag{33}$$

2) $-1 \leq a_2 \leq -\frac{2}{3} b$; $F_2'(a_2) = 5a_2^2 + 4a_2 + 3b^2 + \mu$

The roots of $F_2'(a_2) = 0$ are

$$\beta_1, \beta_2 = -\frac{2}{5} \pm \sqrt{\frac{4}{25} - \frac{3b^2 + \mu}{5}}. \tag{34}$$

At β_2 F_2 has a local maximum

$$F_2(\beta_2) = \frac{22}{25} - \frac{2}{5}(3b^2 + \mu) + \frac{2}{75}\left[4 - 5(3b^2 + \mu)\right]^{3/2}. \tag{35}$$

3) $-2(1-b) \leq a_2 \leq -1$, $b \leq \frac{1}{2}$; $F_3'(a_2) = 3a_2^2 + 3b^2 - 2 + \mu$

$F_3'(a_2)$ vanishes at

$$\gamma_1, \gamma_2 = \pm\sqrt{\frac{2 - 3b^2 - \mu}{3}}. \tag{36}$$

Thus, γ_2 gives a local maximum for F_3:

$$F_3(\gamma_2) = 2\left(\frac{2 - 3b^2 - \mu}{3}\right)^{3/2}. \tag{37}$$

The upper bound in (31) is differentiable even at the points $-2/3\, b$ and -1. Clearly, it has the maximum for $|a_2| \leq 2(1-b)$.

If the maximum is achieved at α_1 the sharpness is guaranteed, provided σ and $\alpha_1 = a_2 = \lambda$ can be determined according to (3) and (4) i.e.

$$\begin{cases} 8\sigma + (6a_2 - 2)\sigma^{-1/2} - (9a_2 + 6b) = 0, \\ \frac{1}{3} - \frac{4}{3}\sigma^{3/2} \leqslant a_2 \leqslant \frac{1}{3} + \frac{8}{3}\sigma^{3/2}, \\ b \leqslant \sigma \leqslant 1. \end{cases} \tag{38}$$

If the maximum is at β_2 the sharpness requires, according to (12) and (13), the existence of σ_1, σ_2 and $\beta_2 = a_2 = \lambda$ such that

$$\begin{cases} \sigma_2 = \left(\frac{1 - 3a_2}{4}\right)^{2/3}, \\ \sigma_1 = \sigma_2 + \frac{3a_2 + 2b}{4}, \\ b \leqslant \sigma_1 \leqslant \sigma_2 \leqslant 1. \end{cases} \tag{39}$$

The sharpness at the maximizing point γ_2 requires only that $-2(1-b) \leqslant \gamma_2 \leqslant -1$, $0 < b \leqslant 1/2$.

As in Sections 3 and 4 also here the result depends on the order of the possible maximizing points $-2(1-b), \gamma_2, \beta_2, \alpha_1$ and $2(1-b)$ as well as on the order of the corresponding F_ν-values. Clearly, a detailed treatment for all values of the parameters μ and b is excessively involved. Therefore, we shall restrict ourselves to some special cases of the parameter μ.

From the expressions of $\alpha_\nu, \beta_\nu, \gamma_\nu$ we see immediately that for a sufficiently large μ the upper bound (31) is monotonously increasing and for μ properly limited from above, monotonously decreasing. Consider the first alternative.

We obtain a lower limit for μ by requiring that

$$2(1-b) \leqslant \alpha_1 = \frac{2}{7}(4-9b) + \sqrt{\frac{4}{49}(4-9b)^2 + \frac{4\mu}{7}}$$

which is equivalent to

$$\mu \geqslant (11b-1)(1-b). \tag{40}$$

Similarly we see that

$$\beta_2 \geqslant -1 \tag{41}$$

if

$$\mu \geq -1-3b^2. \tag{42}$$

For values (40) this requirement is automatically true.

If $b \leq ½$ we have to consider F_3 for $-2(1-b) \leq a_2 \leq -1$. Because in this interval $|a_2| \geq 1$ and (42) holds, we have

$$F_3'(a_2) = 3a_2^2 + 3b^2 - 2 + \mu \geq 3 + 3b^2 - 2 + \mu > 1 + 3b^2 - 1 - 3b^2 = 0.$$

Altogether, if (40) is true the only competing maximizing points are β_2 and $2(1-b)$. The former one of these exists so far as $\mu \leq 4/5 - 3b^2$. Thus, the comparison is to be performed as far as

$$r_1 = (11b-1)(1-b) \leq \mu \leq \frac{4}{5} - 3b^2 = r_2, \quad b \leq \frac{3}{4}(1-\sqrt{0.6}) = 0.169\ldots. \tag{43}$$

For values $b > ¾(1-\sqrt{0.6})$ there holds

$$\frac{4}{5} - 3b^2 < (11b-1)(1-b) \leq \mu$$

which implies that β_2 is non-existent and F_2 is monotonously increasing. Hence, for these values of b

$$\max(a_4 + \mu a_2) = F_1(2(1-b)) = 4 - 20b + 30b^2 - 14b^3 + 2(1-b)\mu.$$

It remains to compare the values $F_2(\beta_2)$ and $F_1(2(1-b))$ in the cases (43). The number $F_2(\beta_3)$ of (35) is maximized in μ at the point $\mu = r_1$ because $-\mu \leq -r_1$. $F_1(2(1-b))$ is minimized in μ at the point $\mu = r_1$ because $r_1 \leq \mu$. For these values we have finally:

$$\max_\mu F_2(\beta_2) = \frac{1}{5}(6.4 - 24b + 16b^2) + \frac{10}{3}(0.36 - 2.4b + 1.6b^2)^{3/2} \leq$$

$$\leq 2(1-b)(1+4b-4b^2) = \min_\mu F_1(2(1-b))$$

if $0 < b < ¾(1-\sqrt{0.6})$. Equality is reached only at $b = 0$, $\mu = -1$. We thus have:

Result. In $S_R(b)$ the linear combination $a_4 + \mu a_2$ is maximized by the left radial-slit-mapping if

$$\mu \geq (11b-1)(1-b); \tag{44}$$

$$\max(a_4 + \mu a_2) = 4 - 20b + 30b^2 - 14b^3 + 2(1-b)\mu. \tag{45}$$

In the case $b = 0$, $\mu = -1$ there exists also another extremal function F of the type 2:2,

$$F(z) = \frac{z}{1+z+z^2}.$$

The existence of the second extremal function follows from (39); $b = 0$, $\mu = -1$; $\beta_2 = -1$; $\sigma_2 = 1$, $\sigma_1 = 1/4$. Thus F is obtained from (14) as a limit case of

$$b\,(f + f^{-1} - 2) = z + z^{-1} + 1 .$$

From (40) we see that if $\mu \geqslant 25/11$ then $a_4 + \mu a_2$ is maximized by the left radial-slit-mapping on the whole interval $0 < b < 1$. Similarly if $0 \leqslant \mu < 25/11$ the same radial-slit-mapping preserves its role for

$$0 < b \leqslant \frac{6 - \sqrt{25 - 11\mu}}{11} \quad \text{and} \quad \frac{6 + \sqrt{25 - 11\mu}}{11} \leqslant b < 1 . \tag{46}$$

If $-1 < \mu < 0$ the former interval (45) preserves its meaning. Thus, in the present case of two even coefficients, the Zyskowska-type radial-slit maximization ([9]) continues even on the negative side of μ.

Next, try to limit μ from above so that the monotonously decreasing upper bound (31) gives the maximum $F_3\,(-2\,(1-b))$. This, however, requires that F_3 is available i.e. $b \leqslant 1/2$.

Suppose that $0 < b \leqslant 1/2$ and consider those values of μ for which

$$\gamma_2 = -\sqrt{\frac{2}{3} - b^2 - \frac{\mu}{3}} \leqslant -2\,(1-b)$$

$$\Longleftrightarrow$$

$$\mu \leqslant -10 + 24\,b - 15\,b^2 . \tag{47}$$

For these values of μ the discriminant of α_1, α_2 is estimated:

$$\frac{4}{49}\,(4 - 9\,b)^2 + \frac{4}{7}\mu \leqslant \frac{24}{49}\,[7 - 4\,(b-2)^2] < 0$$

for $0 < b < 2 - (\sqrt{7}/2) = 0.677\ldots$. Thus for $0 < b \leqslant 1/2$ $F_1' < 0$. Because $F_2'\,(-2/3\,b) = F_1'\,(-2/3\,b)$, also $F_2'\,(-2/3\,b) < 0$.

The requirement

$$\beta_2 \leqslant -1$$

holds if

$$\mu \leqslant -1 - 3\,b^2$$

which, again, is true for (47). Altogether, the derivative of the upper bound in (31) is negative and $F_3\,(-2\,(1-b))$ the maximum.

Result. In $S_R(b)\, a_4 + \mu a_2$ is maximized for the right radial-slit-mapping if

$$0 < b \leq \frac{1}{2}, \quad \mu \leq -10 + 24b - 15b^2; \tag{48}$$

$$\max(a_4 + \mu a_2) = -4 + 20b - 30b^2 + 14b^3 - 2(1-b)\mu. \tag{49}$$

If $b > ½$ the upper bound F_3 is no more avaible. For these values of b the limitations (16) hold in the whole coefficient body (a_2, a_3) (the upper limit $a_2^2 + b^2$ lies in the complement of (a_2, a_3)). This means that both conditions (6) and (15) are avaible in the whole (a_2, a_3). As mentioned above, these upper bounds are maximized on the paraboloe 1° and 2° as far as these lie in the corresponding algebraic part I and II of (a_2, a_3). Outside these the maximum is to be found on the upper boundary arc of (a_2, a_3). This is seen by considering the upper bounds as functions of a_3. By aid of lengthly numerical checking we find:

If $\mu \leq -2 + 8b - 15b^2$ then $a_4 + \mu a_2$ is maximized by the right radial-slit-mapping in the interval $½ < b \leq 0.746\,414\,311$. From this limit onwards our methods fails; elliptic extremal functions are beyond the reach of our method, Similarly, the limit $-2 + 8b - 15b^2$, obtained from our unsharp estimate, is not sharp either.

The functional $a_4 + \mu a_2$ can be maximized by aid of (31) for all those values of μ which lead to algebraic extremal functions controlled by (38) and (39). The checking and comparisons involved can be passed on to computer. However, exact use of inequalities is by no means excluded.

In Figure 1 there is presented the distribution of the types of extremal functions in the $b\mu$-plane. The letters $A, \ldots, E$ indicate the following types of functions and mappings:

A = left radial-slit mapping,

B = 2 : 2 with two-radial slits along the real axis,

C = 2 : 3,

D = 3 : 3 or 1 : 3,

E = right radial-slit mapping,

The arcs on which the types of the extremal functions do change can be distinguished by aid of the points

P = (0.5,–1.75),

Q = (0.718 782 448,–0.781 298 556).

R = (0.6,–0.28),

S = (0,–1),

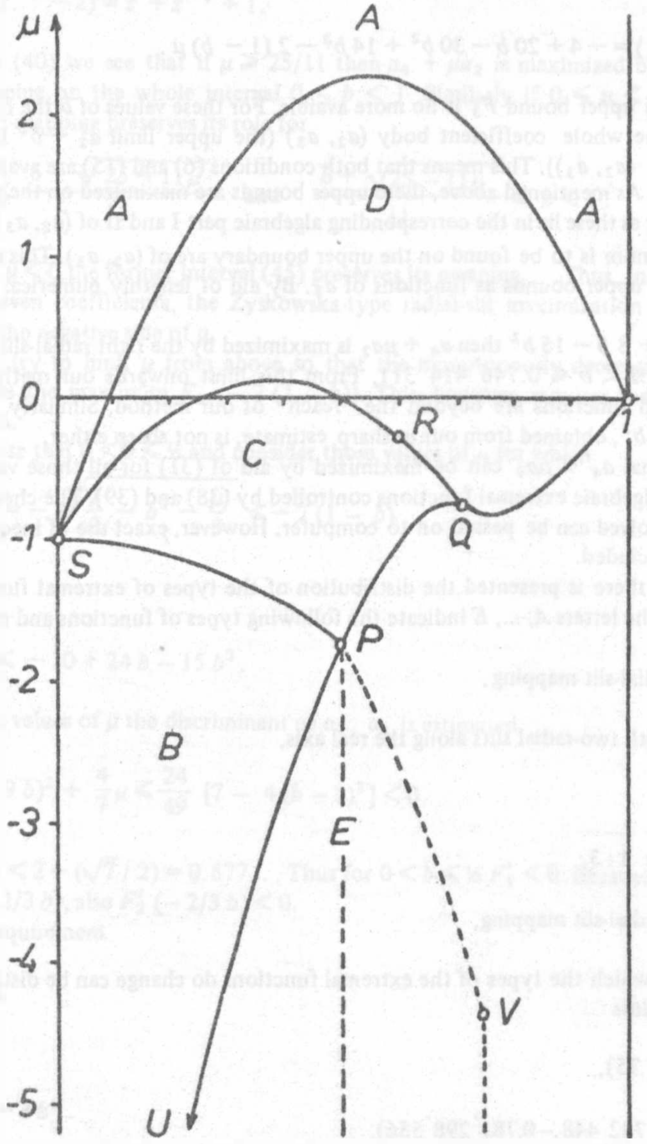

Figure 1.

$T = (1,0)$,

$U = (0, -10)$,

$V = (0.746\,414\,311, -4.385\,700\,368)$.

The arc ST belongs to the parabola

$$\mu = (11\,b-1)(1-b) = -11\,b^2 + 12\,b - 1.$$

The arc RS is obtained from $F_1(\alpha_1) = F_2(\beta_2)$ and reduces to the form

$$\mu = -8\,b^2 + 6\,b - 1. \tag{50}$$

On QR there holds $\alpha_1 = \beta_2 = -2/3\,b$, yielding

$$\mu = -\frac{47}{9}\,b^2 + \frac{8}{3}\,b\,. \tag{51}$$

On this arc both types C and D exists as the same limit case. Thus, on QR the extremal function is unique, whereas on RS there exist two different simultaneous extremal functions.

Crossing the arc TQ means that the type D reduces to an elliptic case so that σ decreases below the limit b. Thus we read out from (38) that on TQ

$$\begin{cases} \sigma = b\,, \\ 8\,\sigma + (6\,a_2 - 2)\,\sigma^{-1/2} - (9\,a_2 + 6\,b) = 0\,; \end{cases}$$

$$\begin{cases} \alpha_1 = a_2 = 2\,\dfrac{b^{3/2}-1}{9\,b^{1/2}-6}\,, \\ 7\,\alpha_1^2 - 4\,(4 - 9\,b)\,\alpha_1 - 4\,\mu = 0\,; \end{cases}$$

$$\mu = 7\,\Big(\frac{b^{3/2}-1}{9\,b^{1/2}-6}\Big)^2 + (18\,b - 8)\,\frac{b^{3/2}-1}{9\,b^{1/2}-6}\,. \tag{52}$$

Crosing PQ means similarly that the type C is shifted on the elliptic region in such a way that σ_1 decreases below b. (Observe that the upper limit F_2 yielding C is defined on $-1 \leqslant a_2 \leqslant -2/3\,b$. This implies the order $\sigma_1 \leqslant \sigma_2$.) From (39) we see that on PQ

$$\begin{cases} b = \sigma_1 = \Big(\dfrac{1-3\,a_2}{4}\Big)^{2/3} + \dfrac{3\,a_2 + 2\,b}{4}\,, \\ a_2 = \beta_2\,; \end{cases}$$

$$\begin{cases} (2-6\beta_2)^2-(2b-3\beta_2)^3=0\,, \\ \beta_2=-0.4-\sqrt{0.16-\dfrac{3b^2+\mu}{5}}\,. \end{cases} \tag{53}$$

The range of B requires that

$$-2(1-b)\leqslant \gamma_2=-\sqrt{\frac{2-3b^2-\mu}{3}}\leqslant -1\,.$$

The left equality case yields PU:

$$\mu=-10+24b-15b^2 \tag{54}$$

and the right one gives PS:

$$\mu=-1-3b^2\,. \tag{55}$$

As was mentioned above, the equation of PV follows from the condition that for $b > ½$ the unsharp upper bound (31) lies below the limit belonging to the type E. Thus, the question of the exact region of elliptic types requires more extended analyss of the extremal elliptic cases and lies outside the scope of results available until now.

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STRESZCZENIE

Niech $S(b)$ oznacza rodzinę funkcji

$$f(z) = bz + a_2 z^2 + \ldots, \quad 0 < b < 1, \; |f(z)| < 1$$

holomorficznych i jednolistnych w kole $|z| < 1$.

Autor rozwiązuje problem

$$\sup \left\{ a_4 + \mu a_2 : f \in S(b) \right\}$$

dla rzeczywistych wartości parametru μ.

РЕЗЮМЕ

Пусть $S(b)$ обозначает класс функций

$$f(z) = bz + a_2 z^2 + \ldots, \quad 0 < b < 1, \; |f(z)| < 1$$

голоморфных и однолистных в круге $|z| < 1$.

Автор решает проблему

$$\sup \; a_4 + \mu a_2 : f \in S(b)$$

для вещественных значений параметра μ.

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