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An Inequality for Asymmetric Entire Functions

Nierówność dla funkcji całkowitych asymetrycznych

Неравенство для ассиметрических целых функций

It is a simple consequence of the maximum principle (see [5], p. 346 or [4], p. 158, problem III 269) that if $p_n(z)$ is a polynomial of degree n, then

$$\max_{|z|=R>1} |p_n(z)| \leq R^n \max_{|z|=1} |p_n(z)|, \tag{1}$$

or equivalently

$$\max_{|z|=\rho<1} |p_n(z)| \leq \rho^n \max_{|z|=1} |p_n(z)|. \tag{2}$$

More precise estimates can be made if $p_n(z)$ has no zeros in $|z|<1$. In fact, if $p_n(z) \neq 0$ in $|z|<1$, then [1]

$$\max_{|z|=R>1} |p_n(z)| \leq \tfrac{1}{2}(R^n+1) \max_{|z|=1} |p_n(z)|, \tag{3}$$

whereas [6]

$$\max_{|z|=\rho<1} |p_n(z)| \geq \left(\frac{1+\rho}{2}\right)^n \max_{|z|=1} |p_n(z)|. \tag{4}$$

Since $p_n(e^{iz})$ is an entire function of exponential type these inequalities suggest generalizations to such functions. It is indeed well known that if $f(z)$ is an entire function of exponential type τ with $|f(x)| < \infty$ on the real axis, then for all real y (see [2], p. 82)

$$\sup_{-\infty < x < \infty} |f(x+iy)| \leq e^{\tau|y|} \sup_{-\infty < x < \infty} |f(x)|, \tag{1'}$$

or equivalently

$$\sup_{-\infty < x < \infty} |f(x+iy)| \geq e^{-\tau|y|} \sup_{-\infty < x < \infty} |f(x)|. \tag{2'}$$

These two latter inequalities are generalizations of (1) and (2) respectively.

If $p_n(z) \neq 0$ in $|z| < 1$, then $f(z) := p_n(e^{iz})$ has no zeros in $y > 0$. Besides, if

$$h_f(\theta) := \limsup_{r \to \infty} r^{-1} \log |f(re^{i\theta})|$$

is its indicator function (see [2], Chapter 5) then $h_f(\pi/2) = 0$. Accordingly, Boas considered the family $\mathfrak{F}_\tau$ of entire functions f of exponential type τ with $|f(x)| < \infty$ for real x, $f(z) \neq 0$ for $y > 0$ and $h_f(\pi/2) = 0$. He generalized (3) to entire functions of exponential type by proving [3] that if $f \in \mathfrak{F}_\tau$ then for $y < 0$,

$$\sup_{-\infty < x < \infty} |f(x+iy)| \leq \tfrac{1}{2}(e^{\tau|y|} + 1) \sup_{-\infty < x < \infty} |f(x)|. \tag{3'}$$

In view of this result one might think that (4) would admit an extension of the form

$$\max_{-\infty < x < \infty} |f(x+iy)| \geq \left(\frac{1+e^{-y}}{2}\right)^{\tau} \sup_{-\infty < x < \infty} |f(x)| \quad \text{for } y > 0 \text{ and } f \in \mathfrak{F}_\tau. \tag{4*}$$

It turns out that (4*) does not hold for all $f \in \mathfrak{F}_\tau$. For an arbitrary $\tau > 0$ let

$$f_T(z) := \left(\frac{1+e^{iz/T}}{2}\right)^{\tau T}$$

where T is a positive number such that $\tau T \in \mathbb{N}$. Clearly, $f_T \in \mathfrak{F}_\tau$ and

$$\sup_{-\infty < x < \infty} |f_T(x)| = 1.$$

Besides, for fixed y

$$|f_T(x+iy)| \leq \left(\frac{1+e^{-y/T}}{2}\right)^{\tau T} \longrightarrow e^{-(\tau/2)y} \quad \text{as } T \to \infty,$$

i.e. (4*) cannot hold for f_T if T is large (and, of course, $\tau T \in \mathbf{N}$). However, instead of (4*) we do have

Theorem 1 A. *If $f \in \mathfrak{Z}_\tau$, then for $y > 0$*

$$\sup_{-\infty < x < \infty} |f(x+iy)| \geq e^{-(\tau/2)y} \sup_{-\infty < x < \infty} |f(x)|. \tag{4'}$$

The above example shows that (4') is best possible.

Proof of Theorem 1 A. We shall prove that if $f \in \mathfrak{Z}_\tau$ then for $y > 0$ and all $x \in \mathbf{R}$

$$|f(x+iy)| \geq |f(x)|\, e^{-(\tau/2)y}, \tag{5}$$

from which (4') would follow immediately.

Case (i). If f is of order < 1 then it must be a constant since otherwise $|f(x)|$ cannot be bounded on the real axis (see [2], pp. 82–83). Hence (5) is trivially true in this case.

Case (ii). Let f be of order 1 type $t \leq \tau$. Since $h_f(\pi/2) = 0$ and $|f(x)|$ is bounded on the real axis $h_f(-\pi/2)$ is necessarily equal to t. Let y_0 be an arbitrary but fixed positive number and put

$$g(z) := f(z + \tfrac{1}{2} i y_0) \exp\left\{-i(t/2)(z + \tfrac{1}{2} i y_0)\right\}.$$

Then g is of exponential type $t/2$; moreover the indicator h_g of g satisfies $h_g(-\pi/2) = h_g(\pi/2)$. Since $g(z) \neq 0$ for $y > -\tfrac{1}{2} y_0$ and a fortiori for $y > 0$, by a theorem of B. Ya. Levin (see [2], p. 129) we have $|g(z)| \geq |g(\bar{z})|$ for $\operatorname{Im} z > 0$. In particular $|g(x + \tfrac{1}{2} i y_0)| \geq |g(x - \tfrac{1}{2} i y_0)|$, or equivalently

$$|f(x+iy_0)| \geq |f(x)|\, e^{-(t/2)y_0} \geq |f(x)|\, e^{-(\tau/2)y_0} \tag{6}$$

which is what we wanted to prove.

As an immediate consequence of Theorem 1 A, we have

Corollary 1. *Let $f(z)$ be an entire function of exponential type τ such that*

(i) $\sup_{-\infty < x < \infty} |f(x)| \leq 1$,

(ii) $h_f(\pi/2) = 0$,

and

(iii) $f(z) \neq 0$

for $y > -k$ where k is some positive number. Then for $0 > y \geq -k$,

$$|f(x+iy)| \leq e^{(\tau/2)|y|}. \tag{7}$$

We can, in fact, prove

Theorem 2. *If $f(z)$ is an entire function of exponential type τ satisfying the conditions of Corollary 1, then* (7) *holds for* $0 > y \geqslant -2k$.

Proof of Theorem 2. It only remains to prove that (3) holds for $-k > y \geqslant -2k$.

Again, the result is trivial if $f(z)$ is of order < 1. If $f(z)$ is of order 1 type $t \leqslant \tau$ then we may apply the theorem of Levin (loc. cit.) to

$$G(z) := f(z-ik)\ \exp\left\{-i\,(t/2)\,(z-ik)\right\}$$

to deduce that

$$|G(z)| \leqslant |G(\bar{z})| \quad \text{for} \quad \operatorname{Im} z < 0\,.$$

Thus, for $\delta > 0$

$$|f(x-i(k+\delta))|\, e^{-(t/2)(k+\delta)} = |G(x-i\delta)| \leqslant |G(x+i\delta)| = |f(x-i(k-\delta))| e^{-(t/2)(k-\delta)},$$

i.e.

$$|f(x-i\,(k+\delta))| \leqslant |f(x-i\,(k-\delta))|\, e^{t\delta}\,.$$

In particular, if $0 < \delta \leqslant k$, then from (7) we obtain

$$|f(x-i\,(k+\delta))| \leqslant |f(x)|\, e^{(t/2)(k-\delta)} e^{t\delta} = |f(x)|\, e^{(t/2)(k+\delta)}$$

from which the desired result follows.

For all $T > 0$ such that $\tau T \in \mathrm{N}$ the function

$$f_{k,\,T}(z) := \left(\frac{1+e^{i(z+ik)/T}}{1+e^{-k/T}}\right)^{\tau T}$$

satisfies the conditions of Corollary 1, whereas

$$f_{k,\,T}(iy) = \left(\frac{1+e^{-(y+k)/T}}{1+e^{-k/T}}\right)^{\tau T} \longrightarrow e^{-(\tau/2)y} \quad \text{as} \quad T \to \infty\,.$$

Hence inequality (7) is best possible for all $y \in [-2k, 0)$.

The example

$$f(z) := (e^{i\tau z} + e^{\tau k})\,/\,(1+e^{\tau k})$$

shows that for a function satisfying the conditions of Corollary 1 inequality (7) may not hold for $y < -2k$.

From Theorem 2 we readily deduce the following generalized version of Theorem 1 A.

Theorem 1. *Let $f(z)$ be an entire function of exponential type τ such that*

(i) $\sup\limits_{-\infty < x < \infty} |f(x)| = 1$,

(ii) $h_f(\pi/2) = 0$,

and

(iii) $f(z) \neq 0$

for $y > k \geq 0$. Then for $y \geq 2k$,

$$\sup_{-\infty < x < \infty} |f(x+iy)| \geq e^{-(\pi/2)y} . \tag{8}$$

Inequality (8) is best possible for all $y \geq 2k$ and may not hold for $y < 2k$.

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STRESZCZENIE

Rozważa się problem znalezienia najlepszego oszacowania od dołu wyrażenia

$$\sup_{-\infty < x < +\infty} |f(x+iy)|, \ y > 0$$

w klasie funkcji całkowitych danego typu wykładniczego.

РЕЗЮМЕ

Рассматривается проблема отыскания наилучшей оценки снизу выражения

$$\sup_{-\infty < x < +\infty} |f(x+iy)|, \ y > 0$$

в классе целых функций даного экспоненциального типа.