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On Subordination and Majorization *

O podporządkowaniu i majoryzacji

О подчинению и махорации

1. Introduction. Let L denote the set of functions f analytic and locally one-to-one in $U = \{z: |z| < 1\}$, which are normalized by $f(0) = 0, f'(0) = 1$. Also, let S and $S^*(\alpha)$ denote the subclasses of L consisting of univalent and starlike functions of order α $(0 \leqslant \alpha \leqslant 1)$, respectively. In this paper we continue the investigation begun by Ruscheweyh [2]. Among his results is the following interesting theorem.

Theorem A. *Let $f \in S$. Then, for* $0 \leqslant t \leqslant 1$,

$$\frac{(1+t)^2}{4t} f(tz) \prec f(z). \tag{1}$$

Ruscheweyh then asks whether (1) is also sufficient for f to belong to S. Using the notion of subordination chain, we offer another proof of Theorem A, one which also shows that many functions not in S may satisfy (1) (see Theorem A').

Definition. Let ϕ be defined on $U \times [a, b]$. We say $\phi(z, t)$ is a subordination chain on $[a, b]$ if $a \leqslant t_1 < t_2 \leqslant b \Rightarrow \phi(z, t_1) \prec \phi(z, t_2)$ in U. Similarly, $\phi(z, t)$ is a majorization chain on $[a, b]$ if $a \leqslant t_1 < t_2 \leqslant b \Rightarrow |\phi(z, t_1)| \leqslant |\phi(z, t_2)|$ in U.

We require the following:

Lemma 1. *Let ϕ be defined on $U \times [a, b]$, $\phi(0, t) = 0$, $\phi'_z(0, t) > 0$, $\phi(z, t)$ analytic in U for fixed t, and $\phi(z, t)$ continuously differentiable in $[a, b]$ for fixed z. Then*

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(i) $\phi(z, t)$ *is a subordination chain on* $[a, b]$ *if and only if for* $t \in [a, b]$, $0 < |z| < 1$,

$$\left|\arg \frac{\phi'_t}{z\phi'_z}\right| \leq \pi/2 \ \text{or} \ \phi'_t = 0 .$$

(ii) $\phi(z, t)$ *is a majorization chain on* $[a, b]$ *if and only if for* $t \in [a, b]$, $0 < |z| < 1$,

$$\left|\arg \frac{\phi'_t}{\phi}\right| \leq \pi/2 \ \text{or} \ \phi'_t = 0 .$$

For $f \in L$, *define* g_α *on* $U \times (-1, 1]$ *by*

$$g_\alpha(z, t) = \begin{cases} \dfrac{(1+t)^{2(1-\alpha)}}{4^{1-\alpha}\, t} \, f(tz), & t \neq 0 , \\ z/4^{1-\alpha}, & t = 0 . \end{cases}$$

We will show that under certain conditions $g_\alpha(z, t)$ is a subordination (majorization) chain on $(-1, 1]$. By this we mean that $g_\alpha(z, t)$ is a subordination (majorization) chain on $[-1+\epsilon, 1]$ for each ϵ, $0 < \epsilon < 1$. Here, the conditions of the Lemma are met. In fact, $g_\alpha(z, t)$ may be regarded, for fixed z, as an analytic function of the complex variable t, for $|t| < 1$.

We shall also prove the following theorem.

Theorem B. *Let* $f \in L$.

(i) $f \in S^*(\alpha)$ *if and only if* $g_\alpha(z, t)$ *is a majorization chain on* $(-1, 1]$.

(ii) *if* $f \in S^*(\alpha)$, *then* $g_\alpha(z, t)$ *is a subordination chain on* $(-1, 1]$.

(iii) *the converse of* (ii) *holds in the case* $\alpha = 0$.

We do not know if the converse of (ii) holds for other values of α.

2. Proofs of theorems. In this section we first state and prove a more precise version of Theorem A.

Theorem A′. *Let* $f \in L$. *Then* $g_0(z, t)$ *is a subordination chain on* $[0, 1]$ *if and only if*

$$\operatorname{Re}\left(\frac{f(z)}{zf'(z)}\right) \leq \frac{1+|z|}{1-|z|}, \ z \in U. \tag{2}$$

Proof. A calculation for $0 < t < 1$ yields

$$\frac{g'_{0t}}{zg'_{0z}} = \frac{1}{t}\left[\frac{t-1}{t+1}\,\frac{f(tz)}{tzf'(tz)} + 1\right]. \tag{3}$$

Thus, g'_{0t}/zg'_{0z} has non-negative real part in U if and only if

$$\operatorname{Re}\left[\frac{f(tz)}{tzf'(tz)}\right] \leq \frac{1+t}{1-t}, \ z \in U, \ 0 < t < 1. \tag{4}$$

By the maximum principle, (4) is equivalent to

$$\mathrm{Re}\left[\frac{f(tz)}{tzf'(tz)}\right] \leqslant \frac{1+t}{1-t},\ |z|=1, 0<t<1,$$

and the substitution $w = tz$ gives the desired result.

Remark. Since a function in S satisfies

$$\left|\frac{zf'(z)}{f(z)}\right| \geqslant \frac{1-|z|}{1+|z|}, \tag{5}$$

it must also satisfy (2), so that Theorem A is contained in Theorem A'.

Example. We now define a function $f \notin S$ (in fact, of infinite valence) which satisfies (5) and hence (1). Let

$$Q(z) = e^{-1}\exp\left(\frac{1+z}{1-z}\right),\ z \in U,$$

and define f by

$$f(z) = z\exp\int_0^z \frac{Q(w)-1}{w}\,dw.$$

We observe that $zf'(z)/f(z) = Q(z)$, and one easily verifies that (5) holds. On the other hand, f grows too rapidly along the positive axis $(0 < r < 1)$ to be p-valent for any p.

Proof of Theorem B. A calculation for $t \neq 0$ gives

$$\frac{g'_{\alpha t}}{zg'_{\alpha z}} = \frac{1}{t}\left[\frac{t(1-2\alpha)-1}{t+1}\ \frac{f(tz)}{tzf'(tz)} + 1\right] \tag{6}$$

and

$$\frac{g'_{\alpha t}}{g_\alpha} = \frac{1}{t}\left[\frac{t(1-2\alpha)-1}{t+1} + \frac{tzf'(tz)}{f(tz)}\right]. \tag{7}$$

With an argument analogous to that of Theorem A', we obtain the following relations (8) – (15):

$$\mathrm{Re}\ \frac{g'_{\alpha t}}{zg'_{\alpha z}} \geqslant 0,\ -1<t<0 \tag{8}$$

if and only if

$$\left| \frac{zf'(z)}{f(z)} - \frac{1+(1-2\alpha)|z|}{2(1-|z|)} \right| < \frac{1+(1-2\alpha)|z|}{2(1-|z|)}. \tag{9}$$

$$\mathrm{Re}\left(\frac{g'_{\alpha t}}{zg'_{\alpha z}}\right) \geq 0,\ 0 \leq t \leq 1 \tag{10}$$

if and only if

$$\left| \frac{zf'(z)}{f(z)} - \frac{1-(1-2\alpha)|z|}{2(1+|z|)} \right| \geq \frac{1-(1-2\alpha)|z|}{2(1+|z|)}. \tag{11}$$

$$\mathrm{Re}\left(\frac{g'_{\alpha t}}{g_\alpha}\right) > 0,\ -1 < t < 0 \tag{12}$$

if and only if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) < \frac{1+(1-2\alpha)|z|}{1-|z|}. \tag{13}$$

$$\mathrm{Re}\left(\frac{g'_{\alpha t}}{g_\alpha}\right) > 0,\ 0 < t \leq 1 \tag{14}$$

if and only if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \frac{1-(1-2\alpha)|z|}{1+|z|}. \tag{15}$$

Now, if $f \in S^*(\alpha)$, then the value of $zf'(z)/f(z)$, for z fixed, lies in the disk with center on the positive real axis and having diameter endpoints at

$$\frac{1 \pm (1-2\alpha)|z|}{1 \mp |z|}.$$

Thus, (9), (11), (13), and (15) are satisfied, and $g_\alpha(z, t)$ is both a majorization and subordination chain on $(-1, 1]$. Here we have used the fact that $g'_{\alpha t}/zg'_{\alpha z}$ and $g'_{\alpha t}/g_\alpha$ are continuous at $t = 0$. We observe also from (14) and (15) that if $g_\alpha(z, t)$ is a majorization chain on $[0, 1]$, then $f \in S^*(\alpha)$. Finally, from (8) and (9) it follows that if $g_\alpha(z, t)$ is a subordination chain on $(-1, 0]$, then $\mathrm{Re}\,[zf'(z)/f(z)] > 0$ so that $f \in S^*(0)$. The proof of Theorem B is complete.

REFERENCES

[1] Bielecki, A., Lewandowski, Z., *Sur certaines familles de fonctions α-etoilées*, Ann. Univ. Mariae Curie-Skłodowska, Sect. A, 15 (1961), 45–55.

[2] Ruscheweyh, St., *A subordination theorem for Φ-like functions*, J. London Math. Soc. 2, 13 (1976), 275–280.

STRESZCZENIE

Niech L oznacza klasę funkcji holomorficznych i lokalnie jednolistnych w kole $|z| < 1$. Kilka lat temu St. Ruscheweyh dowiódł, że w klasie S ma miejsce relacja podporządkowania

$$(1+t)^2\, 4t^{-1} f(tz) \prec f(z), \quad 0 < t < 1 .$$

Autorzy dokonują pewnego uogólnienia tego wyniku w przypadku $f \in L$.

РЕЗЮМЕ

Пусть L обозначает класс функций голоморфных и локально однолистных в круге $|z| < 1$. Несколько лет тому назад Ст. Рушевей доказал, что в классе S имеет место реляция подчинения

$$(1+t)^2\, 4t^{-1} f(tz) \prec f(z), \quad 0 < t < 1.$$

Авторы обобщают этот результат в случае $f \in L$.