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The Rate of Convergence of Randomly Indexed Sums of Independent Random Variables to a Stable Law

Abstract. This paper presents uniform and nonuniform rates of convergence of randomly indexed sums of independent random variables to a stable law. The presented results extend to the case of randomly indexed sums those given by A. Krajka and Z. Rychlik (1993).

1. Introduction. Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables whose distribution functions $\{F_n, n \geq 1\}$ belong to the domain of attraction of a stable law of exponent α, $0 < \alpha < 2$. Then, under some additional assumptions, there exist constants $\{\tau_n, n \geq 1\}$ and $\{\varsigma_n, n \geq 1\}$ such that $\sum_{j=1}^{n}(X_j - \tau_j)/\varsigma_n$ converges to a stable law, as $n \to \infty$. The speed of this convergence is given in [4]. The main results of [4], Theorem 1 and 2 are repeated here. On the base of this results we derive uniform and nonuniform rates of convergence of randomly indexed sums to a stable law. Uniform rate of convergence is presented in Theorem 3, while nonuniform in Theorem 4. It should be mentioned here that we do not assume anything concerning the interdependence between random indices and random variables $\{X_n, n \geq 1\}$. Theorems 3 and 4 seem to be new in the context of stable convergence.

We close this section with some notations. Let $\{\phi_n, n \geq 1\}$ be the sequence of the characteristic functions of $\{X_n, n \geq 1\}$, and let $\{c_{i,j}, j \geq 1\}$, $i = 1, 2$, be sequences of nonnegative numbers such that $c_{1,j} + c_{2,j} > 0$, $j \geq 1$. Define

$$h_j(x) = 1 - F_j(x) + F_j(-x) - (c_{1,j} + c_{2,j})(x^{-\alpha} \wedge 1)\,,$$
$$d_j(x) = 1 - F_j(x) - F_j(-x) - (c_{1,j} - c_{2,j})(x^{-\alpha} \wedge 1)\,,$$
$$H_n(x) = \sum_{j=1}^{n} h_j(x)\,, \qquad D_n(x) = \sum_{j=1}^{n} d_j(x)\,,$$
$$\overline{H}_n(x) = \sum_{j=1}^{n} |h_j(x)|\,, \qquad \overline{D}_n(x) = \sum_{j=1}^{n} |d_j(x)|\,,$$

Here, and in what follows, $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$. Let us put

$$e_1 = \int_0^\infty u^{-\alpha}\sin(u)\,du\,, \quad e_2 = \begin{cases} -\int_0^\infty u^{-\alpha}\cos(u)du & , \text{ if } \alpha \in (0,1), \\ 1 & , \text{ if } \alpha = 1, \\ \int_0^\infty u^{-\alpha}(1-\cos(u))du & , \text{ if } \alpha \in (1,2), \end{cases}\,,$$

$$b_{1,j} = \int_0^\infty (1-\cos(tx))\,dh_j(x) \quad \text{or} \quad b_{1,j}(t) = t\int_0^\infty \sin(tx)h_j(x)\,dx\ ,$$

$$b_{2,j}(t) = \begin{cases} \int_0^\infty \sin(tx)\,dd_j(x) & , \text{ for } \alpha \in (0,1) \\ t\int_0^\infty (1-\cos(tx))d_j(x)dx & , \text{ for } \alpha \in [1,2), \end{cases}$$

$$a_{1,j} = (c_{1,j}+c_{2,j})e_1\ , \quad a_{2,j} = (c_{1,j}+c_{2,j})e_2\ ,$$

(1) $$\mu_j = \int_0^1 (1-F_j(x)-F_j(-x))dx + \int_1^\infty d_j(x)dx - (c_{1,j}-c_{2,j})\gamma\ ,$$

$$\tau_j = \begin{cases} \mu_j + \sum_{i=1}^{j-1} a_{2,i}\log(\varsigma_j^\alpha/\varsigma_{j-1}^\alpha) + a_{2,j}\log(\varsigma_j^\alpha) & , \text{ if } \alpha = 1, \\ 0 & , \text{ otherwise}\ ,\ j \geq 1, \end{cases}$$

$$\gamma = \lim_{n\to\infty}\left(\sum_{i=1}^n j^{-1} - \ln n\right) \qquad \text{(Euler's constant)},$$

$$C_{1,n} = \sum_{j=1}^n c_{1,j}\ , \qquad C_{2,n} = \sum_{j=1}^n c_{2,j},$$

$$\varsigma_n^\alpha = (C_{1,n}+C_{2,n})e_1\ , \quad \varsigma_0 = 0\ , \quad n \geq 1$$

Throughout the paper, we assume that for some real number λ

(2) $$\lambda = \lim_{n\to\infty}(C_{1,n}-C_{2,n})e_2/(C_{1,n}+C_{2,n})e_1\ ,$$
$$\lim_{n\to\infty}\varsigma_n = +\infty\ ,$$

and for $j \geq 1$, $EX_j = 0$, iff it exists.

Let $G_{\alpha,\lambda}(\,\cdot\,)$ denote the stable law with the characteristic function

(3) $$\psi(t) = \begin{cases} \exp\{-|t|^\alpha(1+i\lambda\,\mathrm{sign}\,(t))\} & , \text{ if } \alpha \neq 1, \\ \exp\{-|t|(1+i\lambda\,\mathrm{sign}\,(t)\ln|t|)\} & , \text{ if } \alpha = 1. \end{cases}$$

Let us define

$$\Delta_n(x) = |P[S_n < x\varsigma_n] - G_{\alpha,\lambda}(x)|\ ,$$

where $S_n = \sum_{j=1}^n (X_j - \tau_j)$.

Let $\{N_n, n \geq 1\}$ be a sequence of positive integer-valued random variables such that

(4) $$\varsigma_{N_n}^\alpha/\varsigma_n^\alpha \xrightarrow{P} \zeta\ , \quad \text{as } n \to \infty\ ,$$

where ζ is a positive random variable independent of $\{N_n, n \geq 1\}$ and $\{X_n, n \geq 1\}$. Let us define

$$T_n = \max\{k : \varsigma_k^\alpha < \zeta\varsigma_n^\alpha\}$$
$$\Delta(N_n, x) = |P(S_{N_n} < x\varsigma_{N_n}) - G_{\alpha,\lambda}(x)|\ ,$$
$$\Delta(T_n, x) = |P(S_{N_n} < x\varsigma_{T_n}) - G_{\alpha,\lambda}(x)|\ ,$$
$$\Delta(n,\zeta,x) = |P(S_{N_n} < x\zeta^{1/\alpha}\varsigma_n) - G_{\alpha,\lambda}(x)|\ , \quad n \geq 1\ .$$

In what follows C, denotes the positive constant which may only be dependent on α and λ.

2. The rates of convergence to a stable law. In [4] it is proved:

Theorem 1. *Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables, and let $\{\varepsilon_n, n \geq 1\}$ be a sequences of positive numbers such that for every $0 \leq t \leq \eta$ and $n \geq 1$,*

$$(5)\quad \begin{aligned} &4\max_{1\leq j}\{t^{\alpha}(|a_{1,j}| + |a_{2,j}|) + |b_{1,j}(t)| + |b_{2,j}(t)|\} \leq 1\,, \quad \text{if } \alpha \neq 1\,,\\ &4\max_{1\leq j}\{t(|a_{1,j}| + |a_{2,j}(\log t)^2| + |\mu_j|) + |b_{1,j}(t)| + |b_{2,j}(t)|\} \leq 1\,, \quad \text{if } \alpha = 1\,, \end{aligned}$$

$$(6)\quad \Big|\sum_{j=1}^{n} a_{2,j}/\varsigma_n^{\alpha} - \lambda\Big| = \varepsilon_n\,,$$

where η is some positive number.
Assume

$$(7)\quad \begin{aligned} &\vartheta(x) = \sup_n\{\overline{H}_n(x) \vee \overline{D}_n(x)\}\varsigma_n^{-\alpha} = o(x^{-\alpha}) \quad \text{as } x \to \infty\,,\\ &\text{and } \sup_x x^{\alpha}\vartheta(x) < \infty\,, \end{aligned}$$

and, in addition, if $\alpha = 1$ then

$$(8)\quad \max_{1\leq j}|b_{j,2}(t)|\,|\ln(t)| < \infty\,,$$

for $0 < t < \eta$.

(a). *If $\{h_n, n \geq 1\}$ and $\{d_n, n \geq 1\}$ are sequences of uniformly ultimately monotone functions on $[0,\infty)$ and $0 < \alpha < 1$, then*

$$(9)\quad \begin{aligned} \sup_x \Delta_n(x) \leq C\Big\{&\varsigma_n^{-2}\int_0^{\varsigma_n} x|H_n(x)|\,dx + \varsigma_n^{-1}\int_0^{\varsigma_n}|D_n(x)|\,dx +\\ &+\int_{\varsigma_n}^{\infty} x^{-1}\{|H_n(x)| + |D_n(x)|\}\,dx + \varepsilon_n + \varsigma_n^{-(2\alpha\wedge 1)}\Big\} = U_a(n)\,. \end{aligned}$$

(b). *If $\int_0^\infty \vartheta(x)\,dx < \infty$, then for $\alpha = 1$*

$$(10)\quad \begin{aligned} \sup_x \Delta_n(x) \leq C\Big\{&\varsigma_n^{-2}\int_0^{\varsigma_n} x|H_n(x)|\,dx + \varsigma_n^{-3}\int_0^{\varsigma_n} x^2|D_n(x)|\,dx +\\ &+\varsigma_n^{-1}\int_{\varsigma_n}^{\infty}\{\,|H_n(x)| + |D_n(x)|\,\}\,dx + \varepsilon_n + \varsigma_n^{-2}(\log\varsigma_n)^2\Big\} = U_b(n)\,. \end{aligned}$$

(c). *If (b) holds and $\{h_n, n \geq 1\}$ is a sequence of uniformly ultimately monotone functions on $[0,\infty)$, then for $\alpha = 1$*

(11)
$$\sup_x \Delta_n(x) \leq C\Big\{\varsigma_n^{-2}\int_0^{\varsigma_n} x|H_n(x)|\,dx + \varsigma_n^{-3}\int_0^{\varsigma_n} x^2|D_n(x)|\,dx+$$
$$+\int_{\varsigma_n}^{\infty} x^{-1}|H_n(x)|\,dx + \varsigma_n^{-1}\int_{\varsigma_n}^{\infty}|D_n(x)|\,dx + \varepsilon_n + \varsigma_n^{-2}(\log\varsigma_n)^2\Big\} = U_c(n)\ .$$

(d). *If $\{h_n, n \geq 1\}$ is a sequence of uniformly ultimately monotone functions, then for $1 < \alpha < 2$*

(12)
$$\sup_x \Delta_n(x) \leq C\Big\{\varsigma_n^{-2}\int_0^{\varsigma_n} x|H_n(x)|\,dx + \varsigma_n^{-3}\int_0^{\varsigma_n} x^2|D_n(x)|\,dx+$$
$$+\int_{\varsigma_n}^{\infty} x^{-1}|H_n(x)|\,dx + \varsigma_n^{-1}\int_{\varsigma_n}^{\infty}|D_n(x)|\,dx + \varepsilon_n + \varsigma_n^{-2}\Big\} = U_d(n)\ .$$

(e). *For $1 < \alpha < 2$*

$$\sup_x \Delta_n(x) \leq C\Big\{\varsigma_n^{-2}\int_0^{\varsigma_n} x|H_n(x)|\,dx + \varsigma_n^{-3}\int_0^{\varsigma_n} x^2|D_n(x)|\,dx+ \tag{13}$$
$$+\varsigma_n^{-1}\int_{\varsigma_n}^{\infty}(|H_n(x)| + |D_n(x)|)\,dx + \varepsilon_n + \varsigma_n^{-2}\Big\} = U_e(n)\ .$$

Theorem 2. *Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables satisfying the assumptions* (5)–(8) *of* Theorem 1 *If for every $n \geq 1$, the functions $\overline{H}_n(x)$ and $\overline{D}_n(x)$ are ultimately monotone on $[0,\infty)$ and $1 < \alpha < 2$, then for every $x \in R$*

(14)
$$(1+|x|)\Delta_n(x) \leq C\Big\{\varsigma_n^{-1}\int_{\varsigma_n}^{\infty}(1 + |\ln(x/\varsigma_n)|)[\overline{D}_n(x) + \overline{H}_n(x)]\,dx+$$
$$+\overline{H}_n(\varsigma_n) + \varsigma_n^{-3}\int_0^{\varsigma_n} x^2\overline{D}_n(x)\,dx + \varsigma_n^{-2}\int_0^{\varsigma_n} x\overline{H}_n(x)\,dx + \varepsilon_n + \varsigma_n^{\alpha-2}\Big\} =$$
$$= U(n)\ .$$

The following theorem presents the uniform rate of convergence of randomly indexed sums of independent nonidentically distributed random variables to a stable law.

Theorem 3. *Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables satisfying the assumptions* (5)–(8) *of* Theorem 1 *with $h_n(x) \geq 0$, $d_n(x) \geq 0$, $n \geq 1$, such that for every $0 \leq p \leq q$, $p, q \in N$,*

$$\sup_x |H_q(x) - H_p(x)|\,|x^\alpha \vee 1| \leq C(\varsigma_q^\alpha - \varsigma_p^\alpha)\ , \quad H_0(x) = 0\ , \tag{15}$$

and, in case $\alpha = 1$,

$$\int_0^\infty |D_q(x) - D_p(x)|\,dx \le C(\varsigma_q(\ln(\varsigma_q) \vee 1) - \varsigma_p(\ln(\varsigma_p) \vee 1)) .$$

Assume, for some $i \in \{a, b, c, d, e\}$*, the assumption* (i) *of* Theorem 1 *holds and for some nonincreasing sequence* $\{\varepsilon(n), n \ge 1\}$ $U_i(n) \le \overline{\varepsilon}(n) \to 0$ *as* $n \to \infty$*, where*

$$\overline{\varepsilon}(n) = \begin{cases} \varepsilon(n) & , \text{ if } \alpha \in (0,1], \\ \varepsilon(n)^{1/2} & , \text{ if } \alpha \in (1,2). \end{cases}$$

If $\{N_n, n \ge 1\}$ *is a sequence of positive integer-valued random variables such that*

(16)
$$P[\,|\varsigma_{N_n}^{\alpha}/\varsigma_{T_n}^{\alpha} - 1| > C\varepsilon(n)] \le C\overline{\varepsilon}(n) , \quad n \ge 1 , \text{ if } \alpha \ne 1$$
$$P[\,|\varsigma_{N_n}(\ln(\varsigma_{N_n}) \vee 1) - \varsigma_{T_n}(\ln(\varsigma_{T_n}) \vee 1)| > C\varepsilon(n)\varsigma_{T_n}] \le C\overline{\varepsilon}(n) , \ n \ge 1, \text{ if } \alpha = 1$$

and

$$P[\sigma_\zeta(U) < C/\overline{\varepsilon}(n)] \le C\overline{\varepsilon}(n) , \quad n \ge 1 , \tag{17}$$

where

$$\sigma_n(U) = \sum_{k=1}^{\infty} (U_i(k))^{-1} I(\varsigma_k^{\alpha} < \zeta\varsigma_n^{\alpha} \le \varsigma_{k+1}^{\alpha}) , \quad n \ge 1 ,$$

then

$$\sup_x \Delta(T_n, x) \le C\overline{\varepsilon}(n) , \quad n \ge 1 , \tag{18}$$

and

$$\sup_x \Delta(N_n, x) \le C\overline{\varepsilon}(n) , \quad n \ge 1 . \tag{19}$$

If, additionally, there exists a constant c_0 *such that*

$$\sup_k (c_{1,k+1} + c_{2,k+1})\varsigma_k^{-\alpha} U_i(k)^{-1} \le c_0 ,$$

then

$$\sup_x \Delta(n, \zeta, x) \le C\overline{\varepsilon}(n) , \quad n \ge 1 . \tag{20}$$

Let us observe, that assumptions $h_n(x) \ge 0$, $d_n(x) \ge 0$, $n \ge 1$ in Theorem 3 may be replaced by the following one:

There exists a positive number ρ such that the sequence $\{\varsigma_k^{\rho} U_i(k), k \ge 1\}$ is nondecreasing.

The nonuniform bounds are the following

Theorem 4. *Let $1 < \alpha < 2$ be given and let $\{X_n, n \geq 1\}$ be a sequence of independent random variables satisfying the assumptions (5)–(8) of Theorem 1, and (15). Assume, for some nonincreasing sequence $\{\varepsilon(n), n \geq 1\}$ $U(n) \leq \varepsilon^{1/2}(n) \to 0$ as $n \to \infty$. If $\{N_n, n \geq 1\}$ is a sequence of positive integer-valued random variables such that for every $|x| \geq 1$*

$$(21)\quad \begin{aligned} &P[\,|\varsigma_{N_n}^{\alpha}/\varsigma_{T_n}^{\alpha} - 1| > c_1\varepsilon(n)|x|^{\alpha-1}] \leq C\varepsilon^{1/2}(n)/(1+|x|)\,, \\ &\qquad \text{if } \varepsilon(n)|x|^{\alpha-1} \leq 1\,, \\ &P[1 - c_2 \leq \varsigma_{N_n}^{\alpha}/\varsigma_{T_n}^{\alpha} \leq 1 + C\varepsilon^{1/2}(n)|x|^{\alpha-1}] \leq C\varepsilon^{1/2}(n)/(1+|x|)\,, \\ &\qquad \text{if } \varepsilon(n)|x|^{\alpha-1} > 1\,, \end{aligned}$$

and

$$(22)\quad P[\sigma_\zeta(U) < C/\varepsilon^{1/2}(n)] \leq C\varepsilon^{(\alpha+1)/(2(\alpha-1))}(n)\,, \quad \text{if } \varepsilon(n)|x|^{\alpha-1} \leq 1\,,$$

where

$$\sigma_\zeta(U) = \sum_{k=1}^{\infty}(U(k))^{-1} I(\varsigma_k^{\alpha} < \zeta\varsigma_n^{\alpha} \leq \varsigma_{k+1}^{\alpha})\,, \quad n \geq 1\,,$$

then

$$(23)\quad \Delta(T_n, x) \leq C\varepsilon^{1/2}(n)/(1+|x|)\,, \quad n \geq 1,$$

and

$$(24)\quad \Delta(N_n, x) \leq C\varepsilon^{1/2}(n)/(1+|x|)\,, \quad n \geq 1\,.$$

If, additionally, in case $\varepsilon(n)|x|^{\alpha-1} < 1$ there exists a constant c_0 such that

$$\sup_k (c_{1,k+1} + c_{2,k+1})\varsigma_k^{-\alpha} U(k)^{-1} \leq c_0\,,$$

then for every $x \in R$

$$(25)\quad \Delta(n, \zeta, x) \leq C\varepsilon^{1/2}(n)/(1+|x|)\,, \quad n \geq 1\,.$$

3. Auxiliary lemmas and proofs. In the proofs of Theorems 1 and 2 we need the following three Lemmas. Lemma 1 is proved in [4],Lemma 2 extends Lemma 10 [6] to the case $\alpha \neq 2$ and Lemma 3 follows from [2] and [8].

Lemma 1. *For every $p > 0$, $0 < \alpha \leq 2$ and $\lambda_1 \in [-1,1]$, $\lambda_2 \in [-1,1]$*

$$(26)\quad \sup_x |G_{\alpha,\lambda_1}(x) - G_{\alpha,\lambda_2}(x)| \leq (\Gamma(1+1/\alpha)/(\pi\alpha))|\lambda_1 - \lambda_2|$$

(27)
$$(1+|x|)|G_{\alpha,\lambda_1}(x) - G_{\alpha,\lambda_2}(x)| \leq 640(2\Gamma((2\alpha-1)/\alpha) + (2+1/\alpha)\Gamma(1-1/\alpha))$$
$$|\lambda_1 - \lambda_2|/\pi \qquad (\text{if } 1 < \alpha < 2)$$

(28) $$\sup_x |G_{\alpha,\lambda_1}(x+p) - G_{\alpha,\lambda_1}(x)| \le (p\Gamma(1/\alpha)/(\pi\alpha)) \wedge 1$$

(29) $$|G_{\alpha,\lambda_1}(x+p) - G_{\alpha,\lambda_1}(x)| \le \Gamma(\alpha)|\,|x+p|^{-\alpha} - |x|^{-\alpha}| \quad (\text{for } 0 < \alpha \le 2,\ \alpha \ne 1)$$

(30) $$\sup_x |G_{\alpha,\lambda_1}(px) - G_{\alpha,\lambda_1}(x)| \le \begin{cases} \frac{2}{\pi\alpha}(p^\alpha \vee p^{-\alpha} + |\lambda_1|)|1 - p^\alpha \wedge p^{-\alpha}|\ , & \text{if } \alpha \ne 1\ , \\ \frac{2}{\pi}|1 - p^{-1} \wedge p|(p \wedge p^{-1} + |\lambda_1|\,|\gamma - 2ei(-1)|)+ \\ \quad + \frac{2}{\pi}|\lambda_1|\,|\ln p|\ , & \text{if } \alpha = 1\ , \end{cases}$$

and

(31) $$|G_{\alpha,\lambda_1}(px) - G_{\alpha,\lambda_1}(x)| \le \Gamma(\alpha)|x|^{-\alpha}|p^{-\alpha} \vee p^\alpha - 1|/\pi\ .$$

where

$$ei(x) = \int_{-\infty}^{x} e^t/t\,dt\ .$$

Lemma 2. *Let X be a random variable. If Y is a nonnegative random variable such that for some $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $0 < \alpha \le 2$*

$$P[\,|Y^\alpha - 1| > \varepsilon_1] < \varepsilon_2\ ,$$

then

(32) $$\sup_x |P[X \le xY] - G_{\alpha,\lambda}(x)| \le \sup_x |P[X \le x] - G_{\alpha,\lambda}(x)| + \varepsilon_2 + C_1\varepsilon_1\ ,$$

where

$$C_1 = \begin{cases} \frac{2}{\pi\alpha}(2 + |\lambda|) + 1 & , \text{ if } \alpha \ne 1\ , \\ \frac{2}{\pi}(3 + |\lambda|(\gamma - 2ei(-1) + 1)) + 1 & , \text{ if } \alpha = 1\ . \end{cases}$$

If $\varepsilon_1|x|^{\alpha-1} \le 1/2$, $1 < \alpha < 2$ and

$$P[\,|Y^\alpha - 1| > \varepsilon_1|x|^{\alpha-1}] < \varepsilon_2/(1+|x|)\ ,$$

then

(33)
$$\begin{aligned} |P[X \le xY] - G_{\alpha,\lambda}(x)| &\le |P[X \le x(1 - \varepsilon_1|x|^{\alpha-1})^{1/\alpha}] \\ &\quad - G_{\alpha,\lambda}(x(1 - \varepsilon_1|x|^{\alpha-1})^{1/\alpha})| \vee |P[X \le x(1 + \varepsilon_1|x|^{\alpha-1})^{1/\alpha}] \\ &\quad - G_{\alpha,\lambda}(x(1 + \varepsilon_1|x|^{\alpha-1})^{1/\alpha})| + 4\varepsilon_1\Gamma(\alpha)/(\alpha(1+|x|)) + \varepsilon_2/(1+|x|)\ . \end{aligned}$$

Proof. Assume $0 < \varepsilon_1 < 1$, as in case $\varepsilon_1 \ge 1$ (32) is obvious. One can easily observe that if $x \ge 0$, then

$$P[X < x(1-\varepsilon_1)^{1/\alpha}] - \varepsilon_2 \le P[X < xY] \le P[X < x(1+\varepsilon_1)^{1/\alpha}] + \varepsilon_2\ .$$

On the other hand if $x<0$, then

$$P[X<x(1+\varepsilon_1)^{1/\alpha}]-\varepsilon_2\le P[X<xY]\le P[X<x(1-\varepsilon_1)^{1/\alpha}]+\varepsilon_2\,.$$

Hence

$$\sup_x|P[X\le xY]-G_{\alpha,\lambda}(x)|\le\sup_x|P[X\le x]-G_{\alpha,\lambda}(x)|+$$
$$+\sup_x|G_{\alpha,\lambda}(x(1-\varepsilon_1)^{1/\alpha})-G_{\alpha,\lambda}(x)|\vee\sup_x|G_{\alpha,\lambda}(x(1+\varepsilon_1)^{1/\alpha})-G_{\alpha,\lambda}(x)|+\varepsilon_2\,.$$

Thus by Lemma 1 one can easily get (32). On the other hand, using the inequalities obtained above with ε_1 replaced by $\varepsilon_1|x|^{\alpha-1}$, we get

$$|P[X\le xY]-G_{\alpha,\lambda}(x)|\le|P[X\le x(1-\varepsilon_1|x|^{\alpha-1})^{1/\alpha}]-G_{\alpha,\lambda}(x(1-\varepsilon_1|x|^{\alpha-1})^{1/\alpha})|\vee$$
$$\vee|P[X\le x(1+\varepsilon_1|x|^{\alpha-1})^{1/\alpha}]-G_{\alpha,\lambda}(x(1+\varepsilon_1|x|^{\alpha-1})^{1/\alpha})|+|G_{\alpha,\lambda}(x)-$$
$$-G_{\alpha,\lambda}(x(1-\varepsilon_1|x|^{\alpha-1})^{1/\alpha})|\vee|G_{\alpha,\lambda}(x)-G_{\alpha,\lambda}(x(1+\varepsilon_1|x|^{\alpha-1})^{1/\alpha})|+\varepsilon_2/(1+|x|)\,.$$

Thus Lemma 1 ends the proof of (33).

Lemma 3. *Let $\{X_n,n\ge1\}$ be a sequence of independent random variables. Then, for every $x>0$,*

$$P[\,|S_n|>x\varsigma_n]\le P[\max_{1\le i\le n}|S_i/\varsigma_i|>x]\le$$
$$\exp\{-C_1((x-x_0)/(2x_0))^{(\ln 2)/(\ln 3)}\}+H_n(x\varsigma_n)+x^{-\alpha}$$

and

$$\max\{1-G_{\alpha,\lambda}(x),G_{\alpha,\lambda}(x)\}\le C|x|^{-\alpha}\,,$$

where $C_1=(\ln 3)/2$ and x_0 is defined by

$$P[\,|S_n|>x_0]\le\frac{1}{24}\,.$$

Proof of Theorem 3. At first assume that $\alpha\ne1$. Using the fact that ζ is independent of $\{X_n,n\ge1\}$, we get

$$|P[S_{T_n}<x\varsigma_{T_n}]-G_{\alpha,\lambda}(x)|\le\sum_{k=1}^{\infty}P[T_n=k]\,|P[S_k<x\varsigma_k]-G_{\alpha,\lambda}(x)|\le$$
$$\le P[\sigma_n(U)<C/\overline{\varepsilon}(n)]+\sum_{k\in D(n)}P[T_n=k]\,|P[S_k<x\varsigma_{T_n}]-G_{\alpha,\lambda}(x)|$$

where here, and in what follows $D(n)=\{k:U_i(k)<\overline{\varepsilon}(n)/C\}$. Thus by Theorem 1,

$$|P[S_{T_n}<x\varsigma_{T_n}]-G_{\alpha,\lambda}(x)|\le C\overline{\varepsilon}(n)\,,\quad n\ge1\,. \tag{34}$$

Let $I_n = \{k \geq 1 : |\varsigma_k^\alpha - \varsigma_{T_n}^\alpha| < C\varepsilon(n)\varsigma_{T_n}^\alpha\}$. Then, according to (16)

$$(35) \qquad P[N_n \notin I_n] \leq C\overline{\varepsilon}(n)\,.$$

Let us put

$$A_n(x) = [\max_{k\in I_n} S_k < x\varsigma_{T_n}]\,, \quad B_n(x) = [\min_{k\in I_n} S_k < x\varsigma_{T_n}]\,, \ n \geq 1\,.$$

Then, by (35)

$$(36) \qquad P[A_n(x)] - C\overline{\varepsilon}(n) \leq P[S_{N_n} < x\varsigma_{T_n}] \leq P[B_n(x)] + C\overline{\varepsilon}(n)\,, \quad n \geq 1\,.$$

On the other hand

$$(37) \qquad P[A_n(x)] \leq P[S_{N_n} < x\varsigma_{T_n}] \leq P[B_n(x)]\,, \quad n \geq 1\,,$$

so that (34), (36) and (37) yield

$$(38) \qquad \sup_x \Delta(T_n, x) \leq \sup_x\{P[B_n(x)] - P[A_n(x)]\} + C\overline{\varepsilon}(n)\,, \quad n \geq 1\,.$$

Thus (38) yields (18) if it is shown that

$$(39) \qquad \sup_x\{P[B_n(x)] - P[A_n(x)]\} \leq C\overline{\varepsilon}(n)\,, \quad n \geq 1\,.$$

But (39) is bounded, from above, by

(40)

$$P[\sigma_n(U) < C/\overline{\varepsilon}(n)] + \sup_x \sum_{k\in D(n)} P[T_n = k]\,|P[\min_{i\in I(k,n)} S_i < x\varsigma_k] - P[\max_{i\in I(k,n)} S_i < x\varsigma_k]\,|\,,$$

where

$$I(k,n) = \{i : \varsigma_k^\alpha(1 - C\varepsilon(n)) \leq \varsigma_i^\alpha \leq \varsigma_k^\alpha(1 + C\varepsilon(n))\}\,.$$

Furthermore, for every $p \in I(k,n)$, we have

$$(41) \qquad \begin{aligned} &P[\min_{i\in I(k,n)} S_i < x\varsigma_k] - P[\max_{I\in I(k,n)} S_i < x\varsigma_k] \\ &= P[\max_{i\in I(k,n)} S_i > x\varsigma_k > S_p] - P[\min_{i\in I(k,n)} S_i < x\varsigma_k < S_p]\,. \end{aligned}$$

We give the evaluation of the last term in (41) only, since the second one can be evaluated similarly. Let us put $p = p(k,n) = \min\{i : i \in I(k,n)\}$ and $q = q(k,n) = \max\{i : i \in I(k,n)\}$. Then, by Theorem 1 and Fubini's theorem, we get

$$(42) \qquad \begin{aligned} P[\min_{i\in I(k,n)} S_i > x\varsigma_k > S_p] &= \int F_1 \ldots F_p\{(z_1,\ldots,z_p) : \\ & x\varsigma_k/\varsigma_p - \max_{p\leq j\leq q} \sum_{i=p+1}^{j} z_i/\varsigma_p \leq \sum_{i=1}^{p} z_i/\varsigma_p \leq x\varsigma_k/\varsigma_p\}dF_{p+1}\ldots F_q \\ & (z_{p+1},\ldots,z_q) \leq CU_i(p) + \int |G_{\alpha,\lambda}(x\varsigma_k/\varsigma_p) - G_{\alpha,\lambda}(x\varsigma_k/\varsigma_p - \\ & - \max_{p\leq j\leq q} \sum_{i=p+1}^{j} z_i/\varsigma_p)|dF_{p+1}\ldots F_q(z_{p+1},\ldots,z_q) \leq \\ & \leq CU_i(n) + E(\max_{p\leq j\leq q} \sup_x |G_{\alpha,\lambda}(x) - G_{\alpha,\lambda}(x - (S_j - S_p)/\varsigma_p)|)\,. \end{aligned}$$

Furthermore, by our assumptions, we have

$$U_i(p) \le (1 - C\varepsilon(n))^{-3/\alpha} C U_i(k) \le \overline{C}\varepsilon(n) . \tag{43}$$

For example, in case $i = a$, (43) follows from inequalities

$$\int_{\varsigma_k(1-c\varepsilon(n))^{1/\alpha}}^{\varsigma_k} (x^{-1} - x\varsigma_k^{-2}(1 - c\varepsilon(n))^{-2/\alpha})|H_p(x)|\, dx < 0 ,$$

$$\int_{\varsigma_k(1-c\varepsilon(n))^{1/\alpha}}^{\varsigma_k} (x^{-1} - x\varsigma_k^{-1}(1 - c\varepsilon(n))^{-1/\alpha})|D_p(x)|\, dx < 0 ,$$

$$\varepsilon_p \le \varepsilon_k + \Big|\sum_{j=p}^{k} a_{2,j}\Big| / \varsigma_p^\alpha \le \varepsilon_k + (\varsigma_k^\alpha - \varsigma_p^\alpha)/\varsigma_p^\alpha \le \varepsilon_k + C\varepsilon(n) ,$$

and, for each $x \ge 0$,

$$|H_p(x)| \le |H_k(x)| , \quad |D_p(x)| \le |D_k(x)| .$$

If $0 < \alpha < 1$, then we apply $q - p$ times Lemma 1 and get

$$\begin{aligned} I &= E\big(\max_{p \le j \le q} \sup_x |G_{\alpha,\lambda}(x) - G_{\alpha,\lambda}(x - (S_j - S_p)/\varsigma_p|\big) \le \\ &\le E \sum_{j=p+1}^{q} |X_j| I(|X_j| \le \varsigma_p)/\varsigma_p + \sum_{j=p+1}^{q} P[\,|X_j| > \varsigma_p] \le \\ &\le - \int_0^{\varsigma_p} x\, d(H_q - H_p)(x)/\varsigma_p - (\varsigma_q^\alpha - \varsigma_p^\alpha) \int_1^{\varsigma_p} x\, dx^{-\alpha}/\varsigma_p + \\ &\quad + (H_q(\varsigma_p) - H_p(\varsigma_p)) + (\varsigma_q^\alpha - \varsigma_p^\alpha)/\varsigma_p^\alpha \le \\ &\le \int_0^{\varsigma_p} (H_q(x) - H_p(x))\, dx/\varsigma_p + C(\varsigma_q^\alpha - \varsigma_p^\alpha)/\varsigma_p^\alpha \le \\ &\le C(\varsigma_q^\alpha - \varsigma_p^\alpha)/\varsigma_p^\alpha . \end{aligned}$$

On the other hand, if $1 < \alpha < 2$, then Holder's inequality and Lemma 1 (also applied $q-p$ times) give

$$I = E(\max_{p\le j\le q}\sup_x |G_{\alpha,\lambda}(x) - G_{\alpha,\lambda}(x-(S_j-S_p)/\varsigma_p|) \le$$
$$\le E\Big|\sum_{j=p+1}^{q} X_j I(|X_j|\le \varsigma_p)|/\varsigma_p + \sum_{j=p+1}^{q} P[\,|X_j|>\varsigma_p] \le$$
$$\le \Big\{\sum_{j=p+1}^{q} E(X_j/\varsigma_p)^2 I(|X_j|\le\varsigma_p)\Big\}^{1/2} + \sum_{j=p+1}^{q} E|X_j|/\varsigma_p I(|X_j|>\varsigma_p)+$$
$$+\Big\{\sum_{j=p+1}^{q} E|X_j|/\varsigma_p I(|X_j|>\varsigma_p)\Big\}^{1/2} \le$$
$$\Big\{-\int_0^{\varsigma_p} x^2\, d(H_q-H_p)(x)/\varsigma_p^2 - (\varsigma_q^\alpha-\varsigma_p^\alpha)\int_1^{\varsigma_p} x^2\, dx^{-\alpha}/\varsigma_p^2\Big\}^{1/2} +$$
$$+\Big\{-\int_{\varsigma_p}^{\infty} x\, d(H_q-H_p)(x)/\varsigma_p - (\varsigma_q^\alpha-\varsigma_p^\alpha)\int_{\varsigma_p}^{\infty} x\, dx^{-\alpha}/\varsigma_p\Big\}^{1/2} -$$
$$-\int_{\varsigma_p}^{\infty} x\, d(H_q-H_p)(x)/\varsigma_p - (\varsigma_q^\alpha-\varsigma_p^\alpha)\int_{\varsigma_p}^{\infty} x\, dx^{-\alpha}/\varsigma_p \le$$
$$\le \Big\{-(H_q(\varsigma_p)-H_p(\varsigma_p)) + \alpha(\varsigma_q^\alpha-\varsigma_p^\alpha)\int_1^{\varsigma_p} x^{1-\alpha}\, dx/\varsigma_p^2 +$$
$$+2\int_0^{\varsigma_p} x(H_q(x)-H_p(x))\, dx/\varsigma_p^2\Big\}^{1/2} + \Big\{\int_{\varsigma_p}^{\infty}(H_q-H_p)(x)\, dx/\varsigma_p +$$
$$+(H_q(\varsigma_p)-H_p(\varsigma_p)) + \alpha(\varsigma_q^\alpha-\varsigma_p^\alpha)\int_{\varsigma_p}^{\infty} x^{-\alpha}\, dx/\varsigma_p\Big\}^{1/2} +$$
$$+(H_q(\varsigma_p)-H_p(\varsigma_p)) + \alpha(\varsigma_q^\alpha-\varsigma_p^\alpha)\int_{\varsigma_p}^{\infty} x^{-\alpha}\, dx/\varsigma_p +$$
$$+\int_{\varsigma_p}^{\infty}(H_q-H_p)(x)\, dx/\varsigma_p \le C(\varsigma_q^\alpha-\varsigma_p^\alpha)^{1/2}/\varsigma_p^{\alpha/2}.$$

Thus, by the estimations given above, we obtain

$$I \le C\bar{\varepsilon}(n)\,. \tag{44}$$

Hence, in case $\alpha \ne 1$, (18) is a consequence of (38)–(44).

If $\alpha = 1$, then

$$|\varsigma_{N_n}(\ln(\varsigma_{N_n})\vee 1) - \varsigma_{T_n}(\ln(\varsigma_{T_n})\vee 1)|/\varsigma_{T_n} \ge |\varsigma_{N_n}/\varsigma_{T_n} - 1| \tag{45}$$

so that

$$I^*(k,n) = \{i : \varsigma_k(\ln(\varsigma_k)\vee 1 - C\varepsilon(n)) \le \varsigma_i(ln(\varsigma_i)\vee 1) < \varsigma_k(ln(\varsigma_k)\vee 1 + C\varepsilon(n))\} \subset$$
$$\subset \{i : \varsigma_k(1-C\varepsilon(n)) \le \varsigma_i < \varsigma_k(1+C\varepsilon(n))\}\,.$$

Thus the estimations (34)–(43) also hold with $I(k,n)$ and I_n replaced by $I^*(k,n)$ and

$$I_n^* = \{k\ge 1 : |\varsigma_k \ln(\varsigma_k)\vee 1 - \varsigma_{T_n}\ln(\varsigma_{T_n}\vee 1| < C\varepsilon(n)\varsigma_{T_n}\}\,,$$

respectively. Furthermore the term I, in the case $\alpha = 1$, can be estimated as follows

$$I \le C(\varsigma_q - \varsigma_p)\ln(\varsigma_p)/\varsigma_p + \sum_{j=p+1}^{q} |\tau_j|/\varsigma_p \le C\varepsilon(n) + \Big\{ \int_0^{\infty} |D_q(x) - D_p(x)|\, dx +$$
$$+ (\varsigma_q - \varsigma_p)(1+\gamma) + (\varsigma_q - \varsigma_p)\ln(\varsigma_p) + \varsigma_p \ln(\varsigma_q/\varsigma_p)\Big\}/\varsigma_p \le$$
$$\le C\varepsilon(n) + \ln(\varsigma_q/\varsigma_p) \le C\overline{\varepsilon}(n) .$$

Thus, in the case $\alpha = 1$ (44) also holds, so that (18) holds for every $0 < \alpha < 2$.

Now (19) follows from (18), (16) and Lemma 2. In the case $\alpha = 1$ we also use (45). On the other hand if

$$\sup_k \{(c_{1,k+1} + c_{2,k+1})\varsigma_k^{-\alpha} U_i(k)^{-1}\} \le c_0 ,$$

then

$$P[\,|\zeta\varsigma_n^\alpha/\varsigma_{T_n}^\alpha - 1| \ge c_0 C^{-1}\overline{\varepsilon}(n)] = \sum_{k=1}^{\infty} P[\zeta\varsigma_n^\alpha > \varsigma_k^\alpha + c_0 C^{-1}\overline{\varepsilon}(n)\varsigma_k^\alpha, \varsigma_k^\alpha < \zeta\varsigma_n^\alpha \le \varsigma_{k+1}^\alpha] \le$$
$$\le \sum_{k=1}^{\infty} P[\zeta\varsigma_n^\alpha > \varsigma_k^\alpha + (c_{1,k+1} + c_{2,k+1})C^{-1}U_i(k)^{-1}\overline{\varepsilon}(n), \varsigma_k^\alpha < \zeta\varsigma_n^\alpha \le \varsigma_{k+1}^\alpha] \le$$
$$\le P[\sigma_n(U) \le C/\overline{\varepsilon}(n)] \le C\overline{\varepsilon}(n) .$$

Thus (20) is also a consequence of (18), (16) and Lemma 2, so that the proof of Theorem 3 is completed.

Proof of Theorem 4. At first let us consider the case $\varepsilon(n)|x|^{\alpha-1} < 1$. Since, in this case for $|x| > 1$

$$\varepsilon^{(\alpha+1)/(2(\alpha-1))}(n) = \varepsilon^{1/2}(n)\varepsilon^{1/(\alpha-1)}(n) \le \varepsilon^{1/2}(n)|x|^{-1} ,$$

so that (22) may be replaced by

$$P[\sigma_\zeta(U) < C/\varepsilon^{1/2}(n)] \le C\varepsilon^{1/2}(n)/(1+|x|) .$$

Now, putting

$$I_n(x) = \{k \ge 1 : |\varsigma_k^\alpha - \varsigma_{T_n}^\alpha| < c_1\varepsilon(n)|x|^{\alpha-1}\varsigma_{T_n}^\alpha\} ,$$
$$I(k,n,x) = \{i \ge 1 : \varsigma_k^\alpha(1 - c_1\varepsilon(n)|x|^{\alpha-1}) \le \varsigma_i^\alpha \le \varsigma_k^\alpha(1 + c_1\varepsilon(n)|x|^{\alpha-1})\}$$

and proceeding step by step as in the proof of Theorem 3 we obtain

$$\Delta(T_n, x) \le C\varepsilon^{1/2}(n)/(1+|x|) + P[\max_{i\in I(k,n,x)} S_i > x\varsigma_k > S_p] + P[\min_{i\in I(k,n,x)} S_i < x\varsigma_k < S_p] .$$

Furthermore, the sum of two last terms, by Theorem 2, may be bounded by

$$C\frac{U(p)}{1+|x|}+E(\max_{p<j\le q}|G_{\alpha,\lambda}(x)-G_{\alpha,\lambda}(x-(S_j-S_p)/\varsigma_p)|)\le$$

$$\le C\frac{\varepsilon^{1/2}(n)}{1+|x|}+E\Big(\max_{p<j\le q}\Big|G_{\alpha,\lambda}\Big(x-\sum_{i=p+1}^{j}X_iI(|X_i|<\varsigma_p)/\varsigma_p\Big)-G_{\alpha,\lambda}(x)\Big|\Big)+$$

$$+E\Big(\max_{p<j\le q}\Big|G_{\alpha,\lambda}\Big(x-\sum_{i=p+1}^{j}X_iI(|X_i|<\varsigma_p)/\varsigma_p\Big)-G_{\alpha,\lambda}(x-(S_j-S_p)/\varsigma_p)\Big|\Big)\le$$

$$\le C\frac{\varepsilon^{1/2}(n)}{1+|x|}+C(1+|x|^{\alpha+1})^{-1}\Big[\Big\{\sum_{i=p+1}^{q}EX_i^2I(|X_i|<\varsigma_p)/\varsigma_p^2\Big\}^{1/2}+$$

$$+\sum_{i=p+1}^{q}E|X_i|I(|X_j|>\varsigma_p)/\varsigma_p\Big]+P[\max_{p\le j\le q}|S_j-S_p|/\varsigma_p>|x|/4]+$$

$$+P\Big[\max_{p\le j\le q}\Big|\sum_{i=p+1}^{q}X_iI[|X_i|<\varsigma_p]/\varsigma_p\Big|>|x|/4\Big]+$$

$$=C\varepsilon^{1/2}(n)/(1+|x|)+C(1+|x|^{\alpha+1})^{-1}[I_1+I_2]+I_3+I_4\ ,$$

where $p=p(x)=\min I(k,n,x)$, $q=q(x)=\max I(k,n,x)$.
The estimation of I_1 and I_2 are given in the proof of Theorem 3. On the other hand, the estimations of I_3 and I_4 one can get from Lemma 3. Thus (23) follows. Inequalities (24) and (25) follow from (23) and the assumptions similarly as in the proof of Theorem 3.

Assume $\varepsilon(n)|x|^{\alpha-1}>1$. Then

$$(1-G_{\alpha,\lambda}(|x|))\vee G_{\alpha,\lambda}(-|x|)\le C|x|^{-\alpha}\le C\varepsilon(n)/(1+|x|)\ . \tag{46}$$

Furthermore, choosing an appropriate constant C, by Lemma 3 for every $k\ge 1$ we get

$$P[\,|S_k|>|x|\varsigma_k]\le C|x|^{-\alpha}\le C\varepsilon(n)/(1+|x|)\ .$$

and

$$P[\max_{i\in I(k,n,x)}|S_i|>|x|\varsigma_k>S_p]\le C(\varsigma_q^\alpha-\varsigma_p^\alpha)/\varsigma_k^\alpha|x|^\alpha$$

$$\le C|x|^{-\alpha}+C\varepsilon^{1/2}(n)|x|^{\alpha-1}/(1+|x|^\alpha)\le C\varepsilon^{1/2}(n)/(1+|x|)\ .$$

where

$$I(k,n,x)=\{i\ge 1:\varsigma_k^\alpha(1-c_2)\le\varsigma_i^\alpha\le\varsigma_k^\alpha(1+\varepsilon^{1/2}(n)|x|^{\alpha-1})\}\ .$$

Hence

$$P[|S_{N_n}|>|x|\varsigma_{T_n}]\le C|x|^{-\alpha}\le C\varepsilon^{1/2}(n)/(1+|x|)$$

what with (46) gives (23). Inequalities (24) and (25) follows from (23) and

$$P[\,|S_{N_n}|>|x|\varsigma_{N_n}/(1-c_2)^{1/\alpha}]\vee P[\,|S_{N_n}|>C|x|\varsigma_n\zeta^{1/\alpha}]\le$$

$$\le P[\,|S_{N_n}|>|x|\varsigma_{T_n}]\le C|x|^{-\alpha}\le C\varepsilon^{1/2}(n)/(1+|x|)\ .$$

The first inequality is a consequence of the definition of T_n, i.e., $\varsigma_{T_n} < \zeta^{1/\alpha}\varsigma_n$, and (21). By (21) we get $\varsigma_{T_n} < \varsigma_{N_n}(1-c_2)^{1/\alpha}$ provided $1-c_2 < \varsigma_{N_n}^{\alpha}/\varsigma_{T_n}^{\alpha} < 1+c_1\varepsilon^{1/2}(n)|x|^{\alpha-1}$.

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