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Periodic Solutions of Equations of Higher Order with Right Invertible Operators, Induced by Functional Shifts

Abstract. This paper is a continuation of the author's earlier work [1]. Here, we shall look for periodic solutions induced by functional shifts of initial value problems for linear equations of order $N \geq 2$ in a right invertible operators. Sufficient and necessary conditions for the unique solvability of the problems in the spaces of these periodic elements are given. Functional shifts for right invertible operators have been considered by the author [2]-[9] (cf. related results of D.Przeworska-Rolewicz [12],[13]).

0. Let X be a linear space over the field $\mathbb{C}$ of complex numbers. Denote by $L(X)$ the set of all linear operators with domains and ranges in X and by $L_0(X)$ the set of those operators from $L(X)$ which are defined on the whole space X. An operator $D \in L(X)$ is said to be right invertible if there exists an operator $R \in L(X)$ such that $DR = I$. The set of all right invertible operators belonging to $L(X)$ will be denoted by $R(X)$. For a $D \in R(X)$ we denote by $\mathcal{R}_D$ the set of all its right inverses. In the sequel we shall assume that $\dim \ker D > 0$, i.e. D is right invertible but not invertible and that right inverses belong to $L_0(X)$. An operator $F \in L_0(X)$ is said to be an initial operator for D corresponding to an $R \in \mathcal{R}_D$ if

$$F^2 = F\,, \quad FX = \ker D \quad \text{and} \quad FR = 0\,.$$

This definition implies that F is an initial operator for D if and only if there is an operator $R \in \mathcal{R}_D$ such that $F = I - RD$ on dom D. The set of all initial operators for a given $D \in R(X)$ is denoted by $\mathcal{F}_D$. One can prove that any projection onto $\ker D$ is an initial operator for D. If we know at least one right inverse R, we can determine the set $\mathcal{R}_D$ of all right inverses and the set $\mathcal{F}_D$ of all initial operators for a given $D \in R(X)$. The theory of right invertible operators and its applications is presented by D.Przeworska-Rolewicz in [13].

Here and in the sequel we admit that $0^0 := 1$. We also write: $\mathbb{N}$ for the set of all positive integers and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$.

For a given operator $D \in R(X)$ we shall write (cf. [13], [14]):

$$S := \bigcup_{i=1}^{\infty} \ker D^i\,. \tag{0.1}$$

If $R \in \mathcal{R}_D$ then the set S is equal to the linear span $P(R)$ of all D-monomials, i.e.

$$S = P(R) := \operatorname{lin}\{R^k z : z \in \ker D\,,\ k \in \mathbb{N}_0\}\,. \tag{0.2}$$

Evidently, the set $P(R)$ is independent of the choice of the right inverse R.

In the sequel, K will stand either for the disk $K_\rho := \{h \in \mathbb{C} : |h| < \rho,\ 0 < \rho < \infty\}$, or for the complex plane $\mathbb{C}$. Denote by $H(\Omega)$ the class of all functions analytic on a set $\Omega \subseteq \mathbb{C}$. Suppose that a function $f \in H(K)$ has the following expansion

$$f(h) = \sum_{k=0}^{\infty} a_k h^k \quad \text{for all } h \in K. \tag{0.3}$$

Definition 0.1. Suppose that $D \in R(X)$ and $\dim\ker D > 0$. A family $T_{f,K} = \{T_{f,h}\}_{h\in K} \subset L_0(X)$ is said to be a family of functional shifts for the operator D induced by the function f if

$$T_{f,h}x = [f(hD)]x := \sum_{k=0}^{\infty} a_k h^k D^k x \quad \text{for all } h \in K;\ x \in S, \tag{0.4}$$

where S is defined by Formula (0.1) .

We should point out that by definition of the set S, the last sum has only a finite number of members different than zero.

Proposition 0.1. (cf. [6]) *Suppose that $D \in R(X)$ and $\dim\ker D > 0$, F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$ and a family $T_{f,K} = \{T_{f,h}\}_{h\in K} \subset L_0(X)$. Then following two conditions are equivalent:*

a) $T_{f,K}$ is a family of functional shifts for the operator D induced by the function f,

$$\text{b)}\quad T_{f,h}R^kF = \sum_{j=0}^{k} a_j h^j R^{k-j}F \quad \textit{for all } h \in K;\ k \in \mathbb{N}_0 . \tag{0.5}$$

Formula (0.5) implies

Proposition 0.2. (cf. [5]) *Suppose that $D \in R(X)$, $\dim\ker D > 0$ and $T_{f,K} = \{T_{f,h}\}_{h\in K}$ is a family of functional shifts for the operator D induced by the function f . Then for all $h \in K$; $z \in \ker D$*

$$T_{f,h}R^kz = \sum_{j=0}^{k} a_j h^j R^{k-j}z , \quad \textit{where } k \in \mathbb{N}_0 .$$

We denote by $X_{T_{f,h}}$ the space of $T_{f,h}$-periodic elements, i.e.

$$X_{T_{f,h}} := \{x \in X : T_{f,h}x = x\} , \quad h \in K . \tag{0.6}$$

Suppose, that $D \in R(X)$ and an operator $R \in \mathcal{R}_D$. Denote by $X_m \subset P(R)$, $m \in \mathbb{N}_0$ span of D-monomials

$$X_m = \operatorname{lin}\{R^kz : z \in \ker D ,\ 0 \le k \le m\} . \tag{0.7}$$

Clearly,

$$X_0 = \ker D\ .$$

Note , the set X_m $(m \in \mathbb{N}_0)$ is independent of the choice of a right inverse, i.e. if $R_1, R_2 \in \mathcal{R}_D$, $R_1 \neq R_2$ then

$$\text{(0.8)} \qquad \operatorname{lin}\{R_1^k z : z \in \ker D, 0 \le k \le m\} = \operatorname{lin}\{R_2^k z : z \in \ker D, 0 \le k \le m\}\ ,$$

(cf.[13]).

The general form of the solution of the equation

$$\text{(I)} \qquad D^n x = y\ , \quad y \in X\ ,$$

is given by the formula

$$\text{(II)} \qquad x = R^n y + \sum_{k=0}^{n-1} R^k z_k\ ,$$

where $z_0, z_1, \ldots, z_{n-1} \in \ker D$ are arbitrary and $n > 1$, $R \in \mathcal{R}_D$ are arbitrarily fixed (cf. [13]).

1. In the present section we shall look for solutions of the equation (I) belonging to the space $X_{T_{f,h}}$ in the case $f(0) \neq 0$, where $X_{T_{f,h}}$ is defined by Formula (0.6) . In the sequel, K will stand either for the disk K_ρ $(0 < \rho < \infty)$ or for $\mathbb{C}$. As before, the function $f \in H(K)$ has the expansion (0.3).

Proposition 1.1. *Suppose that $f \in H(K)$, $f(0) = 1$, $D \in R(X)$ and* $\dim\ker D > 0$. *Let $T_{f,K} = \{T_{f,h}\}_{h \in K} \subset L_0(X)$ be a family of functional shifts for D induced by the function f and let $R \in \mathcal{R}_D$ be arbitrarily fixed. If equation* (I) *has a solution belonging to the space $X_{T_{f,h}}$ $(0 \neq h \in K)$, then*

$$\text{(1.1)} \qquad (I - T_{f,h})R^n y \in X_{n-2}\ ,$$

where X_m $(m \in \mathbb{N}_0)$ is determined by formula (0.7).

Proof. Fix $0 \neq h \in K$ and let $x \in X_{T_{f,h}}$ be a solution of equation (I). Then there exist $z_0, z_1, \ldots, z_{n-1} \in \ker D$ such that $x = R^n y + \sum_{k=0}^{n-1} R^k z_k$. Our assumptions and Proposition 0.2 together imply

$$T_{f,h} x = T_{f,h} R^n y + T_{f,h}\Big\{\sum_{k=0}^{n-1} R^k z_k\Big\} = T_{f,h} R^n y + \sum_{k=0}^{n-1} T_{f,h} R^k z_k =$$

$$= T_{f,h} R^n y + \sum_{k=0}^{n-1}\sum_{j=0}^{k} a_{k-j} h^{k-j} R^j z_k = T_{f,h} R^n y + a_0 z_0 +$$

$$+ \sum_{k=1}^{n-1}\sum_{j=0}^{k-1} a_{k-j} h^{k-j} R^j z_k + a_0 \sum_{k=1}^{n-1} R^k z_k = T_{f,h} R^n y + x - R^n y +$$

$$+ \sum_{k=1}^{n-1}\sum_{m=1}^{k} a_{k-m+1} h^{k-m+1} R^{m-1} z_k = x + (T_{f,h} - I) R^n y +$$

$$+ \sum_{m=1}^{n-1}\sum_{k=m}^{n-1} a_{k-m+1} h^{k-m+1} R^{m-1} z_k\ .$$

Hence,

$$T_{f,h}x = x + (T_{f,h} - I)R^n y + \sum_{k=0}^{n-2} R^k z'_k \,, \tag{1.2}$$

where

$$z'_k = \sum_{j=k+1}^{n-1} a_{j-k} h^{j-k} z_j \,, \quad k = 0, 1, \ldots, n-2. \tag{1.3}$$

This and the equality $T_{f,h}x = x$ together imply

$$(I - T_{f,h})R^n y = \sum_{k=0}^{n-2} R^k z'_k \in X_{n-2} \,. \tag{1.4}$$

The following proposition is weaker then the proposition inverse to Proposition 1.1.

Proposition 1.2. *Suppose that all assumptions of* Proposition 1.1. *are satisfied and* $f'(0) \neq 0$. *Let condition* (1.1) *be satisfied and let formula* (1.4) *hold. Then all solutions of equation* (I) *are given by formula* (II) *with an arbitrary* $z_0 \in \ker D$ *and* $z_1, z_2, \ldots, z_{n-1} \in \ker D$, *determined by the recursion formula*

$$\tag{1.5} \begin{aligned} z_{n-1} &= a_1^{-1} h^{-1} z'_{n-2} \\ z_{n-1-m} &= a_1^{-1} h^{-1} \Big[z'_{n-2-m} - \sum_{j=n-m}^{n-1} a_{j-n+m+2} h^{j-n+m+2} z_j \Big] , \end{aligned}$$

and belong to the space $X_{T_{f,h}}$ $(0 \neq h \in K)$.

Proof. Let $0 \neq h \in K$ be arbitrarily fixed. Consider the equalities (1.3) as a linear system of equations with respect to $z_1, z_2, \ldots, z_{n-1}$. Since by our assumptions the determinant of this system

$$\begin{vmatrix} a_1 h & a_2 h^2 & \ldots & \ldots & \ldots & a_{n-1} h^{n-1} \\ 0 & a_1 h & a_2 h^2 & \ldots & \ldots & a_{n-2} h^{n-2} \\ 0 & 0 & a_1 h & a_2 h^2 & \ldots & a_{n-3} h^{n-3} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots\ 0 & a_1 h \end{vmatrix} = (a_1 h)^{n-1} \neq 0 \,,$$

therefore we conclude that the system (1.3) has a unique solution. It is easy to verify that this solution can be determined by formula (1.5). Take

$$x = R^n y + \sum_{k=0}^{n-1} R^k z_k \,, \tag{1.6}$$

where $z_0 \in \ker D$ is arbitrarily fixed, $z_1, z_2, \ldots, z_{n-2}$ are determined by formula (1.5). Obviously, the element x given in (1.6) satisfies the equation (I). Moreover, $x \in X_{T_{f,h}}$. Indeed,

$$T_{f,h}x = x - R^n y + T_{f,h}R^n y + \sum_{k=1}^{n-1}\sum_{j=0}^{k-1} a_{k-j}h^{k-j}R^j z_k =$$

$$= x + (T_{f,h} - I)R^n y + \sum_{k=0}^{n-2} R^k z_k' ,$$

(cf.the proof of Proposition 1.1.). The equality $T_{f,h}x = x$ follows from equality (1.4).

Clearly, the proof of Proposition 1.1. shows that any solution of equation (I) belongig to $X_{T_{f,h}}$ must be of the form (II) with $z_1, z_2, \ldots, z_{n-2} \in \ker D$ determined by formula (1.5).

Proposition 1.1. and Proposition 1.2. together imply

Corollary 1.1. *Suppose that all assumptions of* Proposition 1.1. *are satisfied. If the equation*

(1.7) $$D^2 x = y , \quad y \in X ,$$

has a solution belonging to the space $T_{f,h}$ $(0 \neq h \in K)$ *then*

$$(I - T_{f,h})R^2 y \in \ker D .$$

If this condition is satisfied, $f'(0) \neq 0$ and $(I - T_{f,h})R^2 y = z'$, where $z' \in \ker D$, then the following formula

$$x = R^2 y + z + a_1^{-1}h^{-1}Rz' ,$$

where $z \in \ker D$ is arbitrary, determines all solutions of equation (1.7) which belong to the space $X_{T_{f,h}}$.

As an immediate consequnce of Proposition 1.1. and Proposition 1.2. we obtain

Proposition 1.3. *Suppose that all assumptions of* Proposition 1.2. *are satisfied and* $F \in \mathcal{F}_D$ *is an initial operator for* D *corresponding to* $R \in \mathcal{R}_D$. *Then a necessary and sufficient condition for the initial value problem*

$$D^n x = y , \quad y \in X , \; n > 1 ,$$
$$Fx = z_0 , \quad z_0 \in \ker D$$

to have solutions in the space $X_{T_{f,h}}$, $0 \neq h \in K$, *is that condition* (1.1) *is satisfied. If this condition is satisfied and formula* (1.4) *holds, then a unique solution of this problem exists and is of the form*

$$x = R^n y + z_0 + \sum_{k=1}^{n-1} R^k z_k ,$$

where $z_1, z_2, \dots, z_{n-1} \in \ker D$ are determined by formulae (1.5).

Proposition 1.4. *Suppose that all assumptions of* Proposition 1.1. *are satisfied. Then the condition* (1.1) *is independent of the choice of a right inverse, i.e. if $R_1, R_2 \in \mathcal{R}_D$, $R_1 \neq R_2$ and $(I - T_{f,h})R_1^n y \in X_{n-2}$ then $(I - T_{f,h})R_2^n y \in X_{n-2}$.*

Proof. Indeed, for $R_1, R_2 \in \mathcal{R}_D$ we have the formula

$$R_2^k z = R_1^k z + \sum_{j=0}^{k-1} R_1^j z_j \quad \text{for all } z \in \ker D\ ,\ k \in \mathbb{N}\ , \tag{1.8}$$

where $z_0, z_1, \dots, z_{k-1} \in \ker D$ (cf. [13]). Let x be a solution of equation (I). Then formula (II) implies

$$x = R_1^n + \sum_{k=0}^{n-1} R_1^k z_k^{(1)}$$

and

$$x = R_2^n y + \sum_{k=0}^{n-1} R_2^k z_k^{(2)}\ ,$$

where $z_0^{(1)}, z_0^{(2)}, z_1^{(1)}, z_1^{(2)}, \dots, z_{n-1}^{(1)}, z_{n-1}^{(2)} \in \ker D$. This and formula (1.8) together imply

$$R_2^n y = R_1^n y + \sum_{k=0}^{n-1} R_1^k z_k^{*}\ , \tag{1.9}$$

where $z_0^{*}, z_1^{*}, \dots, z_{n-1}^{*} \in \ker D$. Let $(I - T_{f,h})R_1^n y = \sum_{k=0}^{n-2} R_1^k z_k'$, where $z_0', z_1', \dots, z_{n-2}' \in \ker D$ (this is possible by formula 0.8), $0 \neq h \in K$. Proposition 0.1. and formula (1.9) together imply for $0 \neq h \in K$

$$(I - T_{f,h})R_2^n y = (I - T_{f,h})R_1^n y + \sum_{k=0}^{n-1} R_1^k z_k^* - \sum_{k=0}^{n-1} T_{f,h} R_1^k z_k^* =$$

$$= \sum_{k=0}^{n-2} R_1^k z_k' + \sum_{k=0}^{n-1} R_1^k z_k^* - \sum_{k=0}^{n-1} \sum_{j=0}^{k} a_{k-j} h^{k-j} R_1^j z_k^* =$$

$$= \sum_{k=0}^{n-2} R_1^k z_k' + \sum_{k=0}^{n-1} R_1^k z_k^* - a_0 z_0^* - \sum_{k=1}^{n-1} a_0 h^0 R_1^k z_k^* +$$

$$- \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} a_{k-j} h^{k-j} R_1^j z^* = \sum_{k=0}^{n-2} R_1^k z_k' +$$

$$- \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} a_{k-j} h^{k-j} R_1^j z_k^*$$

$$= \sum_{k=0}^{n-2} R_1^k z_k' - \sum_{k=0}^{n-2} R_1^k \Big[\sum_{j=k+1}^{n-1} a_{j-k} h^{j-k} z_j^* \Big] =$$

$$= \sum_{k=0}^{n-2} R_1^k z_k^{**},$$

where $z_0^{**}, z_1^{**}, \ldots, z_{n-2}^{**} \in \ker D$. This and formula (0.8) together imply the conclusion.

In a similar way as Proposition 1.1. we prove

Theorem 1.1. *Suppose that all assumptions of* Proposition 1.1. *are satisfied. Let $a_m \neq 0$ $(m > 0)$ be a coefficient in the expansion* (0.3) *of the function f such that $a_j = 0$ for $0 < j < m$; $m > 1$. If the equation* (I) *has a solution belonging to the space $X_{T_{f,h}}$ $(0 \neq h \in K)$. Then*

(1.10) a) $(I - T_{f,h})R^n y \in X_{n-1-m}$ *for* $1 \leq m \leq n-1$;

(1.11) b) $T_{f,h} R^n y = R^n y$ *for* $m \geq n$.

In a similar way as Proposition 1.2. we prove

Theorem 1.2. *Suppose that all assumptions of* Theorem 1.1. *are satisfied and $0 \neq h \in K$. Let either the condition* (1.10) *and*

(1.12) $$(I - T_{f,h})R^n y = \sum_{k=0}^{n-1-m} R^k z_k', \quad \text{for } 1 \leq m \leq n-1,$$

where $z_0', z_1', \ldots, z_{n-1-m}' \in \ker D$, or the condition (1.11) *be satisfied for $m \geq n$. Then*

(i) *For* $1 \le m < n$ *all solutions of equation* (I) *belonging to* $X_{T_{f,h}}$ *are given by formula* (II) *with arbitrary* $z_0, z_1, \ldots, z_{m-1} \in \ker D$ *and* $z_m, z_{m+1}, \ldots, z_{n-1} \in \ker D$ *determined by the recursion formula*
(1.12')

$$z_{n-1} = a_m^{-1} h^{-m} z'_{n-m-1}$$

$$z_{n-1-r} = a_m^{-1} h^{-m}\left[z'_{n-m-1-r} - \sum_{j=n-m-r+1}^{n-1} a_{j-n+m+1+r} h^{j-n+m+1+r} z_j\right],$$

$$r = 1, 2, \ldots, n-1-m .$$

(ii) *For* $m \ge n$ *an arbitrary solution of equation* (I) *belongs to the space* $X_{T_{f,h}}$.

Theorem 1.2. implies (cf. Proposition 1.3.)

Theorem 1.3. *Suppose that all assumptions of* Theorem 1.2. *are satisfied and* $m \in <1, n>$ $(n > 1)$. *Let* F *be an initial operator for* D *corresponding to* $R \in \mathcal{R}_D$. *Then the initial value problem*

(I) $$D^n x = y \quad , \; y \in X \; ;$$

(1.13) $$F D^k x = z_k \quad , \; z_k \in \ker D \; ; \; k = 0, 1, \ldots, m-1 ,$$

has a unique solution belonging to the space $X_{T_{f,h}}$ $(0 \ne h \in K)$ *which is of the form* (II) *with* $z_m, z_{m+1}, \ldots, z_{n-1} \in \ker D$ *determined by formula* (1.12') *for* $m < n$.

We have also a theorem which is inverse to Theorem 1.3.

Theorem 1.4. *Suppose that all assumptions of* Theorem 1.1. *are satisfied and* $m \in <1, n>$, $n > 1$. *Let the initial value problem* (I), (1.13) *has an unique solution belonging to* $X_{T_{f,h}}$ $(0 \ne h \in K)$. *Then the element* $y \in X$ *satisfies condition* (1.10) *for* $1 \le m < n-1$ *or condition* (1.11) *for* $m = n$.

Example 1.1. Suppose that $D \in R(X)$ and $\dim \ker D > 0$, $R \in \mathcal{R}_D$ is arbitrarily fixed. Let $S_K = \{S_h\}_{h \in K} \subset L_0(X)$ be a family of functional shifts for D induced by the function $f(h) = e^h$. Then Proposition 1.1 and Proposition 1.2. together imply that equation (I) has a solution belonging to the space $X_{S_h} := \{x \in X : S_h x = x\}$ $(0 \ne h \in K)$,if and only if

(1.14) $$(I - S_h) R^n y \in X_{n-2} .$$

If this condition is satisfied and

(1.15) $$(I - S_h) R^n y = \sum_{k=0}^{n-2} R^k z'_k ,$$

where $z'_0, z'_1, \ldots, z'_{n-2} \in \ker D$. Then formula (II) determines all solutions of equation (I) which belong to the space X_{S_h} with an arbitrary $z_0 \in \ker D$ and

(1.16) $$z_{n-1-k} = \sum_{j=0}^{k} (-1)^j \frac{h^{j-1}}{(j+1)!} z'_{n-2+k-j} ,$$

where $k = 0, 1, \ldots, n-2$. Proposition 1.3. implies that the equation (I) with the initial condition

$$Fx = z_0\ , \quad z \in \ker D$$

where F is an initial operator for D corresponding to $R \in \mathcal{R}_D$ has a solution belonging to X_{S_h}, $0 \neq h \in K$, if and only if y satisfies condition (1.14). If this condition is satisfied and equality (1.15) holds, then a unique solution which belongs to $X_{T_{j,h}}$ exists and is of the form (II) where $z_1, z_2, \ldots, z_{n-1} \in \ker D$ are determined by formula (1.16).

Example 1.2. Suppose that $D \in R(X)$ and $\dim \ker D > 0$, F is an initial operator for D corresponding to $R \in \mathcal{R}_D$. Let $c_K = \{c_h\}_{h\in K}$, $ch_K = \{ch_h\}_{h\in K} \subset L_0(X)$ be families of functional shifts for D induced by the functions cosinus, hyperbolic cosinus, respectively. Then the equation (1.7) has a solution belonging to the space $X_{c_h} := \{x \in X : c_h x = x\}(X_{ch_h})$ if and only if $R^2 y \in X_{c_h}$ $(R^2 y \in X_{ch_h})$ for $0 \neq h \in K$. If this condition is satisfied then formula (II) $\{n = 2\}$ determines all solutions of equation (1.7) which belong to X_{c_h} (X_{ch_h}). This follows from Theorem 1.1. and Theorem 1.2.

As a conseqence of these theorems we conclude that the equation (I) with $n > 2$ has a c_h-periodic solution $(0 \neq h \in K)$ if and only if

$$(I - c_h)R^n y \in X_{n-3}\ . \tag{1.17}$$

If this condition is satisfied, $n > 2$, $0 \neq h \in K$ and

$$(I - c_h)R^n y = \sum_{k=0}^{n-3} R^k z'_k\ , \tag{1.18}$$

where $z'_0, z'_1, \ldots, z'_{n-3} \in \ker D$, then formula (II) determines all c_h-periodic solutions of equation (I) with arbitrary $z_0, z_1 \in \ker D$ and $z_2, z_3, \ldots, z_{n-1} \in \ker D$ which are determined by the following recursion formula

(1.19)

$$z_{n-1-2k} = \begin{cases} -2h^{-2} z'_{n-3} & \text{for } k = 0 \\ -2h^{-2} z'_{n-3-2k} & +2\sum_{j=1}^{k} (-1)^{j+1} \frac{h^{2j}}{(2j+2)!} z_{n-1-2(k-j)} \\ & \text{for } k = 1, 2, \ldots, [(n-3)/2]\ , \end{cases}$$

$$z_{n-2-2m} = \begin{cases} -2h^{-2} z'_{n-4} & \text{for } k = 0 \\ -2h^{-2} z'_{n-4-2m} & +2\sum_{j=1}^{m} (-1)^{j+1} \frac{h^{2j}}{(2j+2)!} z_{n-2-2(m-j)} \\ & \text{for } m = 1, 2, \ldots, [(n-4)/2]\ . \end{cases}$$

Equation (I) with $n > 2$ has a ch_h-periodic solution $(0 \neq h \in K)$ if and only if

$$(I - ch_h)R^n y \in X_{n-3}\ . \tag{1.20}$$

If this condition is satisfied and

$$(1.21)\qquad (I - ch_h)R^n y = \sum_{k=0}^{n-3} R^k z'_k ,$$

where $z'_0, z'_1, \ldots, z'_{n-3} \in \ker D$. Then all ch_h-periodic solutions ($K \ni h \neq 0$) of equation (I) are given by formula (II) with arbitrary $z_0, z_1 \in \ker D$ and $z_2, z_3, \ldots, z_{n-1} \in \ker D$ which are determined by the following recursion formula

$$z_{n-1-2k} = \begin{cases} 2h^{-2} z'_{n-3} & \text{for } k = 0 \\ 2h^{-2} z'_{n-3-2k} - 2\sum_{j=1}^{k} \frac{h^{2j}}{(2j+2)!} z_{n-1-2(k-j)} \end{cases}$$

$$(1.22)\qquad \text{for } k = 1, 2, \ldots, [(n-3)/2] ,$$

$$z_{n-2-2m} = \begin{cases} 2h^{-2} z'_{n-4} & \text{for } k = 0 \\ 2h^{-2} z'_{n-4-2m} - 2\sum_{j=1}^{m} \frac{h^{2j}}{(2j+2)!} z_{n-2-2(m-j)} \end{cases}$$

$$\text{for } m = 1, 2, \ldots, [(n-4)/2] .$$

The equation (1.7) with the initial condition

$$FD^p x = z_p , \quad z_p \in \ker D , \quad (p = 0, 1)$$

has a c_h-periodic (ch_h-periodic) solution which is of the form

$$x = z_0 + Rz_1 + R^2 y .$$

This follows from Theorem 1.3.

Theorem 1.3. imply that for arbitrary integer $n > 2$ the equation (I) with the initial condition $FD^p x = z_p$, $z_p \in \ker D$ $(p = 0, 1)$ has a c_h-periodic (ch_h-periodic) solution if and only if y satisfies condition (1.17) (condition (1.20)). If this condition is satisfied and equality (1.18) (equality (1.21)) holds, then the above initial value problem has a unique c_h-periodic (ch_h-periodic) solution which is of the form (II), where $z_2, z_3, \ldots, z_{n-1} \in \ker D$ are determined by formula (1.19) (formula (1.22)).

2. In this section we shall look for solutions of the equation (I) belonging to the space $X_{T_{f,h}}$ ($0 \neq h \in K$) in the case $f(0) = 0$, where $X_{T_{f,h}}$ defined by formula (0.6). Here, we still assume that the function $f \in H(K)$ has the expansion

$$(2.1)\qquad f(h) = \sum_{k=0}^{\infty} a_k h^k \quad \text{for all } h \in K ,$$

where K will stand either for the disk K_ρ ($0 < \rho < \infty$) or for $\mathbf{C}$.

Theorem 2.1. *Suppose that $D \in R(X)$, $\dim \ker D > 0$ and $R \in \mathcal{R}_D$ is arbitrarily fixed. Let $a_m \neq 0$ ($m \in \mathbf{N}$) be a coefficient in the expansion (2.1) of the function $f \in H(K)$ such that $a_j = 0$ for $j < m$. Let $T_{f,K} = \{T_{f,h}\}_{h \in K} \subset L_0(X)$ be a* *family of functional shifts for D induced by f. If equation* (I) *has a solution belonging to the space $X_{T_{f,h}}$ $(0 \neq h \in K)$ then*

$$(2.2) \qquad (I - T_{f,h})R^n y \in X_{n-1},$$

where X_k $(k \in \mathbb{N}_0)$ is determined by formula (0.7).

Proof. Fix $0 \neq h \in K$ and let $x \in X_{T_{f,h}}$ be a solution of equation (I). Then there exist $z_k \in \ker D$ $(k = 0, 1, 2, \dots)$ such that the formula (II) holds i.e. $x = R^n y + \sum_{k=0}^{n-1} R^k z_k$. Our assumptions and Proposition 0.2 together imply for $m < n$

(2.3)

$$T_{f,h}x = T_{f,h}R^n y + T_{f,h}\left\{\sum_{k=0}^{n-1} R^k z_k\right\} = T_{f,h}R^n y + \sum_{k=0}^{n-1} T_{f,h}R^k z_k =$$

$$= T_{f,h}R^n y + \sum_{k=0}^{n-1}\sum_{j=0}^{k} a_{k-j}h^{k-j}R^j z_k =$$

$$= T_{f,h}R^n y + \sum_{j=0}^{n-1}\sum_{k=j}^{n-1} a_{k-j}h^{k-j}R^j z_k = T_{f,h}R^n y + \sum_{j=0}^{n-1} R^j\left\{\sum_{k=j}^{n-1} a_{k-j}h^{k-j}z_k\right\} =$$

$$= T_{f,h}R^n y + \sum_{j=0}^{n-1} R^j\left\{\sum_{l=0}^{n-1-j} a_l h^l z_{l+j}\right\} = T_{f,h}R^n y + \sum_{j=0}^{n-m-1} R^j\left\{\sum_{l=m}^{n-1-j} a_l h^l z_{l+j}\right\} =$$

$$= T_{f,h}R^n y + \sum_{j=0}^{n-m-1} R^j z_j^*,$$

where $z_j^* = \sum_{l=m}^{n-1-j} a_l h^l z_{l+j}$. Write

$$0 = T_{f,h}x - x = T_{f,h}R^n y - R^n y + \sum_{j=0}^{n-m-1} R^j z_j^* - \sum_{k=0}^{n-1} R^k z_k .$$

Hence,

$$(2.4) \qquad (T_{f,h} - I)R^n y = \sum_{k=0}^{n-1} R^k z_k' \in X_{n-1},$$

where

$$(2.5) \qquad \begin{cases} z_k' = z_k - \sum_{l=m}^{n-1-k} a_l h^l z_{l+k} & \text{for } k = 0, 1, \dots, n-1-m \\ z_k' = z_k & \text{for } k = n-m, n-m+1, \dots, n-1 \end{cases}$$

For $m \geq n$ we have

$$T_{f,h}x = T_{f,h}R^n y + \sum_{k=0}^{n-1}\sum_{j=0}^{k} a_j h^j R^{k-j} z_k = T_{f,h}R^n y .$$

Hence,

$$(2.6)\qquad (T_{f,h}-I)R^n y=\sum_{k=0}^{n-1} R^k z_k \in X_{n-1}\ .$$

This completes the proof.

We have the following theorem which is inversed to Theorem 2.1.

Theorem 2.2. *Suppose that all assumptions of* Theorem 2.1. *are satisfied. Let an element y in the equation* (I) *satisfy the condition* (2.2) *and let formula* (2.4) *hold. Then the equation* (I) *has an unique solution belonging to the space $X_{T_{f,h}}$* $(0\neq h\in K)$, *determined by the formula* (II) *with*

$$(2.7)\qquad z_k=z'_k \quad \text{for } m\geq n \quad (k=0,1,\ldots,n-1)$$

or

$$(2.7')\qquad z_k=\begin{cases} z'_k+\sum_{l=m}^{n-1-k} a_l h^l z_{l+k} & (k=0,1,\ldots,n-1-m)\\ z'_k & (k=n-m,n-m+1,\ldots,n-1)\end{cases}$$

for $0<m<n$.

Proof. The theorem for $m\geq n$ follows from formula (2.6). For $0<m<n$ we consider the equalities (2.5) as a linear system of equations with respect to $z_0,z_1,\ldots,z_{n-1}$. It is easy to show that the determinant of this system is equal to 1. Therefore we conclude that the system (2.5) has a unique solution. One can prove that this solution is determined by the recursion formula (2.7').

Formula (1.8) implies (cf.the proof of Proposition 1.4)

Proposition 2.1. *Suppose that all assumptions of* Theorem 2.1 *are satisfied. Then the condition* (2.2) *is independent of the choice of the right inverse $R\in\mathcal{R}_D$.*

Theorem 2.2 implies

Proposition 2.2. *Suppose that all assumptions of* Theorem 2.2. *are satisfied. Then the initial value problem*

$$(\mathrm{I})\qquad \begin{aligned} &D^n x=y\ ,\quad y\in X\ ,\ n>1\ ,\\ &FD^k x=z_k\ ,\quad z_k\in\ker D\ ,\ k=0,1,\ldots,p\quad (p\leq n-1)\end{aligned}$$

has a solution belonging to the set $X_{T_{f,h}}$ $(0\neq h\in K)$ if and only if the elements $z_0,z_1,\ldots,z_p$ satisfy the equalities (2.7').

Example 2.1 Let $D\in R(X)$, $\dim\ker D>0$ and let $R\in\mathcal{R}_D$ be arbitrarily fixed. Suppose that we are given a family $s_K=\{s_h\}_{h\in K}\subset L_0(X)$ of functional shifts for the operator D induced by the function sinus. Theorem 2.1 and Theorem 2.2 together imply that the equation (I) has a solution belonging to the set

$$X_{T_{\sin,h}} := X_{s_h} \quad (0 \neq h \in K) \quad \text{if and only if } (I - s_h)R^n y \in X_{n-1}.$$

If this condition is satisfied and $(I - s_h)R^n y = \sum_{k=0}^{n-1} R^k z'_k$, where $z'_0, z'_1, \ldots, z'_{n-1} \in$ ker D. Then the unique solution of equation (I) which beiongs to X_{s_h} is given by formula (II) with $z_0, z_1, \ldots, z_{n-1} \in \ker D$ which are determined by the following recursion formula

$$z_{n-1} = z'_{n-1}$$

$$z_{n-2-k} = z'_{n-2-k} - \sum_{j=1}^{k+1} \frac{h^j}{j!} \sin(j\pi/2) z_{n-2+j-k},$$

where $k = 0, 1, \ldots, n-2$.

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