ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN–POLONIA
VOL. XXXV, 9 SECTIO A 1981

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On Some Riemannian Structure on Linearized Tangent Bundle of Second Order

O pewnej riemanowskiej strukturze na uliniowionej wiązce stycznej drugiego rzędu

Об некоторой римановной структуре на линеаризованом касательном расслоении второго порядка

The purpose of this paper is a construction of Riemannian metric of Sasaki type and Riemannian connection, and its geodesics on a total space of linearized tangent bundle of second order.

1. Let M be an n-dimensional C^∞, Hausdorff manifold with given linear connection Γ. Let ${}^1_0\pi : TM \to M$ be a tangent bundle over M and $T_\pi : TTM \to TM$ be a tangent bundle over TM. We consider the tangent bundle of second order ${}^2_0\pi : {}^2M \to M$, where

$$ {}^2M = \left\{ A \in TTM : {}^1_0\pi_* A = T_\pi A \right\}, \quad {}^2_0\pi = {}^1_0\pi \cdot {}^1_0\pi_{*}|_{{}^2M} . \tag{1.1}$$

A local chart (U, x^{0i}) on M induces a local chart $({}^1_0\pi^{-1}(U), x^{0i}, x^{1i})$ on TM and a local chart $({}^2_0\pi^{-1}(U), x^{0i}, x^{1i}, x^{2i})$ on ${}^2M$. Then the tangent bundle of second order ${}^2_0\pi : {}^2M \to M$ has a vector bundle structure with coordinates:

$$ z^{0i} = x^{0i}, \; z^{1i} = x^{1i}, \; z^{2i} = x^{2i} + \Gamma^i_{jk} x^{1j} x^{1k}, \tag{1.2}$$

with respect to bases of the local sections

$$ E^0_{1i} = \frac{\partial}{\partial x^{0i}} - \Gamma^k_{il}\, \delta^l_i \frac{\partial}{\partial x^{1k}}\Big|_{(x^{0j},\, \delta^j_l)}, \quad E^0_{2i} = \frac{\partial}{\partial x^{1i}}\Big|_{(x^{0j},\, 0)} . \tag{1.3}$$

A linear connection Γ in the tangent bundle ${}^1_0\pi : TM \to M$ induces a linear connection $\hat\Gamma$ in the tangent bundle of second order ${}^2M \to M$. The connection Γ and $\hat\Gamma$ may be considered as a left splitting of exact sequences of bundles over TM and ${}^2M$ respectively:

$$\begin{tikzcd} 0 \arrow[r] & V(^2M) \arrow[r, "i"] \arrow[d, "{}^{2}_{0}\pi_*"'] & T(^2M) \arrow[r] \arrow[l, bend right, "\Gamma"'] \arrow[d, "{}^{2}_{1}\pi_*"] & {}^2M \times_M TM \arrow[r] & 0 \\ 0 \arrow[r] & V(TM) \arrow[r, "i"] & T(TM) \arrow[r] \arrow[l, bend left, "\Gamma"] & TM \times_M TM \arrow[r] & 0 \end{tikzcd} \tag{1.4}$$

$$\Gamma(x^{0i}, x^{1i}; y^{0i}, y^{1i}) = (x^{0i}, x^{1i}; 0, y^{1i} + \Gamma^i_{jk}\, x^{1k}\, y^{0j})$$

$$\hat{\Gamma}(z^{0i}, z^{1i}, z^{2i}; y^{0i}, y^{1i}, y^{2i}) = (z^{0i}, z^{1i}, z^{2i}; 0, y^{1i} + \Gamma^i_{jk}\, z^{1k}\, y^{0j}, y^{2i} + \Gamma^i_{jk}\, z^{2k}\, y^{0j}) . \tag{1.5}$$

A connection map for the connections Γ and $\hat{\Gamma}$ are of the form

$$D = p_2 \cdot i_{VTM} \cdot \Gamma, \quad \hat{D} = p_2 \cdot i_{V(^2M)} \cdot \hat{\Gamma} , \tag{1.6}$$

where $i_{V(TM)} : V(TM) \to TM \times_M TM$, $i_{V(^2M)} : V(^2M) \to {}^2M \times_M {}^2M$ is an isomorphism into the Whitney sum and p_2 is a projection on second factor. Then, on the total space ^2M there exists an adopted frame and coframe which in a local chart $({}^2_0\pi^{-1}(U), z^{0i}, z^{1i}, z^{2i})$ have the following form:

$$D_{0i} = \frac{\partial}{\partial z^{0i}} - \Gamma^k_{ij} z^{1i} \frac{\partial}{\partial z^{1k}} - \Gamma^k_{ij} z^{2j} \frac{\partial}{\partial z^{2k}} , \quad D_{1i} = \frac{\partial}{\partial z^{1i}} , \quad D_{2i} = \frac{\partial}{\partial z^{2i}} ,$$

$$\omega^{0i} = dz^{0i}, \; \omega^{1i} = dz^{1i} + \Gamma^i_{jk}\, z^{1k}\, dz^{0j}, \; \omega^{2i} = dz^{2i} + \Gamma^i_{jk}\, z^{2k}\, dz^{0j} . \tag{1.7}$$

Let g be a metric tensor on a manifold M.

Definition 1. The tensor $\hat{g}$ induced by the metric tensor g in the following way:

$$\hat{g}(A, B) = g({}^1_0\pi_* A, {}^1_0\pi_* B) + g(DA, DB) , \quad A, B \in {}^2M , \tag{1.8}$$

is called a metric tensor of Sasaki type in a fibre of the tangent bundle of second order ${}^2_0\pi : {}^2M \to M$.

Definition 2. The tensor G induced by the metric tensor g on M and the metric tensor $\hat{g}$ in fibre $^2M \to M$ and defined in the following way:

$$G(\tilde{X}, \tilde{Y}) = g({}^2_0\pi_* \tilde{X}, {}^2_0\pi_* \tilde{Y}) + \hat{g}(\hat{D}\tilde{X}, \hat{D}\tilde{Y}) , \quad \tilde{X}, \tilde{Y} \in T(^2M) . \tag{1.9}$$

is called a metric tensor of Sasaki type on the total space ^2M of the tangent bundle of second order ${}^2_0\pi : {}^2M \to M$. Thus we have:

Proposition 1. *Let ${}^2_0\pi : {}^2M \to M$ be the linearized tangent bundle of second order with given a connection $\hat{\Gamma}$ induced by connection Γ in $TM \to M$. The metric tensor G, (1.9) of Sasaki type on the total space ^2M induced by the metric tensor g on M has in adopted* *frame and natural frame in local chart* $({}^2_0\pi^{-1}(U), z^{0i}, z^{1i}, z^{2i})$ *the following forms respectively:*

$$G = g_{ij}\,\omega^{0i}\otimes\omega^{0j} + g_{ij}\,\omega^{1i}\otimes\omega^{1j} + g_{ij}\,\omega^{2i}\otimes\omega^{2j} \tag{1.10}$$

$$G = G_{JK}\,dz^J\otimes dz^K,\quad J = 0i,\,1i,\,2i,\;\; K = 0k,\,1k,\,2k, \tag{1.11}$$

$$G_{0i\,0j} = g_{ij} + g_{kl}\,\Gamma^k_{ip}\,\Gamma^l_{jq}\,z^{1p}z^{1q} + g_{kl}\,\Gamma^k_{jp}\,\Gamma^l_{iq}\,z^{2p}z^{2q},\; G_{0i\,1j} = g_{kj}\,\Gamma^k_{ip}\,z^{1p}$$

$$G_{0i\,2j} = g_{kj}\,\Gamma^k_{ip}\,z^{2p},\; G_{1i\,1j} = g_{ij},\; G_{2i\,2j} = g_{ij},\; G_{1i\,2j} = 0.$$

2. Let Γ be the Riemannian connection with respect to the metric tensor g on M. Then, the tensor $\hat{g}$, (1.8), of Sasaki type in fibres on the bundle ${}^2M \to M$ is parallel with respect to induced connection $\hat{\Gamma}$ in ${}^2M \to M$.

We construct the Christoffel symbols $\tilde{\Gamma}^I_{JK}$ for the tensor G, (1.11) of Sasaki type on total space ${}^2M$ in natural frame in local chart $({}^2_0\pi^{-1}(U), z^{0i}, z^{1i}, z^{2i})$ by formulas:

$$\tilde{\Gamma}_{I,JK} = \frac{1}{2}(\partial_J G_{IK} + \partial_K G_{JI} - \partial_I G_{JK}),\; \tilde{\Gamma}^I_{JK} = G^{IH}\,\Gamma_{H,JK}\,. \tag{2.1}$$

A coordinates G^{IJ}, such that $G^{IJ}G_{JK} = \delta^I_K$, has the form:

$$\begin{gathered} G^{0i\,0j} = g^{ij},\quad G^{0i\,1j} = -g^{ik}\,\Gamma^j_{kp}\,z^{1p},\; G^{0i\,2j} = -g^{ik}\,\Gamma^j_{kp}\,z^{2p},\\ G^{1i\,1j} = g^{ij} + g^{kl}\,\Gamma^i_{kp}\,\Gamma^j_{lq}\,z^{1p}z^{1q},\; G^{1i\,2j} = g^{kl}\,\Gamma^i_{kp}\,\Gamma^j_{lq}\,z^{1p}z^{2q},\\ G^{2i\,2j} = g^{ij} + g^{kl}\,\Gamma^i_{kp}\,\Gamma^j_{lq}\,z^{2p}z^{2q}\,. \end{gathered} \tag{2.2}$$

The Christoffel symbols $\tilde{\Gamma}^I_{JK}$ have the following form:

$$\begin{aligned} \tilde{\Gamma}^{0i}_{0j\,0k} &= \Gamma^i_{jk} + \frac{1}{2}g^{ia}\,(R^b_{jap}\,\Gamma^c_{kq} + R^b_{kap}\,\Gamma^c_{jq})\,g_{bc}\,(z^{1p}z^{1q} + z^{2p}z^{2q}),\\ \tilde{\Gamma}^{1i}_{0j\,0k} &= -\Gamma^a_{jk}\,\Gamma^i_{ap}\,z^{1p} - \frac{1}{2}g^{ab}\,\Gamma^i_{ar}\,z^{1r}\,(R^c_{jbp}\,\Gamma^d_{kq} + R^c_{kbp}\,\Gamma^d_{jq})\,g_{cd}\,(z^{1p}z^{1q} +\\ &\quad + z^{2p}z^{2q}) + \frac{1}{2}(\partial_j\Gamma^i_{kp} + \partial_k\Gamma^i_{jp} + \Gamma^r_{kp}\,\Gamma^i_{jr} + \Gamma^i_{kr}\,\Gamma^r_{jp})\,z^{1p},\\ \tilde{\Gamma}^{2i}_{0j\,0k} &= -\Gamma^a_{jk}\,\Gamma^i_{ap}\,z^{2p} - \frac{1}{2}g^{ab}\,\Gamma^i_{ar}\,z^{2r}\,(R^c_{jbp}\,\Gamma^d_{kq} + R^c_{kip}\,\Gamma^d_{jq})\,g_{cd}\,(z^{1p}z^{1q} +\\ &\quad + z^{2p}z^{2q}) + \frac{1}{2}(\partial_j\Gamma^i_{kp} + \partial_k\Gamma^i_{jp} + \Gamma^r_{kp}\,\Gamma^i_{jr} + \Gamma^r_{jp}\,\Gamma^i_{kr})\,z^{2p},\\ \tilde{\Gamma}^{1i}_{1j\,0k} &= \Gamma^i_{jk} - \frac{1}{2}g^{bc}\,R^a_{kbp}\,\Gamma^i_{cq}\,g_{aj}\,z^{1p}z^{1q},\\ \tilde{\Gamma}^{2i}_{2j\,0k} &= \Gamma^i_{jk} - \frac{1}{2}g^{bc}\,R^a_{kbp}\,\Gamma^i_{cq}\,g_{aj}\,z^{2p}z^{2q}, \end{aligned} \tag{2.3}$$

$$\tilde{\Gamma}^{0i}_{0j\,1k} = \frac{1}{2} R^{l}_{jrp}\, g^{ir}\, g_{lk}\, z^{1p}\,,$$

$$\tilde{\Gamma}^{0i}_{0j\,2k} = \frac{1}{2} R^{l}_{jrp}\, g^{ir}\, g_{lk}\, z^{2p}\,, \tag{2.3}$$

$$\tilde{\Gamma}^{0i}_{\alpha\beta} = 0\,, \quad \tilde{\Gamma}^{\delta}_{\alpha\beta} = 0\,, \qquad \alpha, \beta, \delta = 1i, 2i\,.$$

We consider a geodesic $\tilde{\gamma}$ on the total space ${}^2M$, $\tilde{\gamma} : R \supset I \to {}^2M$ for the Riemannian connection $\breve{\Gamma}$ with respect to the metric tensor G, (1.11) of Sasaki type on ${}^2M$. In local chart $({}^2_0\pi^{-1}(U), z^{0i}, z^{1i}, z^{2i})$ we have for the geodesic $\tilde{\gamma} : t \to (z^{0i}(t), z^{1i}(t), z^{2i}(t))$

$$\frac{d^2 z^I}{dt^2} + \Gamma^I_{JK} \frac{dz^J}{dt} \frac{dz^K}{dt} = 0, \; I = 0i, 1i, 2i\,. \tag{2.4}$$

Thus, using the formulas (2.3) for $\tilde{\Gamma}^J_{JK}$, we get for the geodesic $\tilde{\gamma}$:

$$\frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} + g^{ia} R^{b}_{jap}\, z^{1p} \frac{dx^j}{dt} \frac{Dz^{1r}}{dt} g_{br} + $$
$$+ g^{ia} R^{b}_{jap}\, z^{2p} \frac{dx^j}{dt} \frac{Dz^{2r}}{dt} g_{br} = 0$$

$$\frac{D^2 z^{1i}}{dt^2} - \Gamma^i_{kp} z^{1p} \left(\frac{d^2 x^k}{dt^2} + \Gamma^k_{jl} \frac{dx^j}{dt} \frac{dx^l}{dt}\right) - g^{ia}\, \Gamma^i_{aq} R^{b}_{\cdot jlp} \frac{dx^j}{dt} \frac{Dz^{1r}}{dt} g_{br}\, z^{1p} z^{1q} + $$
$$- g^{la}\, \Gamma^i_{aq} R^{b}_{jlp} \frac{dx^j}{dt} \frac{Dz^{2r}}{dt} g_{br}\, z^{2p} z^{2q} = 0\,, \tag{2.5}$$

$$\frac{D^2 z^{2i}}{dt^2} - \Gamma^i_{kp} z^{2p} \left(\frac{d^2 x^k}{dt^2} + \Gamma^k_{jl} \frac{dx^j}{dt} \frac{dx^l}{dt}\right) - g^{ka}\, \Gamma^i_{aq} R^{b}_{jkp}\, g_{br} \frac{dx^j}{dt} \frac{Dz^{1r}}{dt} z^{1p} z^{1q} - $$
$$- g^{ka}\, \Gamma^i_{aq} R^{b}_{jkp}\, g_{br} \frac{dx^j}{dt} \frac{Dz^{2r}}{dt} z^{2p} z^{2q} = 0.$$

In the above equations (2.5) we denote:

$$\frac{Dz^{1i}}{dt} = \frac{dz^{1i}}{dt} + \Gamma^i_{jk} z^{1k} \frac{dx^j}{dt}\,. \tag{2.6}$$

$$\frac{Dz^{2i}}{dt} = \frac{dz^{2i}}{dt} + \Gamma^i_{jk} z^{2k} \frac{dx^j}{dt} \tag{2.6}$$

$$\frac{D^2 z^{1i}}{dt^2} = \frac{D}{dt}\left(\frac{Dz^{1i}}{dt}\right) = \frac{d^2 x^i}{dt^2} + (\partial_k \Gamma^i_{jl} + \Gamma^i_{km}\Gamma^m_{jl}) z^{1l} \frac{dx^k}{dt}\frac{dx^j}{dt} +$$

$$+ 2\Gamma^i_{jk} \frac{dx^j}{dt}\frac{dx^k}{dt} + \Gamma^i_{jk} z^{1k} \frac{d^2 x^j}{dt^2}$$

We have:

Proposition 2. *Let M be the Riemannian manifold with the metric tensor g and the Riemannian connection Γ. In the linearized tangent bundle of second order ${}^2_0\pi : {}^2M \to M$ there are the induced connection $\hat{\Gamma}$, (1.5), and on total space ${}^2M$ the metric tensor G, (1.9) of Sasaki type induced by the metric tensor g on M.*

A curve $\tilde{\gamma}$ on the total space ${}^2M$ is a geodesic with respect to the Riemannian connection $\tilde{\Gamma}$, (2.1) for the metric tensor G if its projection $\gamma = {}^2_0\pi(\tilde{\gamma})$ on M is a geodesic with respect to the Riemannian connection Γ for g and if it is horizontal with respect to the induced connection $\hat{\Gamma}$ in the bundle ${}^2_0\pi : {}^2M \to M$.

Moreover, we have:

Proposition 3. *Let γ be a geodesic on a manifold M with respect to the Riemannian connection Γ for the metric tensor g.*

Then its canonical lift into the total space ${}^2M$ of the tangent bundle of second order ${}^2_0\pi : {}^2M \to M$ is a geodesic with respect to the Riemannian connection $\tilde{\Gamma}$ for the metric tensor G, (1.9) of Sasaki type on ${}^2M$ induced by the metric tensor g on M.

Proof: The canonical lift of the curve $\gamma : t \to x^i(t)$ on M into ${}^2M$ is the curve $\tilde{\gamma} : t \to z^{0i}(t) = x^i(t), z^{1i}(t) = \dot{x}^i(t), z^{2i}(t) = \ddot{x}^i(t) + \Gamma^i_{jk}\dot{x}^j\dot{x}^k$. If γ is a geodesic:

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0,$$

then its canonical lift $\tilde{\gamma}$ is horizontal:

$$\frac{dz^{1i}}{dt} + \Gamma^i_{jk} z^{1k}\frac{dx^j}{dt} = 0, \quad \frac{dz^{2i}}{dt} + \Gamma^i_{jk} z^{2k}\frac{dx^j}{dt} = 0.$$

Thus $\tilde{\gamma}$ is a geodesic on ${}^2M$.

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STRESZCZENIE

W pracy wyznacza się metrykę riemanowską typu Sasaki, koneksję riemanowską i geodezyjne na przestrzeni totalnej uliniowionej wiązki stycznej drugiego rzędu.

РЕЗЮМЕ

В работе определяется римановая метрика типа Сасаки, римановая связность и геодезичес-кие на пространстве расслоения линеаризованого касательного расслоения второго порядка.