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On a Structure of Tensor Fields of Type (1, 0), (0, 1), (0, 2), (1, 1) on a Linearized Tangent Bundle of Second Order

O strukturze pól tensorowych typu (1, 0), (0, 1), (0, 2), (1, 1) na uliniowionej wiązce stycznej drugiego rzędu

О структуре тензорных полей типа (1, 0), (0, 1), (0, 2), (1, 1) на линеаризованом касательном расслоении второго порядка

The aim of this paper is to find forms of tensor fields of type (1, 0), (0,1), (0,2), (1, 1) in adopted frame on total space of linearized tangent bundle of second order.

Introducing a linear connection Γ on a manifold M allows to define a vector structure for a tangent bundle of second order ${}^2_0\pi : {}^2M \to M$. Moreover a linear connection Γ in tangent bundle ${}^1_0\pi : TM \to M$ allows to induce a linear connection (Γ, Γ) in linearized tangent bundle of second order ${}^2_0\pi : {}^2M \to M$, [1].

In § 1 we introduce an adopted frame and adopted coframe on a total space ${}^2M$ with respect to induced linear conection (Γ, Γ) in a linear bundle ${}^2_0\pi : {}^2M \to M$. There is also used concept of M-tensor on a total space ${}^2M$ with respect to a linear connection (Γ, Γ) in a bundle ${}^2_0\pi : {}^2M \to M$.

In § 2 we shall find forms of horizontal and vertical vectors on ${}^2M$ and horizontal lifts of sections of TM and vertical lifts of sections of ${}^2M$. Next, we describe forms of Yano–Ishihara lifts of type: 0, I, II of vector fields and 1-forms on M into ${}^2M$.

In § 3 we consider forms of tensor fields of type (0, 2) on ${}^2M$. In particular we define a metric tensor of Sasaki type on ${}^2M$. Moreover, we describe forms of Yano–Ishihara lifts of type 0, I, II for tensor type (0, 2).

In § 4, we consider tensor fields of type (1, 1) on ${}^2M$. We find forms of tensor of type (1, 1) in adopted frame on ${}^2M$. Also, we describe forms of Yano–Ishihara lifts of type 0, I, II of tensor fields of type (1, 1).

§ 1. Let M be an n-dimensional manifold of the class C^∞, with a given linear connection $\Gamma : (\Gamma^i_{jk})$ i.e. connection in tangent bundle ${}^1_0\pi : TM \to M$. Then, the tangent bundle of second order ${}^2_0\pi : {}^2M \to M$ has a vector bundle structure with coordinates

$$z^{0i} = x^{0i}, \quad z^{1i} = x^{1i}, \quad z^{2i} = x^{2i} + \Gamma^{i}_{jk}\, x^{1j} x^{1k} \tag{1.1}$$

and basis local sections:

$$E^{0}_{1i} = \frac{\partial}{\partial x^{0i}} - \Gamma^{k}_{ij}\,\delta^{j}_{i}\,\frac{\partial}{\partial x^{1k}}\Big|_{(x^{0j},\,\delta^{j}_{1})}, \quad E^{0}_{2i} = \frac{\partial}{\partial x^{1i}}\Big|_{(x^{0j},\,0)} . \tag{1.2}$$

The tangent bundle of second order ${}^{2}_{0}\pi : {}^{2}M \to M$ with coordinates (x^{0i}, x^{1i}, x^{2i}) induced by coordinates (x^{0i}) on M, is not linear. Moreover, the linear connection Γ in the tangent bundle induces the linear connection $\widetilde{\Gamma}$ in the linearized tangent bundle of second order ${}^{2}_{0}\pi : {}^{2}M \to M$ of the form:

$$\Gamma^{1i}_{j1k} = \Gamma^{i}_{jk}, \quad \Gamma^{2i}_{j2k} = \Gamma^{i}_{jk}, \quad \Gamma^{1i}_{j2k} = 0, \quad \Gamma^{2i}_{j1k} = 0 . \tag{1.3}$$

The connection $\widetilde{\Gamma}$ in the bundle ${}^{2}_{0}\pi : {}^{2}M \to M$ induced by a linear connection Γ in the bundle ${}^{1}_{0}\pi : TM \to M$ may be regarded as a left splitting of an exact sequence of vector bundles on ${}^{2}M$ of the form:

$$\begin{tikzcd}
0 \arrow[r] & V({}^{2}M) \arrow[r, "\widetilde{J}"] \arrow[d, "{}^{2}_{1}\pi_{*}"'] & T({}^{2}M) \arrow[l, bend left, "\widetilde{\Gamma}"] \arrow[r] \arrow[d, "{}^{2}_{1}\pi_{*}"] & {}^{2}M \times_{M} TM \arrow[r] & 0 \\
0 \arrow[r] & V(TM) \arrow[r, "J"] & T(TM) \arrow[l, bend left, "\Gamma"] \arrow[r] & TM \times_{M} TM \arrow[r] & 0
\end{tikzcd} \tag{1.4}$$

$$\widetilde{\Gamma} \cdot \widetilde{J} = id_{V({}^{2}M)},$$

$$\widetilde{\Gamma}(z^{0i}, z^{1i}, z^{2i}; y^{0i}, y^{1i}, y^{2i}) = (z^{0i}, z^{1i}, z^{2i}; 0, z^{1i} + \Gamma^{i}_{jk} z^{1k} y^{0j}, z^{2i} + \Gamma^{i}_{jk} z^{2k} y^{0j}).$$

A connection map for this induced connection $\widetilde{\Gamma}$ is of the form:

$$\widetilde{D} : T({}^{2}M) \to {}^{2}M, \qquad \widetilde{D} = p_2 \cdot i_{V({}^{2}M)} \cdot \widetilde{\Gamma}, \tag{1.4}$$

$$\widetilde{D}\Big(y^{0i} \frac{\partial}{\partial z^{0i}} + y^{1i} \frac{\partial}{\partial z^{1i}} + y^{2i} \frac{\partial}{\partial z^{2i}}\Big) =$$

$$= (y^{1i} + \Gamma^{i}_{jk} z^{1k} y^{0j}) E^{0}_{1i} + (y^{2i} + \Gamma^{i}_{jk} z^{2k} y^{0j}) E^{0}_{2i} .$$

where: $i_{V({}^{2}M)} : V({}^{2}M) \to {}^{2}M \times_{M} {}^{2}M$ is a canonical isomorphism of vertical subbundle $V({}^{2}M)$ into the Whitney sum of ${}^{2}M$, and $p_2 : {}^{2}M \times_{M} {}^{2}M \to {}^{2}M$ is a projection on the second component and ${}^{2}_{1}\pi : {}^{2}M \to TM$ is a projection.

Definition 1. A bundle with a total space $H(^2M) = Ker\ \widetilde{D}$ is said to be horizontal subbundle over ^2M of a bundle $T(^2M) \to {}^2M$.

A bundle with a total space $V(^2M) = Ker\ (^2_0\pi_*)$ is called vertical subbundle over ^2M. Then for any $A \in {}^2M$, we have decomposition:

$$T_A(^2M) = H_A(^2M) \oplus V_A(^2M). \tag{1.5}$$

In a local hart $(^2_0\pi^{-1}(U), z^{0i}, z^{1i}, z^{2i})$ we have:

$$H(^2M) = \left\{ Y \in T(^2M) : Y = y^{0i}\left(\frac{\partial}{\partial z^{0i}} - \Gamma^k_{ij} z^{1i} \frac{\partial}{\partial z^{1k}} - \Gamma^k_{ij} z^{2j} \frac{\partial}{\partial z^{2k}}\right)\right\}, \tag{1.6}$$

$$V(^2M) = \left\{ Y \in T(^2M) : Y = y^{1i} \frac{\partial}{\partial z^{1i}} + y^{2i} \frac{\partial}{\partial z^{2i}} \right\}.$$

For the cotangent bundle $T^*(^2M) \to {}^2M$ we define a decomposition corresponding to (1.5):

$$T^*(^2M) = H(^2M)^{\perp} \oplus V(^2M)^{\perp}, \tag{1.7}$$

where:

$$V(^2M)^{\perp} = \left\{ \omega \in T^*(^2M) : \omega = \alpha_i\, dz^{0i} \right\},$$

$$H(^2M)^{\perp} = \left\{ \omega \in T^*(^2M) : \omega = \alpha_{1i}\,(dz^{1i} + \Gamma^i_{jk} z^{1k} dz^{0j}) + \alpha_{2i}\,(dz^{2i} + \Gamma^i_{jk} z^{2k} dz^{0j}) \right\}. \tag{1.8}$$

Definition 2. A system $3n$-vectors (D_{0i}, D_{1i}, D_{2i}) that span $H(^2M)$ and $V(^2M)$, locally defined by:

$$D_{0i} = \frac{\partial}{\partial z^{0i}} - \Gamma^k_{ij} z^{1j} \frac{\partial}{\partial z^{1k}} - \Gamma^k_{ij} z^{2j} \frac{\partial}{\partial z^{2k}},\ D_{1i} = \frac{\partial}{\partial z^{1i}},\ D_{2i} = \frac{\partial}{\partial z^{2i}} \tag{1.10}$$

is called an adopted frame on ^2M with respect to the induced connection $\widetilde{\Gamma}$ in bundle $^2_0\pi : {}^2M \to M$.

A system of $3n$ 1-forms $(\omega^{0i}, \omega^{1i}, \omega^{2i})$ that span respectively $V(^2M)^{\perp}$, $H(^2M)^{\perp}$ and locally defined by formulas:

$$\omega^{0i} = dz^{0i},\ \omega^{1i} = dz^{1i} + \Gamma^i_{jk} z^{1k} dz^{0j},\ \omega^{2i} = dz^{2i} + \Gamma^i_{jk} z^{2k} dz^{0j} \tag{1.11}$$

is called an adopted coframe on ^2M. For the adopted frame and coframe on ^2M with respect to coordinates (z^{0i}, z^{1i}, z^{2i}) and $(z^{0\bar{i}}, z^{1\bar{i}}, z^{2\bar{i}})$, where:

$$z^{0i'} = z^{0i'}(z^{0i}),\ z^{1i'} = A^{i'}_{i} z^{1i},\ z^{2i'} = A^{i'}_{i} z^{2i}, A^{i'}_{i} = \frac{\partial z^{0i'}}{\partial z^{0i}}. \tag{1.12}$$

we have:

$$[D_{0i}, D_{1i}, D_{2i}] = [D_{0i'}, D_{1i'}, D_{2i'}] \begin{bmatrix} A^{i'}_{i} & 0 & 0 \\ 0 & A^{i'}_{i} & 0 \\ 0 & 0 & A^{i'}_{i} \end{bmatrix},$$

$$\begin{bmatrix} \omega^{0i'} \\ \omega^{1i'} \\ \omega^{2i'} \end{bmatrix} = \begin{bmatrix} A^{i'}_{i} & 0 & 0 \\ 0 & A^{i'}_{i} & 0 \\ 0 & 0 & A^{i'}_{i} \end{bmatrix} \begin{bmatrix} \omega^{0i} \\ \omega^{1i} \\ \omega^{2i} \end{bmatrix}. \tag{1.13}$$

The formulas of adopted frame and coframe in natural frame and coframe on ${}^2M$ with coordinates (z^{0i}, z^{1i}, z^{2i}) can be writen in the from:

$$\begin{bmatrix} \omega^{0i} \\ \omega^{1i} \\ \omega^{2i} \end{bmatrix} = \begin{bmatrix} \delta^{i}_{j} & 0 & 0 \\ \Gamma^{1i}_{j} & \delta^{i}_{j} & 0 \\ \Gamma^{2i}_{j} & 0 & \delta^{i}_{j} \end{bmatrix} \begin{bmatrix} dz^{0j} \\ dz^{1j} \\ dz^{2j} \end{bmatrix},\ [D_{0i}, D_{1i}, D_{2i}] =$$
$$= [\frac{\partial}{\partial z^{0j}}, \frac{\partial}{\partial z^{1j}}, \frac{\partial}{\partial z^{2j}}] \begin{bmatrix} \delta^{j}_{i} & 0 & 0 \\ -\Gamma^{1j}_{i} & \delta^{j}_{i} & 0 \\ -\Gamma^{2j}_{i} & 0 & \delta^{j}_{i} \end{bmatrix}. \tag{1.14}$$

We denote:

$$\Gamma^{1i}_{j} = \Gamma^{i}_{jk} z^{1k},\ \Gamma^{2i}_{j} = \Gamma^{i}_{jk} z^{2k}. \tag{1.15}$$

Definition 3 ([2]) By an M-tensor of type (r, s) on a total space ${}^2M$ we mean an object determined in a local chart $({}^2_0\pi^{-1}(U), z^{0i}, z^{1i}, z^{2i})$ by a set n^{r+s} functions: $F^{i_1 \dots i_r}_{j_1 \dots j_s}(z^{0i}, z^{1i}, z^{2i})$ which transform themselves in the following way:

$$F^{i'_1 \dots i'_r}_{j'_1 \dots j'_s} = A^{i'_1}_{i_1} \dots A^{i_r'}_{i_r} F^{i_1 \dots i_r}_{j_1 \dots j_s} A^{j_1}_{j'_1} \dots A^{j_s}_{j_s'} \tag{1.16}$$

for the change of a local chart (1.12)

Remark. Any tensor F of type (r, s) on the total space ${}^2M$ has in adopted frame and coframe the form

$$F = F^{\alpha_1 \dots \alpha_r}_{\beta_1 \dots \beta_s}(z^{0i}, z^{1i}, z^{2i}) D_{\alpha_1} \otimes \dots \otimes D_{\alpha_r} \otimes \omega^{\beta_1} \otimes \dots \otimes \omega^{\beta_s} \tag{1.17}$$

where: $\alpha_p = 0i, 1i, 2i, \beta_q = 0j, 1j, 2j$. The tensor F is described by 3^{r+s} M-tensors on ${}^2M$.

§ 2. The horizontal vector field $\widetilde{X}$ on ${}^2M$, as a section of horizontal subbundle $H({}^2M) \to {}^2M$ is of the form: $\widetilde{X} = \xi^i(z^{0j}, z^{1j}, z^{2j}) D_{0i}$ and is determined by an M-tensor (ξ^i) on ${}^2M$. The differential of projection ${}^2_0\pi : {}^2M \to M$ i.e. ${}^2_0\pi_* : T({}^2M) \to TM$ is an isomorphism of horizontal subbundle $H({}^2M)$ into tangent bundle TM. Then for a section $X \in TM$ there exists a section $X^{HL} \in H({}^2M)$, so-called horizontal lift of X, such that:

$$ {}^2_0\pi_* (X^{HL}) = X, \quad \widetilde{D} (X^{HL}) = 0 . \tag{2.1}$$

For $X = \xi^i \dfrac{\partial}{\partial x^i}$, the horizontal lift $X^{HL} = \xi^i D_{0i}$ is the horizontal field.

A vertical vector field B on ${}^2M$, as a section of a vertical subbundle $V({}^2M) \to {}^2M)$ is of the form: $B = \xi^{1i}(z^{0j}, z^{1j}, z^{2j}) D_{1j} + \xi^{2i}(z^{0j}, z^{1j}, z^{2j}) D_{2j}$ and is determined by two M-tensors (ξ^{1i}), (ξ^{2i}) on ${}^2M$. For the vector bundle ${}^2M \to M$ there is the canonical isomorphism into vertical subbundle $V({}^2M) \to {}^2M$, and for a section $A \in {}^2M$, $A = A^{1i}(z^0) E^0_{1i} + A^{2i}(z^0) E^0_{2i}$ there is so called vertical lift $B = A^{VL}$, $B = A^{1i}(z^0) D_{1i} + A^{2i}(z^0) D_{2i}$. Let A be a section of class C^∞ of the bundle ${}^2_0\pi : {}^2M \to M$ over U and $X \in TM$. In local chart they have the forms: $A = A^{1i} E^0_{1i} + A^{2i} E^0_{2i}$, $X = \xi^i \dfrac{\partial}{\partial x^i}$ respectively. Then for the value of the differential $A_* : TU \to T({}^2M)$ on the vector field X is:

$$A_* X = \xi^i \frac{\partial}{\partial z^{0i}} + \xi^i \frac{\partial A^{1j}}{\partial x^i} \frac{\partial}{\partial z^{1j}} + \xi^i \frac{\partial A^{2j}}{\partial x^i} \frac{\partial}{\partial z^{2j}} .$$

Thus we have:

Proposition 1. *Any vector field $\widetilde{X}$ on ${}^2M$ is the sum of horizontal and vertical vectors and is determined by three M-tensors on ${}^2M$:*

$$\widetilde{X} = \xi^{0i}(z^0, z^1, z^2) D_{0i} + \xi^{1i}(z^0, z^1, z^2) D_{1i} + \xi^{2i}(z^0, z^1, z^2) D_{2i} . \tag{2.2}$$

For C^∞ section $A \in {}^2M$ and vector $X \in TM$ the value of the differential $A_* X \in T({}^2M)$ is the sum of the horizontal lift of X and the vertical lift of the value of the connection map $\widetilde{D}(A_* X) \in {}^2M$

$$A_* X = X^{HL} + [\widetilde{D}(A_* X)]^{VL} . \tag{2.3}$$

Any section of a subbundle $V({}^2M)^\perp \to {}^2M$ is called a vertical 1-form ω^V on ${}^2M$ and it is of the form: $\omega^V = \alpha_{0i}(z^0, z^1, z^2)\, \omega^{0i}$, where (α_{0i}) is M-tensor of type (0,1) on ${}^2M$.

A map $\pi^* : T^*({}^2M) \to T^*M$ defined by the formula:

$$(\pi^* \omega)(X) = \omega(X^{HL}) , \tag{2.4}$$

is an isomorphism of bundles: $V(^2M)^\perp$, T^*M, where $X \in TM$ and $X^{HL} \in H(^2M)$ being its horizontal lift. This follows from that: $Ker\,\pi^* = H(^2M)^\perp$.

A vertical lift of 1-form $\omega \in T^*M$ is called a section $\omega^{VL} \in V(^2M)^\perp$ such that: $\pi^*\omega^{VL} = \omega$. In a local chart for an 1-form $\omega = \alpha_i dx^i$ the vertical lift has a form: $\omega^{VL} = \alpha_i \omega^{0i}$ and is vertical form.

Any section $\omega^H \in H(^2M)^\perp$ is called a horizontal 1-form on ^2M and is determined by two M-tensors of type (0,1) on ^2M:

$$\omega^H = \alpha_{1i}(z^0, z^1, z^2)\,\omega^{1i} + \alpha_{2i}(z^0, z^1, z^2)\,\omega^{2i}.$$

A map $D^* : T^*(^2M) \to {}^2M^*$ defined by the formula:

$$(D^*\omega)(A) = \omega(A^{VL}), \tag{2.5}$$

is an isomorphism of bundles $H(^2M)^\perp$, $^2M^*$, because: $Ker D^* = V(^2M)^\perp$.

A horizontal lift of covector $\eta \in {}^2M^*$ is called a section $\eta^{HL} \in H(^2M)^\perp$ such that $\eta = D^*\eta^{HL}$. In local chart we have:

$$\eta = \alpha_{1i} E_*^{1i} + \alpha_{2i} E_*^{2i},\ \eta^{HL} = \alpha_{1i}\,\omega^{1i} + \alpha_{2i}\,\omega^{2i},$$

where: E_*^{1i}, E_*^{2i} is a dual basis to E^0_{1i}, E^0_{2i}.

Now, we determine a form of Yano-Ishihara lifts of tensors of type (1,0), (0,1) into linearized tangent bundle of second order ^2M.

Proposition 2. *Let M be n-dimensional manifold of class C^∞ with a given linear connection $\Gamma : (\Gamma^i_{jk})$ in the tangent bundle $TM \to M$. In the linearized tangent bundle of second order $^2M \to M$ with coordinates* (1.1) : (z^{0i}, z^{1i}, z^{2i}) *is given induced connection* (1.4), $\widetilde{\Gamma} = (\Gamma, \Gamma) : (\Gamma^{1i}_{j1k} = \Gamma^i_{jk}, \Gamma^{2i}_{j2k} = \Gamma^i_{jk}, \Gamma^{1i}_{j2k} = 0, \Gamma^{2i}_{j1k} = 0)$ *and on the total space ^2M there are adopted frame and coframe* (1.10), (1.11), (D_{0i}, D_{1i}, D_{2i}), $(\omega^{0i}, \omega^{1i}, \omega^{2i})$. *If $X \in TM$ and $\omega \in T^*M$ are a vector field and an* 1*-form respectively on M with representations in a local chart (U, x^i):*

$$X = \xi^i \frac{\partial}{\partial x^i},\ \omega = \alpha_i dx^i$$

then their Yano-Ishihara lifts of type: 0, I, II *into the tangent bundle of second order ^2M have the form in the adopted frame and coframe:*

$$\begin{aligned} X^0 &= \xi^i D_{2i}, \\ X^I &= \xi^i D_{1i} + 2(z^{1k}\nabla_k \xi^i) D_{2i}, \\ X^{II} &= \xi^i D_{0i} + (z^{1k}\nabla_k \xi^i) D_{1i} + (z^{2k}\nabla_k \xi^i + z^{1k} z^{1l}\nabla_k \nabla_l \xi^i + R^i_{jkl} z^{1k} z^{1l}\xi^j) D_{2i} \end{aligned} \tag{2.6}$$

$$\omega^0 = \alpha_i\, \omega^{0i},$$

$$\omega^{\mathrm{I}} = (z^{1j}\, \nabla_j\, \alpha_i)\, \omega^{0i} + \alpha_i\, \omega^{1i}, \tag{2.7}$$

$$\omega^{\mathrm{II}} = (z^{2j}\, \nabla_j\, \alpha_i + \nabla_j \nabla_k\, \alpha_i\, z^{1j} z^{1k} - R^{l}_{ijk}\, z^{1j} z^{1k}\, \alpha_l)\, \omega^{0i} + 2(z^{1j} \nabla_j\, \alpha_i)\, \omega^{1i} + \alpha_i\, \omega^{2i}.$$

The symbols: ∇_j and R^i_{jkl} denote a covariant derivative and components of a curvature tensor R respectively for the linear connection $\Gamma : (\Gamma^i_{jk})$.

Proof: Because of (1.12) and (1.13) we can observe that locally defined fields (2.6), (2.7) determine the fields on ^2M. Using (1.1) we get:

$$\frac{\partial}{\partial x^{0i}} = \frac{\partial}{\partial z^{0i}} + \partial_i \Gamma^l_{jk}\, z^{1j} z^{1k}\, \frac{\partial}{\partial z^{2l}},$$

$$\frac{\partial}{\partial x^{1i}} = \frac{\partial}{\partial z^{1i}} + 2\Gamma^l_{ij}\, z^{1j}\, \frac{\partial}{\partial z^{2l}},$$

$$\frac{\partial}{\partial x^{2i}} = \frac{\partial}{\partial z^{2i}},$$

$$dz^{0i} = dx^{0i}, \tag{2.8}$$

$$dz^{1i} = dx^{1i},$$

$$dz^{2i} = dx^{2i} + 2\Gamma^i_{jk} x^{1j} dx^{1k} + \partial_l \Gamma^i_{jk} x^{1j} x^{1k} dx^{0l}.$$

The adopted frame and coframe (1.10), (1.11) in the induced coordinates are of the form:

$$D_{0i} = \frac{\partial}{\partial x^{0i}} - \Gamma^k_{ij}\, x^{1j}\, \frac{\partial}{\partial x^{1k}} +$$

$$+ \left\{ -\Gamma^m_{ij}\, x^{2j} + (-\Gamma^m_{ir}\, \Gamma^r_{pq} - \partial_i\, \Gamma^m_{pq} + 2\Gamma^r_{ip}\, \Gamma^m_{rq})\, x^{1p}\, x^{1q} \right\} \frac{\partial}{\partial x^{2r}},$$

$$D_{1i} = \frac{\partial}{\partial x^{1i}} - 2\Gamma^k_{ij}\, x^{1j}\, \frac{\partial}{\partial x^{2k}},$$

$$D_{2i} = \frac{\partial}{\partial x^{2i}},$$

$$\omega^{0i} = dx^{0i}, \tag{2.9}$$

$$\omega^{1i} = dx^{1i} + \Gamma^i_{jk}\, x^{1k}\, dx^{0j},$$

$$\omega^{2i} = dx^{2i} + 2\Gamma^i_{jk}\, x^{1k}\, dx^{1j} + (\Gamma^i_{;j}\, x^{2j} + \partial_l\, \Gamma^i_{jk}\, x^{1j}\, x^{1k} + \Gamma^i_{rl}\, \Gamma^r_{jk}\, x^{1j}\, x^{1k})\, dx^{0l}.$$

The easy calculations, in viev of (1.1), (2.8), (2.9) give thesis.

§ 3. A symmetric tensor field G of type (0,2) on a total space ${}^2M$ in adopted frame and coframe has a form:

$$G = G_{0i\,0j}\,\omega^{0i}\otimes\omega^{0j} + G_{0i\,1j}\,\omega^{0i}\otimes\omega^{1j} + G_{0i\,2j}\,\omega^{0i}\otimes\omega^{2j} + G^T_{0i\,1j}\,\omega^{1i}\otimes\omega^{0j} +$$
$$+ G_{1i\,1j}\,\omega^{1i}\otimes\omega^{1j} + G_{1i\,2j}\,\omega^{1i}\otimes\omega^{2j} + G^T_{0i\,2j}\,\omega^{2i}\otimes\omega^{0j} + \qquad (3.1)$$
$$+ G^T_{1i\,2j}\,\omega^{2i}\otimes\omega^{1j} + G_{2i\,2j}\,\omega^{2i}\otimes\omega^{2j}\,.$$

The tensor G is determined by 3^2 M-tensors on ${}^2M$ and $G_{1i\,0j} = G^T_{0i\,1j}$, $G_{2i\,0j} = G^T_{0i\,2j}$, $G_{2i\,1j} = G^T_{1i\,2j}$ denote the transpose matrices.

Proposition 3. *Let ${}^2_0\pi : {}^2M \to M$ be the linearized tangent bundle of second order with coordinates (1.1), (z^{0i}, z^{1i}, z^{2i}) and with given induced connection (1.4), $\widetilde{\Gamma}$, $(\Gamma^{1i}_{j1k} = \Gamma^i_{jk}, \Gamma^{2i}_{j2k} = \Gamma^i_{jk}, \Gamma^{1i}_{j2k} = 0, \Gamma^{2i}_{j1k} = 0)$ and adopted frame and coframe (1.10), (1.11): (D_{0i}, D_{1i}, D_{2i}), $(\omega^{0j}, \omega^{1j}, \omega^{2j})$. Any symmetric tensor field G of type (0,2) on total space ${}^2M$ has in the natural frame and coframe with respect to coordinates (1.1): $(\frac{\partial}{\partial z^{0i}}, \frac{\partial}{\partial z^{1i}}, \frac{\partial}{\partial z^{2i}})$, $(dz^{0j}, dz^{1j}, dz^{2j})$ the form:*

$$\overline{G}_{0i\,0j} = G_{0i\,0j} + \Gamma^{1i}_k G^T_{0k\,1j} + \Gamma^{2i}_k G^T_{0k\,2j} + G_{0i\,1k}\Gamma^{1k}_j + G_{0i\,2k}\Gamma^{2k}_j +$$
$$+ \Gamma^{1i}_k G_{1k\,1l}\Gamma^{1l}_j + \Gamma^{2i}_k G_{1k\,2l}\Gamma^{1l}_j + \Gamma^{1i}_k G_{1k\,2l}\Gamma^{2l}_j + \Gamma^{2i}_k G_{2k\,2l}\Gamma^{2l}_j\,, \qquad (3.2)$$
$$\overline{G}_{0i\,1j} = G_{0i\,1j} + \Gamma^{1i}_k G_{1k\,1j} + \Gamma^{2i}_k G^T_{1k\,2j},$$
$$\overline{G}_{0i\,2j} = G_{0i\,2j} + \Gamma^{1i}_k G_{1k\,2j} + \Gamma^{2i}_k G_{2k\,2j},$$
$$\overline{G}_{1i\,1j} = G_{1i\,1j},\ \overline{G}_{1i\,2j} = G_{1i\,2j},\ \overline{G}_{2i\,2j} = G_{2i\,2j}\,.$$

Moreover, coordinates: $\overline{G}_{1i\,1j} = G_{1i\,1j} = G(D_{1i}, D_{1j})$, $\overline{G}_{2i\,2j} = G_{2i\,2j}$, $\overline{G}_{1i\,2j} = G_{1i\,2j}$, are independent on the connection. In adopted frame the symmetric tensor field has a matrix: $G = [G_{\alpha\beta}]_{\alpha = 0i,\,1i,\,2i;\ \beta = 0j,\,1j,\,2j}$ and coordinates $G_{\alpha\beta}$, $\alpha = 0i, 1i, 2i$, $\beta = 0j, 1j, 2j$, are M-tensors on ${}^2M$. We denote (1;15), $\Gamma^{1i}_j = \Gamma^i_{jk} z^{1k}$, $\Gamma^{2i}_j = \Gamma^i_{jk} z^{2k}$.

Proof: Using relation (1.14) : $(\omega^\alpha) = N\cdot(dz^\alpha)$ we get for matrix $\overline{G}$ in natural frame: $\overline{G} = N^T\cdot G : N$. Moreover, we have: $\overline{G}_{1i\,1j} = G(\frac{\partial}{\partial z^{1i}}, \frac{\partial}{\partial z^{1j}}) = G(D_{1i}, D_{1j}) = G_{1i\,1j}$, $\overline{G}_{1i\,2j} = G(D_{1i}, D_{2j}) = G_{1i\,2j}$, $\overline{G}_{2i\,2j} = G_{2i\,2j}$.

A symmetric tensor field G of type (0.2) on ${}^2M$ defines at each point $A \in {}^2M$ the symmetric bilinear form: $G_A : T_A({}^2M) \times T_A({}^2M) \to R$ as the inner product on $T_A({}^2M)$. By means G_A we define 'ortogonality' in $T_A({}^2M)$. Subspaces: horizontal $H_A({}^2M)$ and vertical $V_A({}^2M)$ are ortogonal, if $G_{0i\,1j} = G(D_{0i}, D_{1j}) = 0$, $G_{0i\,2j} = G(D_{0i}, D_{2j}) = 0$.

Let g be a metric tensor on a manifold M.

Definition 4. The tensor ${}^2g$ induced by the metric tensor g into ${}^1_0\pi : TM \to M$ in the following way:

$$ {}^2g(A, B) = g({}^1_0\pi_* A, {}^1_0\pi_* B) + g(D(A), D(B)), A, B \in {}^2M \tag{3.3}$$

is called a metric tensor of Sasaki type in fibre of the tangent bundle of second order ${}^2M \to M$, where D is a connection map of Γ.

Definition 5. The tensor $\widetilde{G}$ induced by the metric tensor g and the tensor ${}^2g$ and defined in the following way:

$$\widetilde{G}(\widetilde{X}, \widetilde{Y}) = g({}^2_0\pi_* \widetilde{X}, {}^2_0\pi_* \widetilde{Y}) + {}^2g(\widetilde{D}\widetilde{X}, \widetilde{D}\widetilde{Y}),\ \widetilde{X}, \widetilde{Y} \in T({}^2M), \tag{3.4}$$

is called a metric tensor of Sasaki type on the total space ${}^2M$ of the tangent bundle of second order ${}^2_0\pi : {}^2M \to M$.

Remark: It is easy to see that the definitions justify the names for ${}^2g$ and $\widetilde{G}$. For the metric $\widetilde{G}$ we have:

$$\widetilde{G}(\widetilde{X}, \widetilde{Y}) = g({}^1_0\pi_* \widetilde{X}, {}^2_0\pi_* \widetilde{Y}) + g({}^1_0\pi_* \widetilde{D}\widetilde{X}, {}^1_0\pi_* \widetilde{D}\widetilde{Y}) + g(D\widetilde{D}\widetilde{X}, D\widetilde{D}\widetilde{Y}). \tag{3.5}$$

In adopted frame (1.10) we can writte:

$$\widetilde{G}(D_{0i}, D_{0j}) = g_{ij},\ \widetilde{G}(D_{1i}, D_{1j}) = g_{ij},\ \widetilde{G}(D_{2i}, D_{2j}) = g_{ij}. \tag{3.6}$$

Thus we have:

Proposition 4. *The metric $\widetilde{G}$ of Sasaki type on total space ${}^2M$ has in the adopted frame (1.10), (1.11) with respect to the induced connection $\widetilde{\Gamma}$, (1.14), in the bundle ${}^2M \to M$ the form:*

$$\widetilde{G} = g_{ij}\, \omega^{0i} \otimes \omega^{0j} + g_{ij}\, \omega^{1i} \otimes \omega^{1j} + g_{ij}\, \omega^{2i} \otimes \omega^{2j}. \tag{3.7}$$

The horizontal subbundle $H({}^2M)$ and vertical subbundle $V({}^2M)$ are orthogonal with respect to $\widetilde{G}$.

The tensor $\widetilde{G}$ has in the natural frame with respect (1.1) *the matrix $\overline{G}$:*

$$\begin{gathered}\overline{G}_{0i\,0j} = g_{ij} + g_{kl}\, \Gamma^i_{kp}\, \Gamma^l_{jq}\, z^{1p} z^{1q} + g_{kl}\, \Gamma^l_{kp}\, \Gamma^l_{jq}\, z^{2p} z^{2q},\quad \overline{G}_{1i\,2j} = 0,\\ \overline{G}_{0i\,1j} = g_{kj}\, \Gamma^l_{kp}\, z^{1p},\ \overline{G}_{0i\,2j} = g_{kj}\, \Gamma^i_{kp}\, z^{2p},\ \overline{G}_{1i\,1j} = g_{ij},\ \overline{G}_{2i\,2j} = g_{ij},\end{gathered} \tag{3.8}$$

Proposition 5. *Let g be a symmetric tensor field of type* (0,2) *on n-dimensional manifold M with given linear connection $\Gamma : (\Gamma^l_{jk})$, that is, in a local chart (U, x^i), g is of* *the form $g = g_{ij}\, dx^i \otimes dx^j$. The lifts of Yano-Ishihara [3] : g^0, g^I, g^{II} into the total space ${}^2M$ of the linearized tangent bundle of second order ${}^2_0\pi : {}^2M \to M$ with the induced connection $\widetilde{\Gamma}$, (1.4), have in the adopted frame (1.10), (1.11) in the local chart $({}^2_0\pi^{-1}(U), z^{0i}, z^{1i}, z^{2i})$ the form:*

$$
\begin{aligned}
g^0 &= g_{ij}\, \omega^{0i} \otimes \omega^{0j}, \\
g^I &= (z^{1r} \nabla_r g_{ij})\, \omega^{0i} \otimes \omega^{0j} + g_{ij}\, \omega^{0i} \otimes \omega^{1j} + g_{ij}\, \omega^{1i} \otimes \omega^{0j}, \\
g^{II} &= [z^{2r} \nabla_r g_{ij} + z^{1r} z^{1s} (\nabla_s \nabla_r g_{ij} - g_{ik} R^k_{jsr} - g_{kj} R^k_{isr})]\, \omega^{0i} \otimes \omega^{0j} + \\
&\quad + 2z^{1r} \nabla_r g_{ij}\, \omega^{1i} \otimes \omega^{0j} + 2z^{1r} \nabla_r g_{ij}\, \omega^{0i} \otimes \omega^{1j} + g_{ij}\, \omega^{2i} \otimes \omega^{0j} + \\
&\quad + 2g_{ij}\, \omega^{1i} \otimes \omega^{1j} + g_{ij}\, \omega^{0i} \otimes \omega^{2j}.
\end{aligned}
\tag{3.9}
$$

Proof: Using the formulas (1.1), (2.8), (2.9) and after some calculations we obtain our propositions for the fields g^0, g^I, g^{II} ([3], p. 332).

§ 4. We now consider a tensor fields of type (1.1) on the total space ${}^2M$.

Proposition 6. *Let ${}^2_0\pi : {}^2M \to M$ be a linearized tangent bundle of second order with given induced connection $\widetilde{\Gamma}$, (1.3) and the adopted frame (1.10), (1.11) on ${}^2M$. Any tensor field F of type (1.1) on the total space ${}^2M$ has in the adopted frame the form:*

$$
\begin{aligned}
F &= F^{0i}_{0j} D_{0i} \otimes \omega^{0j} + F^{0i}_{1j} D_{0i} \otimes \omega^{1j} + F^{0i}_{2j} D_{0i} \otimes \omega^{2j} + F^{1i}_{0j} D_{1i} \otimes \omega^{0j} + \\
&+ F^{1i}_{1j} D_{1i} \otimes \omega^{1j} + F^{1i}_{2j} D_{1i} \otimes \omega^{2j} + F^{2i}_{0j} D_{2i} \otimes \omega^{0j} + F^{2i}_{1j} D_{2i} \otimes \omega^{1j} + F^{2i}_{2j} D_{2i} \otimes \omega^{2j}.
\end{aligned}
\tag{4.1}
$$

The tensor F is determined by 3^2 of M-tensors on ${}^2M$: $F^\alpha_\beta(z^{0k}, z^{1k}, z^{2k})$, $\alpha, \beta = 0i, 1i, 2i$ and components: F^{0i}_{1j}, F^{0i}_{2j} are independent on the connection $\widetilde{\Gamma}$. The field F has in the natural frame with respect coordinates (z^{0i}, z^{1i}, z^{2i}) the matrix $\overline{F} = [\overline{F}^\alpha_\beta]$. We denote:

$\Gamma^{1i}_j = \Gamma^i_{jk} z^{1k}$, $\Gamma^{2i}_j = \Gamma^i_{jk} z^{2k}$

$$
\begin{aligned}
&\overline{F}^{0i}_{0j} = F^{0i}_{0j} + F^{0i}_{1k} \Gamma^{1k}_j + F^{0i}_{2k} \Gamma^{2k}_j, \quad \overline{F}^{0i}_{1j} = F^{0i}_{1j}, \quad \overline{F}^{0i}_{2j} = F^{0i}_{2j}, \\
&\overline{F}^{1i}_{0j} = F^{1i}_{0j} - \Gamma^{1i}_k F^{0k}_{0j} - \Gamma^{1i}_k F^{0k}_{0l} \Gamma^{1l}_j + F^{1i}_{1k} \Gamma^{1k}_j - \Gamma^{1i}_k F^{0k}_{2l} \Gamma^{2l}_j + F^{1i}_{2k} \Gamma^{2k}_j, \\
&\overline{F}^{1i}_{1j} = -\Gamma^{1i}_k F^{0k}_{1j} + F^{1i}_{1j}, \quad \overline{F}^{1i}_{2j} = F^{1i}_{2j} - \Gamma^{1i}_k F^{0k}_{2j}, \\
&\overline{F}^{2i}_{0j} = F^{2i}_{0j} - \Gamma^{2i}_k F^{0k}_{0j} - \Gamma^{2i}_k F^{0k}_{1l} \Gamma^{1l}_j + F^{2i}_{1k} \Gamma^{1k}_j - \Gamma^{2i}_k F^{0k}_{2l} \Gamma^{2l}_j + F^{2i}_{2k} \Gamma^{2k}_j, \\
&\overline{F}^{2i}_{1j} = F^{2i}_{1j} - \Gamma^{2i}_k F^{0k}_{1j}, \quad \overline{F}^{2i}_{2j} = F^{2i}_{2j} - \Gamma^{2i}_k F^{0k}_{2j}.
\end{aligned}
\tag{4.2}
$$

Proof: Using (1.13), (1.16) we obtain (4.1). Because of (1.14) $(\omega^{\alpha}) = N(dz^{\alpha}), (D_{\beta}) = = (\frac{\partial}{\partial z^{\beta}}) \cdot N'$, we get for matrix $\bar{F}$ in natural frame: $\bar{F} = N' \cdot F \cdot N$. Moreover, we have: $F^{0\,i}_{1j} = F(D_{1i}, \omega^{0j}), F^{0\,i}_{2j} = F(D_{2i}, \omega^{0j})$.

Now we determine Yano-Ishihara lifts of a tensor field F of type (1.1) on M into the total space ${}^2M$ in the adopted frame (1.10) with respect induced connection $\tilde{\Gamma}$, (1.3), given in the natural frame with respect induced coordinates on ${}^2M$, [3].

Proposition 7. *Let on an n-dimensional manifold M with a linear connection* $\Gamma : (\Gamma^i_{jk})$ *(torsionless) be given a tensor field F of type* (1.1) *having in the local chart* (U, x^i) *the form:* $F = F^i_j \frac{\partial}{\partial x^i} \otimes dx^j$. *Then Yano-Ishihara lifts* ([3], p. 331) : $F^0, F^{\mathrm{I}}, F^{\mathrm{II}}$ *into total space* ${}^2M$ *of linearized tangent bundle of second order* ${}^2_0\pi : {}^2M \to M$ *in the adopted frame* (1.10), (1.11) *with respect to the induced connection* $\tilde{\Gamma} = (\Gamma, \Gamma)$, (1.3), *has in the local chart* $({}^2_0\pi^{-1}(U), z^{0i}, z^{1i}, z^{2i})$, (1.1), *the form:*

$$
\begin{aligned}
F^0 &= F^i_j D_{2i} \otimes \omega^{0j} \\
F^{\mathrm{I}} &= F^i_j D_{1i} \otimes \omega^{0j} + 2F^i_j D_{2i} \otimes \omega^{1j} + 2z^{1k} \nabla_k F^i_j D_{2i} \otimes \omega^{0j} \\
F^{\mathrm{II}} &= F^i_j D_{0i} \otimes \omega^{0j} + F^i_j D_{1i} \otimes \omega^{1j} + F^i_j D_{2i} \otimes \omega^{2j} + \\
&\quad + z^{1k} \nabla_k F^i_j D_{1i} \otimes \omega^{0j} + 2z^{1k} \nabla_k F^i_j D_{2i} \otimes \omega^{1j} + \\
&\quad + [z^{2k} \nabla_k F^i_j + z^{1r} z^{1s} (\nabla_r \nabla_s F^i_j + F^l_j R^i_{lrs} - F^i_l R^l_{jrs})] D_{2i} \otimes \omega^{0j} .
\end{aligned}
\tag{4.3}
$$

Proof: If we use the formulas (1,1), (2.8), (2.9) for the field ([3], p. 331) $F^0, F^{\mathrm{I}}, F^{\mathrm{II}}$ given in the induced coordinates (x^{0i}, x^{1i}, x^{2i}) we obtain our proposition.

REFERENCES

[1] Bowman, R. H., *Second Order Connections*, J. Differential Geom., 7 (1972), 549–561.

[2] Mok, K. P., Patterson, E. M., Wong, Y. C., *Structure of Symmetric Tensors of Type* (0,2) *and Tensors of Type* (1,1) *on the Tangent Bundle*, Trans Amer. Math. Soc., 234 (1977), 253–278.

[3] Yano, K., Ishihara, S., *Tangent and Cotangent Bundles*, M. Dekker, 1973.

[4] Duc, T. V., *Sur la geometrie differentielle des fibres vectoriels*, Kodai Math. Sem. Rep., 26 (1975), 349–408.

STRESZCZENIE

W pracy wyznaczamy postaci pól tensorowych typu (1,0), (0,1), (0,2), (1,1) w reperze adoptowanym na przestrzeni totalnej uliniowionej wiązki stycznej drugiego rzędu ${}^2_0\pi$: ${}^2M \to M$. Wprowadzamy

reper adoptowany na przestrzeni totalnej ${}^2M$ względem koneksji liniowej $\tilde{\Gamma}$ w wiązce ${}^2M \to M$ indukowanej za pomocą koneksji liniowej Γ w wiązce $TM \to M$. Wyznaczamy postaci podniesień horyzontalnych przekrojów $TM \to M$, podniesień wertykalnych przekrojów ${}^2M \to M$ oraz postaci podniesień Yano-Ishihara ([3]) pól typu (1,0), (0,1) do ${}^2M$ w reperze adoptowanym. Definiujemy metrykę riemanowską typu Sasaki na ${}^2M$ i wyznaczamy postaci podniesień Yano-Ishihara dla tensorów typu (0,2) do ${}^2M$. Ponadto znajdujemy postaci tensorów typu (1,1) oraz podniesień Yano-Ishihara dla tensorów typu (1,1) w reperze adoptowanym na ${}^2M$.

РЕЗЮМЕ

В работе определено вид тензорных полей тира (1,0), (0,1), (0,2), (1,1) в адаптированом репере на пространстве расслоения ${}^2M$ линеаризованного касательного расслоения второго порядка ${}^2_0\pi : {}^2M \to M$. Вводим адаптированный репер на пространстве расслоения ${}^2M$ относительно линейной связности (Γ, Γ) в расслоении ${}^2_0\pi : {}^2M \to M$ индуцированой при помощи связности Γ в расслоений $TM \to M$. Определяем вид горизонтального лифта сечений $TM \to M$, вертикального лифта сечений ${}^2M \to M$, а также вид Яно-Исихара лифтов [3] полей типа (1,0), (0,1) в адаптированом репере на ${}^2M$. Определяем риманову метрику типа Сасаки на ${}^2M$, а также вид Яно-Исихара лифтов тензоров типа (0,2) в адаптированом репере. Также находим вид тензоров типа (1,1) и Яно-Исихара лифтов тензоров типа (1,1) в адаптированом репере на ${}^2M$.