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On Some Linear Connections of Second Order

O pewnych koneksjach liniowych drugiego rzędu

О некоторых линейных связностях второго порядка

The purpose of this paper is to investigate special linear connections in linearized tangent bundle of second order ${}^2_0\pi : {}^2M \to M$. These connections are obtained in a natural way from connections in the principal bundle of frames for the vector bundle ${}^2_0\pi : {}^2M \to M$. The resulting connections are the second order connections in the sense of Bowman [1].

1. Let M be an n-dimensional Hausdorff manifold of class C^∞. Moreover, let ${}^1_0\pi : TM \to M$ be the tangent bundle over M and $T_\pi : T(TM) \to TM$ be the tangent bundle over manifold TM. We consider the tangent bundle of second order: ${}^2_0\pi : {}^2M \to M$, where ${}^2M = \left\{ A \in T(TM) : {}^1_0\pi_* A = T_\pi A \right\}$ and ${}^2_0\pi = {}^1_0\pi_* \cdot {}^1_0\pi_*|_{{}^2M}$. Let $\Gamma : (\Gamma^i_{jk})$ be a linear connection on M i.e. a connection in the tangent bundle ${}^1_0\pi : TM \to M$. A local chart (U, x^{0i}) on M induces a local chart $({}^1_0\pi^{-1}(U), x^{0i}, x^{1i})$ on the total space of the tangent bundle TM and a local chart $({}^2_0\pi^{-1}(U), x^{0i}, x^{1i}, x^{2i})$ on the total space of the tangent bundle of second order ${}^2M \subset T(TM)$. Then, for the tangent bundle of second order ${}^2_0\pi : {}^2M \to M$ there exists a vector bundle structure [1] with coordinates:

$$z^{0i} = x^{0i}, z^{1i} = x^{1i}, z^{2i} = x^{2i} + \Gamma^i_{jk} x^{1j} x^{1k} . \tag{1.1}$$

with respect to a basis of the local sections:

$$E^0_{1i\,|\,x} = \frac{\partial}{\partial x^{0i}} - \Gamma^k_{lj} \delta^j_i \frac{\partial}{\partial x^{1k}} \Big|_{(x^{0l}, \delta^l_i)} , E^0_{2i\,|\,x} = \frac{\partial}{\partial x^{1i}} \Big|_{(x^{0l}, 0)} . \tag{1.2}$$

The tangent bundle of second order ${}^2_0\pi : {}^2M \to M$ is the Whitney sum ${}^2M = {}^2M^H \underset{M}{\times} {}^2M^V$

of the horizontal subbundle ${}^2M^H$ and the vertical subbundle ${}^2M^V$. A fibre ${}^2M_x$ is the direct sum of fibres of the horizontal subbundle and the vertical subbundle spanned by $E^0_{1i\,|\,x}, E^0_{2i\,|\,x}$ respectively.

We consider the trivial vector bundle on M: $M \times \mathbf{R}^{2n} \to M$. We construct the principal bundle:

$$r : Isom\,(M \times \mathbf{R}^{2n}, {}^2M) \longrightarrow M\,. \tag{1.3}$$

The fibre $r^{-1}(x)$ over $x \in M$ consists of isomorphisms $u_x : \mathbf{R}^{2n} \to {}^2M_x$ satysfying the conditions:

$$u_{x\,|\,\mathbf{R}^n_1}(\mathbf{R}^n_1) \subset {}^2M^H_x,\ u_{x\,|\,\mathbf{R}^n_2}(\mathbf{R}^n_2) \subset {}^2M^V_x\,. \tag{1.4}$$

for the decompositions: $\mathbf{R}^{2n} = \mathbf{R}^n_1 \oplus \mathbf{R}^n_2$. Moreover, let $G \subset GL\,(2n, R)$ be a subgroup such that for decomposition $\mathbf{R}^{2n} = \mathbf{R}^n_1 \oplus \mathbf{R}^n_2$ subspaces $\mathbf{R}^n_1, \mathbf{R}^n_2$ are invariant with respect to the operation: $G : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$. The subgroup G has the form:

$$G = \left\{ \left[\begin{array}{c|c} A^1 & 0 \\ \hline 0 & A^2 \end{array}\right] : A^1, A^2 \in GL(n, \mathbf{R}) \right\} \tag{1.5}$$

with respect to the canonical basis (e_{1i}, e_{2i}) in $\mathbf{R}^{2n} = \mathbf{R}^n_1 \oplus \mathbf{R}^n_2$. The subgroup G operates on the right on the fibre bundle (1.3)

$$\mathbf{R}^{2n} \xrightarrow[g]{} \mathbf{R}^{2n} \xrightarrow[u_x]{} {}^2M_x \tag{1.6}$$

$$(g, u_x) \longrightarrow u_x \cdot g\,.$$

Proposition 1. *The bundle of isomorphism $r : Isom\,(M \times \mathbf{R}^{2n}, {}^2M) \to M$ is the principal bundle over M with the structure group G, (1.5). This bundle will be denoted ${}^2P(M, G)$ and called the principal bundle of frames of the vector bundle ${}^2_0\pi : {}^2M \to M$.*

Proof: Let (U, x^i) and $(U', x^{i'})$ be local charts on a manifold M and $x^i = x^i(x^{i'})$ be the change of charts on $U \cap U'$. Then a basis of a local section in the bundle ${}^2M$ over U consists of E^0_{1i}, E^0_{2i} and over U' of $E^0_{1i'} = E^0_{1i}\dfrac{\partial x^i}{\partial x^{i'}}$, $E^0_{2i'} = E^0_{2i}\dfrac{\partial x^i}{\partial x^{i'}}$. The following sections: $u_x(e_{1i}) = E^0_{1j}\,A^{1j}_i$, $u_x(e_{2i}) = E^0_{2j}A^{2j}_i$ constitute a basis of a fibre ${}^2M_x$. Thus local coordinates in the bundle ${}^2P(M, G)$ are of the form: $\left(x^i, \left[\begin{array}{c|c} A^{1j}_i & 0 \\ \hline 0 & A^{2j}_i \end{array}\right]\right)$. Transition functions in ${}^2P(M, G)$ corresponding to the charts (U, x^i) and $(U', x^{i'})$ on M have the form: $x^i\,(U \cap U') \times G \to x^{i'}\,(U \cap U') \times G$,

$$\left(x^i, \left[\begin{array}{c|c} A^{1j}_i & 0 \\ \hline 0 & A^{2j}_i \end{array}\right]\right) \longrightarrow \left(x^{i'}(x^i), \left[\begin{array}{c|c} \dfrac{\partial x^{i'}}{\partial x^i} & 0 \\ \hline 0 & \dfrac{\partial x^{i'}}{\partial x^i} \end{array}\right] \left[\begin{array}{c|c} A^{1i}_j & 0 \\ \hline 0 & A^{2i}_j \end{array}\right]\right).$$

This means that if: $h_{(x,U)}, h_{(x,U')} : G \to r^{-1}(x)$ are diffeomorphism then:

$$h^{-1}_{(x,U')} \cdot h_{(x,U)}(A) = \left[\begin{array}{c|c} \dfrac{\partial x^{i'}}{\partial x^{i}} & 0 \\ \hline 0 & \dfrac{\partial x^{i'}}{\partial x^{i}} \end{array}\right] \cdot A\,.$$

Moreover we have:

Proposition 2. *For the principal bundle $^2P\,(M,\,G)$ of frames of the vector bundle $^2_0\pi : {}^2M \to M$ just this vector bundle is associated with the standard fibre R^{2n}.*

Proof: We define the right action of the group G, on the manifold $^2P \times \mathrm{R}^{2n}$: $^2P \times \mathrm{R}^{2n} \times G \to {}^2P \times \mathrm{R}^{2n}$. $((u_x, \xi), g) \to (u_x \cdot g, g^{-1}\xi)$. Let q be a canonical mapping: $q : {}^2P \times \mathrm{R}^{2n} \to {}^2P \underset{G}{\times} \mathrm{R}^{2n}$,

$$q(u_x, \xi) = \left\{(u_x \cdot g, g^{-1}\xi) : g \in G\right\} = q(u^0_x, A\xi)\,. \tag{1.7}$$

In local coordinates, we have:

$$q\,(u_x, \xi) = q\,(u^0_x, A\xi) = (A^{1i}_j\,\xi^{1j})E^0_{1i\,|\,x} + (A^{2i}_j\,\xi^{2j})E^0_{2i\,|\,x} = z^{1i}E^0_{1i\,|\,x} + z^{2i}E^0_{2i}$$

Moreover: $^2P \underset{G}{\times} \mathrm{R}^{2n} = {}^2M$.

2. For the principal bundle $^2P(M, G)$ of frames of vector bundle $^2_0\pi : {}^2M \to M$ the structure group G has the form (1.5). We consider a connection in the principal bundle $^2P(M,\,G)$ defined by the connection form: $\omega : T(^2P) \to T_eG$. The structure group G, (1.5) determines the form of a connection form in the following way:

$$\omega = \left[\begin{array}{c|c} \omega^{1i}_j & 0 \\ \hline 0 & \omega^{2i}_j \end{array}\right]. \tag{2.1}$$

Let (U, x^i) be a local chart on M and $\sigma : U \to {}^2P_U$ a section corresponding to the unity e of the group G:

$$\sigma : x \longrightarrow \sigma(x) = (x^i, E^0_{1i\,|\,x}, E^0_{2i\,|\,x})\,. \tag{2.2}$$

Then, we have so called the local connection forms:

$$\omega^{1i}_{Uj} = \sigma^*\omega^{1i}_j = \Gamma^{1i}_{jk}\,dx^k\,,\ \omega^{2i}_{Uj} = \sigma^*\omega^{2i}_j = \Gamma^{2i}_{jk}\,dx^k\,. \tag{2.3}$$

Definition of a connection in the principal bundle $^2P(M,\,G)$ by using the connection form $\omega : T(^2P) \to T_eG$ is equivalent to the definition given by the left splitting $\Gamma' : T(^2P) \to V(^2P)$ of the exact sequence:

$$0 \longrightarrow V(^2P) \underset{\Gamma'}{\overset{i}{\rightleftarrows}} T(^2P) \xrightarrow{r^!} {}^2P \underset{M}{\times} TM \longrightarrow 0\,, \tag{2.4}$$

$$\omega = j^{-1} \cdot \Gamma' . \tag{2.5}$$

A mapping $j : T_eG \to V(^2P)$ is a linear isomorphism and a mapping $\Gamma' : T(^2P) \to V(^2P)$ is the right invariant: $\Gamma \cdot R_g^* = R_g^* \cdot \Gamma$. We use the method of Duc [2] for the introducing a connection in the associated vector bundle ${}_0^2\pi : {}^2M \to M$ by means of the connection in the principal bundle ${}^2P(M, G)$.

Let us consider the canonical projection (1.7); $q : {}^2P \times \mathrm{R}^{2n} \to {}^2M$ and its differential: $q_* : T(^2P) \times T(\mathrm{R}^{2n}) \to T(^2M)$. Then we have mappings of exact sequences of bundles over ${}^2P$ and ${}^2M$ respectively:

$$\begin{tikzcd} 0 \arrow[r] & V(^2P) \times T\mathrm{R}^{2n} \arrow[r] \arrow[d, "\widetilde{q}_*"] & T(^2P) \times T\mathrm{R}^{2n} \arrow[r] \arrow[d, "q_*"] \arrow[l, bend right, "\Gamma' x id_{T\mathrm{R}^{2n}}"'] & {}^2P \times_M TM \times T\mathrm{R}^{2n} \arrow[r] & 0 , \\ 0 \arrow[r] & V(^2M) \arrow[r, "i"] & T(^2M) \arrow[r, "\pi'"] \arrow[l, bend left, "\widetilde{\Gamma}"] & {}^2M \times_M TM \arrow[r] & 0 . \end{tikzcd}$$

(2.6)

(2.7)

The connections Γ' and $\widetilde{\Gamma}$ in bundles ${}^2P(M, G)$ and ${}^2M \to M$ respectively as the left splitting of the exact sequences (2.6), (2.7) satisfy the relation:

$$\widetilde{\Gamma} \cdot q_* = \widetilde{q}_* \cdot (\Gamma' \times id_{T\mathrm{R}^{2n}}) . \tag{2.8}$$

In local coordinates for the connection $\Gamma' : T(^2P) \to V(^2P)$ we have:

$$\begin{aligned} \Gamma'\left(x^j, \left[\begin{array}{c|c} A_j^{1i} & 0 \\ \hline 0 & A_j^{2i} \end{array}\right]; y^j, \left[\begin{array}{c|c} B_j^{1i} & 0 \\ \hline 0 & B_j^{2i} \end{array}\right]\right) = \\ = \left(x^j, \left[\begin{array}{c|c} A_j^{1i} & 0 \\ \hline 0 & A_j^{2i} \end{array}\right]; 0, \left[\begin{array}{c} B_j^{1i} + \Gamma_{kl}^{1i} A_j^{1k} y^l \\ \hline B_j^{2i} + \Gamma_{kl}^{2i} A_j^{2k} y^l \end{array}\right]\right). \end{aligned} \tag{2.9}$$

Thus:

Proposition 3. *The connection Γ' in the principal bundle ${}^2P(M, G)$ of frames of vector bundle ${}_0^2\pi : {}^2M \to M$ induces in the associated bundle ${}^2M$ with the standard fibre R^{2n} and the canonical projection* (1.7): $q : {}^2P \times \mathrm{R}^{2n} \to {}^2M$ *the connection $\widetilde{\Gamma}$ such that*: $\widetilde{\Gamma} \cdot q_* = \widetilde{q}_* \cdot (\Gamma' \times id_{T\mathrm{R}^{2n}})$. *In local coordinates we have:*

$$\begin{aligned} \widetilde{\Gamma}(z^{0i}, z^{1i}, z^{2i}, y^{0i}, y^{1i}, y^{2i}) = \\ = (z^{0i}, z^{1i}, z^{2i}; 0, y^{1i} + \Gamma_{jk}^{1i} z^{1k} y^{0j}, y^{2i} + \Gamma_{jk}^{2i} z^{2k} y^{0j}) . \end{aligned} \tag{2.10}$$

A connection $\widetilde{\Gamma}$ in vector bundle ${}_0^2\pi : {}^2M \to M$ as the left splitting of the exact sequence (2.7) is of the form:

$$\widetilde{\Gamma} : T(^2M) \to V(^2M),\ \widetilde{\Gamma} \cdot i = id_{V(^2M)}$$

$$\widetilde{\Gamma}\,(z^{0i}, z^{1i}, z^{2i}; y^{0i}, y^{1i}, y^{2i}) =$$
$$= (z^{0i}, z^{1i}, z^{2i}; 0, y^{1i} + \omega_j^{1i}(z^0, z^1, z^2)\, y^{0j}, y^{2i} + \omega_j^{2i}(z^0, z^1, z^2)\, y^{0j})\,. \tag{2.11}$$

The connection map $\widetilde{D}$ for the connection $\widetilde{\Gamma} : T(^2M) \to V(^2M)$ is of the form:

$$\widetilde{D} = p_2 \cdot i_{V(^2M)} \cdot \widetilde{\Gamma}\,, \tag{2.12}$$

where $i_{V(^2M)} : V(^2M) \to {}^2M \underset{M}{\times} {}^2M$ is an isomorphism into the Whitney sum and p_2 is a projection on second factor.

A connection $\widetilde{\Gamma}$ in the bundle ${}^2_0\pi : {}^2M \to M$ is linear if a connection map $\widetilde{D} : T(^2M) \to {}^2M$ is linear in fibres of the bundle ${}^2_0\pi_* : T(^2M) \to TM$.

We have:

Proposition 4. *Let M be an n-dimensional manifold with given linear connection Γ. Let ${}^2_0\pi : {}^2M \to M$ be the linearized tangent bundle of second order with coordinates of vector bundle type* (1.1),(z^{0i}, z^{1i}, z^{2i}) *and a basis of local sections* (1.2), (E^0_{1i}, E^0_{2i}).

The linear connection $\widetilde{\Gamma}$ in the linearized tangent bundle of second order ${}^2_0\pi : {}^2M \to M$ has the connection map $\widetilde{D} : T(^2M) \to {}^2M$ of the following form in local coordinates:

$$\widetilde{D}\,(y^{0i}\frac{\partial}{\partial z^{0i}} + y^{1i}\frac{\partial}{\partial z^{1i}} + y^{2i}\frac{\partial}{\partial z^{2i}}) =$$
$$= (y^{1i} + \Gamma^{1i}_{j1k}(z^0)\, z^{1k} y^{0j} + \Gamma^{1i}_{j2k}(z^0)\, z^{2k} y^{0j})\, E^0_{1i} +$$
$$+ (y^{2i} + \Gamma^{2i}_{j1k}(z^0)\, z^{1k} y^{0j} + \Gamma^{2i}_{j2k}(z^0)\, z^{2k} y^{0j})\, E^0_{2i}\,. \tag{2.13}$$

The objects $\Gamma^{1i}_{j1k}(z^0)$, $\Gamma^{2i}_{j2k}(z^0)$ are Christoffel symbols and $\Gamma^{1i}_{j2k}(z^0)$, $\Gamma^{2i}_{j1k}(z^0)$ are tensors on manifold M.

Proof: For the linear connection $\widetilde{\Gamma}$ in the bundle ${}^2_0\pi$: ${}^2M \to M$ the connection map $\widetilde{D} : T(^2M) \to {}^2M$ is linear in fibres on the bundle ${}^2_0\pi_* : T(^2M) \to TM$. This means that $\widetilde{D}$ is linear in the fibre ${}^2_0\pi_*^{-1}(z^{0i}, y^0)$ with respect to z^{1i}, z^{2i}, y^{1i}, y^{2i}. Thus we have that components: $\omega_j^{1i}(z^0, z^1, z^2)$, $\omega_j^{2i}(z^0, z^1, z^2)$ of the connection $\widetilde{\Gamma}$ are linearly depend on z^{1i}, z^{2i}

$$\omega_j^{1i}(z^0, z^1, z^2) = \Gamma^{1i}_{j1k}(z^0)\, z^{1k} + \Gamma^{1i}_{j2k}\, z^{2k}\,, \quad \omega_j^{2i} = \Gamma^{2i}_{j1k}\, z^{1k} + \Gamma^{2i}_{j2k}\, z^{2k}\,. \tag{2.14}$$

Let us consider two local charts (U, x^i), $(U', x^{i'})$ in M. The change of coordinates $x^i = x^i(x^{i'})$ on $U \cap U'$ gives the following change for coordinates and basis in ${}^2M$:

$$z^{0i} = z^{0i}(z^{0i'}),\, z^{1i} = A^i_{i'}\, z^{1i'},\quad z^{2i} = A^i_{i'}\, z^{2i'}\,, \tag{2.15}$$

$$E^0_{1i'} = E^0_{1i} A^i_{i'},\ E^0_{2i'} = E^0_{2i} A^i_{i'},\ A^i_{i'} = \frac{\partial x^i}{\partial x^{i'}},\ A^i_{i'j'} = \frac{\partial^2 x^i}{\partial x^{i'}\partial x^{j'}}. \tag{2.16}$$

For the coordinates in $T({}^2M)$ we have:

$$y^{0i} = A^i_{i'} y^{0i'},\ \ y^{1i} = A^i_{i'} y^{1i'} + A^i_{i'j'} z^{1j'} y^{0i'},\ \ y^{2i} = A^i_{i'} y^{2i'} + A^i_{i'j'} z^{2j'} y^{0i'}. \tag{2.17}$$

Thus, after having used (2.14), (2.15), (2.16), (2.17) in (2.11) we get the following formulas:

$$\begin{aligned} &\Gamma^{1i'}_{j'1k'} = A^{i'}_l (A^l_{j'k'} + \Gamma^{1l}_{j1k} A^j_{j'} A^k_{k'}),\quad \Gamma^{1i'}_{j'2k'} = A^{i'}_l \Gamma^{1l}_{j2k} A^j_{j'} A^k_{k'},\\ &\Gamma^{2i'}_{j'2k'} = A^{i'}_l (A^l_{j'k'} + \Gamma^{2l}_{j2k} A^j_{j'} A^k_{k'}),\quad \Gamma^{2i'}_{j'1k'} = A^{i'}_l \Gamma^{2l}_{j1k} A^j_{j'} A^k_{k'}. \end{aligned} \tag{2.18}$$

Definition 1: A connection $\widetilde{\Gamma}$ in the linearized tangent bundle of second order ${}^2_0\pi : {}^2M \to M$ is said to be pure-linear, if its connection map $\widetilde{D} : T({}^2M) \to {}^2M$ satisfies:

$$\widetilde{D}_{|T({}^2M^H)}(T({}^2M^H)) \subset {}^2M^H,\ \ \widetilde{D}_{|T({}^2M^V)}(T({}^2M^V)) \subset {}^2M^V. \tag{2.19}$$

Definition 2 [1]: A connection $\widetilde{\Gamma}$ in the tangent bundle of second order ${}^2_0\pi : {}^2M \to M$ for which there exists a connection Γ in the tangent bundle ${}^1_0\pi : TM \to M$ such that the diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & V({}^2M) \arrow[r] \arrow[d, "{}^2_1\pi_*"'] & T({}^2M) \arrow[r] \arrow[l, bend right, "\widetilde{\Gamma}"'] \arrow[d, "{}^2_1\pi_*"] & {}^2M \times_M TM \arrow[r] & 0 \\ 0 \arrow[r] & V(TM) \arrow[r, "i"] & T(TM) \arrow[r] \arrow[l, bend left, "\Gamma"] & TM \times_M TM \arrow[r] & 0, \end{tikzcd} \tag{2.20}$$

is said to be a second order connection on M.

Thus, the connection $\widetilde{\Gamma}$ is of the form in local coordinates:

$$\widetilde{\Gamma}(z^{0i}, z^{1i}, z^{2i}; y^{0i}, y^{1i}, y^{2i}) =$$
$$= (z^{0i}, z^{1i}, z^{2i}; 0, y^{1i} + \omega^{1i}_j(z^0, z^1) y^{0j}, y^{2i} + \omega^{2i}_j(z^0, z^1, z^2) y^{0j})$$

We prove:

Proposition 5. *Let M be an n-dimensional manifold with given linear connection Γ and let ${}^2_0\pi : {}^2M \to M$ be the linearized tangent bundle of second order.*

A pure – linear connection in the bundle ${}^2_0\pi : {}^2M \to M$ is a connection of second order on M and its connection map $\widetilde{D}$ is of the form:

$$\widetilde{D}\left(y^{0i}\frac{\partial}{\partial z^{0i}}+y^{1i}\frac{\partial}{\partial z^{1i}}+y^{2i}\frac{\partial}{\partial z^{2i}}\right)=$$

$$=(y^{1i}+\Gamma^{1i}_{j1k}z^{1i}y^{0j})E^{0}_{1i}+(y^{2i}+\Gamma^{2i}_{j2k}z^{2k}y^{0j})E^{0}_{2i}. \tag{2.21}$$

Moreover, a pure – linear connection $\widetilde{\Gamma}$ in bundle ${}^2_0\pi : {}^2M \to M$ may be obtained from the connection Γ' in the principal bundle ${}^2P(M, G)$ in the following way: $\widetilde{\Gamma}\cdot q_* = \widetilde{q}_*\cdot(\Gamma'\times id_{T\mathbf{R}^{2n}})$

$$\Gamma^{1i}_{j1k}=\Gamma^{1i}_{jk},\quad \Gamma^{2i}_{j2k}=\Gamma^{2i}_{jk}. \tag{2.22}$$

In particular, a given connection $\Gamma : (\Gamma^{i}_{jk})$ in the tangent bundle ${}^1_0\pi : TM \to M$ induces a pure – linear connection $\widetilde{\Gamma}$ in the bundle ${}^2_0\pi : {}^2M \to M$ of the form:

$$\Gamma^{1i}_{j1k}=\Gamma^{i}_{jk},\quad \Gamma^{2i}_{j2k}=\Gamma^{i}_{jk}. \tag{2.23}$$

Proof: For the horizontal subbundle ${}^2M^H$ we have $z^{2i}=0$. The value of the connection map $\widetilde{D}$ on $T({}^2M^H)$ is of the form:

$$\widetilde{D}\left(y^{0i}\frac{\partial}{\partial z^{0i}}+y^{1i}\frac{\partial}{\partial z^{1i}}\right)=(y^{1i}+\Gamma^{1i}_{j1k}z^{1k}y^{0j})E^{0}_{1i}+(\Gamma^{2i}_{j1k}z^{1k}y^{0j})E^{0}_{2i}.$$

Thus for the pure – linear connection we have that $\Gamma^{2i}_{j1k}=0$ identically. Similarly, for the vertical subbundle we have $z^{1i}=0$ and condition 2° for the pure – linear connection gives that $\Gamma^{1i}_{j2k}=0$ identically. Now, from (2.10), (2.21) it is easy to see that $\widetilde{\Gamma}$ satisfies (2.22). In particular we have (2.23).

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STRESZCZENIE

W pracy badamy specjalną koneksję liniową w uliniowionej wiązce stycznej drugiego rzędu ${}^2_0\pi : {}^2M \to M$.

Koneksję tę nazywa się czysto-liniową i otrzymuje się w sposób naturalny z koneksji w wiązce głównej reperów dla wiązki wektorowej ${}^2M \to M$. Otrzymana koneksja jest koneksją drugiego rzędu w sensie Bowmana, [1].

РЕЗЮМЕ

В работе исследуется специальная линейная связность. В линеаризованом касательном расслоении второго порядка ${}^2_0\pi : {}^2M \to M$.

Эта связность называется чисто линейной и получается естественно с связностью в главном расслоении реперов векторного расслоения ${}^2_0\pi : {}^2M \to M$.

Полученная связность является связностью второго порядка в смысле Бовмана, [1].