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Lifts of Almost Paracontact Structures on a Lie Group

Podniesienia prawie para-kontaktowych struktur na grupie Lie'go

Поднятие почти паракоктактных структур на группе Ли

In this paper that is continuation of [2] we deal with lifts of left invariant almost paracontact normal (or weak-normal) structures on a Lie group G to the left invariant almost paracontact normal (or weak-normal) structures on the group TG.

1. Let G be a Lie group and TG be a tangent bundle to G. It is known [3], TG has a natural structure of a Lie group. If Φ is given tensor field of type (p, q) on G, then the symbols Φ^v, Φ^c denote vertical and complete lifts of the field Φ to TG (For details see [3]). It will be useful:

Theorem 1.1 [3]. *Vertical and complete lifts of a left invariant vector field on G are left invariant vector fields on TG. If vector fields $X_1, \dots, X_n$ constitute a basis of left invariant vector fields on G, then the fields $X_1^v, \dots, X_n^v; X_1^c, \dots, X_n^c$ are the vector fields of a basis of left invariant vector fields on TG.*

The following formulas will be useful:

$$\omega^v(X^v) = 0, \quad \omega^c(X^c) = [\omega(X)]^c, \quad \omega^v(X^c) = \omega^c(X^v) = (\omega(X))^v \tag{1.1}$$

for any 1-form ω and any vector field X on G.

$$\phi^c(X^c) = [\phi(X)]^c, \quad \phi^v(X^v) = 0, \quad \phi^c(X^v) = \phi^v(X^c) = [\phi(X)]^v \tag{1.2}$$

for any tensor field ϕ of type (1.1) and any vector field X on G.

$$[X^v, Y^v] = 0, \ [X^c, Y^c] = [X, Y]^c, \ [X^v, Y^c] = [X^c, Y^v] = [X, Y]^v \tag{1.3}$$

for any vector fields X, Y on G.

$$(P \otimes Q)^c = P^c \otimes Q^v + P^v \otimes Q^c \tag{1.4}$$

for any tensor fields P, Q on G.

Let ω be a left invariant form on a group G. If $X_1, .., X_n$ are vector fields of any basis of the left invariant vector fields on G, then $\omega(X_i) = \text{const}$ for $i = 1, .., n$. Conversely, if $\omega(X_i) = \text{const}$ for $i = 1, .., n$, then the form ω is left invariant.

In virtue of (1.1) and Theorem 1.1 it follows that the forms ω^v and ω^c take constant values on $X_1^v, .., X_n^v, X_1^c, .., X_n^c$. Hence they are the left invariant forms on G.

Definition 1.1. Let ϕ be a tensor field of type (1.1) on G. ϕ is said to be left invariant, if $(L_g)_* \circ \phi = \phi \circ (L_g)_*$ for any $g \in G$, where L_g denotes the left translation of G by an element $g \in G$, and $(L_g)_*$ is the induced mapping by L_g for vector fields on G.

Lemma 1.2. *A tensor field ϕ of type* (1.1) *is left invariant if and only if takes left invariant vector fields into left invariant vector fields.*

Proof. In fact, let **G** denote the Lie algebra of the left invariant vector fields on G and let $X \in \mathbf{G}$. Then $(L_g)_*(X) = X$ for every $g \in G$ and $\phi(X) = \phi((L_g)_*(X)) = (L_g)_*(\phi(X))$ hence $\phi(X) \in \mathbf{G}$. Suppose now, that for any $X \in \mathbf{G}$, $\phi(X)$ belongs to **G**. Denote by $X_i(h)$ the value of the left invariant vector fields X_i at a point $h \in G$. Let $Y \in \mathbf{G}$, then $Y(h) = a^i(h)X_i(h)$ where $a^i : G \to R$, and $X_1, .., X_n$ constitute a basis of **G**. We have

$$\phi((L_g)_*(a^i(h)X_i(h))) = \phi(a^i(h)X_i(gh)) = a^i(h)\phi(X_i(gh))$$

and

$$(L_g)_*(\phi(a^i(h)X_i(h)) = a^i(h)(L_g)_*(\phi(X_i(h))) = a^i(h)\phi(X_i(gh))$$

Thus $\phi \circ (L_g)_* = (L_g)_* \circ \phi$ for every $g \in G$.

From Theorem 1.1 and the condition (1.2), it follows that the tensor fields ϕ^v and ϕ^c take left invariant vector fields into left invariant vector fields, thus they are left invariant.

We shal need the following:

Theorem 1.3 ([3] p. 33). *Let $\widetilde{P}$, $\widetilde{Q}$ be tensor fields of type* (0, s) *or* (1, s) *on TG, where $s > 0$, such that*

$$\widetilde{P}(Y_1^c, .., Y_s^c) = \widetilde{Q}(Y_1^c, .., Y_s^c)$$

for any vector fields $Y_1, .., Y_s$ on G. Then $\widetilde{P} = \widetilde{Q}$.

Observe that in particular if $\widetilde{P}(Y_1^c, .., Y_s^c) = 0$ for any $Y_1, .., Y_s$ on G, then $\widetilde{P} = 0$.

Now suppose that F is a left invariant tensor field of an almost product structure on G. Then F^v and F^c are left invariant tensor fields on TG and

$$(F^v)^2 = 0_{TG}, \ (F^c)^2 = Id_{TG}. \tag{1.5}$$

Thus, F^c is a tensor of an almost product structure on TG. If an almost product structure F on G is integrable i.e. the Nijenhuis tensor $[F, F]$ vanishes identically on G, then from Propositions 5.5 and 5.6, pp. 35, 36 [3], it follows that $[F^c, F^c] = 0$. Thus we have:

Theorem 1.4. *The complete lift F^c of a left invariant tensor field F of an almost product structure on G is an integrable left invariant almost product structure on TG if and only if F is integrable.*

Now we shall deal with lifts of left invariant almost paracontact structures from G to TG. Let (ϕ, ξ, η) be such structure on G. We have:

$$\begin{aligned} &\phi^2 = Id_{TG} - \eta \otimes \xi\,, \\ &\eta(\xi) = 1\,, \\ &\eta \circ \phi = 0\,, \\ &\phi(\xi) = 0\,. \end{aligned} \tag{1.6}$$

Thus, using (1.1), (1.2) and (1.4) we obtain:

$$\begin{aligned} &(\phi^c)^2 = Id_{TG} - \eta^v \otimes \xi^c - \eta^c \otimes \xi^v\,, \\ &\eta^v(\xi^v) = 0,\ \eta^v(\xi^c) = \eta^c(\xi^c) = 1,\ \eta^c(\xi^c) = 0, \\ &\phi^c(\xi^v) = 0,\ \phi^c(\xi^c) = 0\,. \end{aligned} \tag{1.7}$$

From (1.7) it follows that $(\phi^c)^3 = \phi^c$. On the other hand:

$$(\phi^c)^2(\phi^c(X)) = \phi^c(X) - \eta^c(\phi^c(X))\,\xi^v - \eta^v(\phi^c(X))\,\xi^c\,.$$

Hence

$$\eta^c(\phi^c(X))\,\xi^v + \eta^v(\phi^c(X))\,\xi^c = 0\,.$$

Since ξ^v and ξ^c are linearly independent, we have:

$$\eta^c \circ \phi^c = 0,\ \eta^v \circ \phi^c = 0. \tag{1.8}$$

Let $f_1 = \phi^c + \epsilon_1 \eta^c \otimes \xi^v$ and $f_2 = \phi^c + \epsilon_2 \eta^v \otimes \xi^c$, where $\epsilon_1 = \pm 1$, $\epsilon_2 = \pm 1$. Then

$$\begin{aligned} (f_1)^2 &= (\phi^c + \epsilon_1 \eta^c \otimes \xi^v)^2 = (\phi^c)^2 + \epsilon_1(\phi^c \circ (\eta^c \otimes \xi^v) + (\eta^c \otimes \xi^v) \circ \phi^c) + (\eta^c \otimes \xi^v)^2 = \\ &= Id_{TG} - \eta^v \otimes \xi^c - \eta^c \otimes \xi^v + \eta^c \otimes \xi^v = Id_{TG} - \eta^v \otimes \xi^c. \end{aligned}$$

Similarly $(f_2)^2 = Id_{TG} - \eta^c \otimes \xi^v$. After having used (1.7) and (1.8) we got:

Theorem 1.5. *If (ϕ, ξ, η) is a left invariant almost paracontact structure on G, then (f_1, ξ^c, η^v) and (f_2, ξ^v, η^c) are the left invariant almost paracontact structures on TG.*

Now suppose that a left invariant almost paracontact structure (ϕ, ξ, η) on G is normal. Then, by virtue of [2] we have:

$$\Phi(X, Y) = \phi[X, Y] - [\phi X, Y] - [X, \phi Y] + \phi[\phi X, \phi Y] = 0 \tag{1.9}$$

for any vector fields X, Y on G.

Conversely, if $\Phi = 0$, then (ϕ, ξ, η) is normal. If we put $Y = \xi$ in (1.9), then we have:

$$\phi[X, \xi] - [\phi X, \xi] = 0 \tag{1.10}$$

for any vector field X on G. As we know [2], for a left invariant normal almost paracontact structure the condition (1.9) implies:

$$\eta[\phi X, \phi Y] + \eta[X, Y] = 0 \tag{1.11}$$

for any vector fields X, Y on G. From (1.10) and (1.11) we have:

$$\phi[\phi X, \xi] - [X, \xi] = 0\,, \tag{1.12}$$

$$\eta[X, \xi] = 0\,, \tag{1.13}$$

for any vector field X on G.

Now we prove the following:

Proposition 1.6. *The following equalities hold good:*

$$\eta^c[\widetilde{X}, \widetilde{Y}] + \eta^c[\phi^c\widetilde{X}, \phi^c\widetilde{Y}] = 0\,, \tag{1.14}$$

$$\eta^v[\widetilde{X}, \widetilde{Y}] + \eta^v[\phi^c\widetilde{X}, \phi^c\widetilde{Y}] = 0\,, \tag{1.15}$$

$$\phi^c[\xi^v, \phi^c\widetilde{Y}] - [\xi^v, \widetilde{Y}] = 0\,, \tag{1.16}$$

$$\phi^c[\xi^c, \phi^c\widetilde{Y}] - [\xi^c, \widetilde{Y}] = 0\,, \tag{1.17}$$

$$\eta^c[\xi^v, \phi^c\widetilde{Y}] = 0\,, \tag{1.18}$$

$$\eta^v[\xi^c, \phi^c\widetilde{Y}] = 0\,, \tag{1.19}$$

$$\phi^c[\widetilde{X}, \widetilde{Y}] - [\phi^c\widetilde{X}, \widetilde{Y}] - [\widetilde{X}, \phi^c\widetilde{Y}] + \phi^c[\phi^c\widetilde{X}, \phi^c\widetilde{Y}] = 0\,, \tag{1.20}$$

for any vector fields $\widetilde{X}$, $\widetilde{Y}$ on TG.

Proof. On account of Theorem 1.3 it suffices to show that tensors of the left hand sides of (1.14) through (1.20) vanish for the complete lifts of all vector fields on G.

Let X, Y be vector fields on G. Using (1.1), (1.2), (1.3), (1.10) – (1.13) we have:

$$\eta^c[X^c, Y^c] + \eta^c[\phi^c X^c, \phi^c Y^c] = (\eta[X, Y] + \eta[\phi X, \phi Y])^c = 0,$$

$$\eta^v[X^c, Y^c] + \eta^v[\phi^c X^c, \phi^c Y^c] = (\eta[X, Y] + \eta[\phi X, \phi Y])^v = 0,$$

$$\phi^c[\xi^v, \phi^c Y^c] - [\xi^v, Y^c] = (\phi[\xi, \phi Y] - [\xi, Y])^v = 0,$$

$$\phi^c[\xi^c, \phi^c Y^c] - [\xi^c, Y^c] = (\phi[\xi, \phi Y] - [\xi, Y])^c = 0,$$

$$\eta^c[\xi^v, \phi^c Y^c] = (\eta[\xi, \phi Y])^v = 0,$$

$$\eta^c[\xi^c, \phi^c Y^c] = (\eta[\xi, \phi Y])^c = 0,$$

$$\phi^c[X^c, Y^c] - [\phi^c X^c, Y^c] - [X^c, \phi^c Y^c] + \phi^c[\phi^c X^c, \phi^c Y^c] =$$

$$= (\phi[X, Y] - [\phi X, Y] - [X, \phi Y] + \phi[\phi X, \phi Y])^c = 0.$$

Now we prove:

Theorem 1.7. *Let (ϕ, ξ, η) be a left invariant normal almost paracontact structure on G. Then (f_1, ξ^c, η^v), (f_2, ξ^v, η^c) are left invariant normal almost paracontact structures on TG.*

Proof. We know (see [2]), that the normality of the structures (f_1, ξ^c, η^v) and (f_2, ξ^v, η^c) is equivalent to the following conditions:

$$\tilde{\Phi}_1(\tilde{X}, \tilde{Y}) = f_1[\tilde{X}, \tilde{Y}] - [f_1\tilde{X}, \tilde{Y}] - [\tilde{X}, f_1\tilde{Y}] + f_1[f_1\tilde{X}, f_1\tilde{Y}] = 0$$

$$\tilde{\Phi}_2(\tilde{X}, \tilde{Y}) = f_2[\tilde{X}, \tilde{Y}] - [f_2\tilde{X}, \tilde{Y}] - [\tilde{X}, f_2\tilde{Y}] + f_2[f_2\tilde{X}, f_2\tilde{Y}] = 0$$

We have:

$$\tilde{\Phi}_1(\tilde{X}, \tilde{Y}) = \phi^c[\tilde{X}, \tilde{Y}] - [\phi^c\tilde{X}, \tilde{Y}] - [\tilde{X}, \phi^c\tilde{Y}] + \phi^c[\phi^c\tilde{X}, \phi^c\tilde{Y}] +$$

$$+ \epsilon_1(\eta^c[\tilde{X}, \tilde{Y}] + \eta^c[\phi^c\tilde{X}, \phi^c\tilde{Y}])\xi^v + \epsilon_1\eta^c(\tilde{X})(\phi^c[\xi^v, \phi^c\tilde{Y}] - [\xi^v, \tilde{Y}]) +$$

$$+ \epsilon_1\eta^c(\tilde{Y})(\phi^c[\phi^c\tilde{X}, \xi^v] - [\tilde{X}, \xi^v]) + \eta^c(\tilde{Y})\eta^c[\phi^c\tilde{X}, \xi^v]\xi^v + \eta^c(\tilde{X})\eta^c[\xi^v, \phi^c\tilde{Y}]\xi^v.$$

$$\tilde{\Phi}_2(\tilde{X}, \tilde{Y}) = \phi^c[\tilde{X}, \tilde{Y}] - [\phi^c\tilde{X}, \tilde{Y}] - [\tilde{X}, \phi^c\tilde{Y}] + \phi^c[\phi^c\tilde{X}, \phi^c\tilde{Y}] +$$

$$+ \epsilon_2(\eta^v[\tilde{X}, \tilde{Y}] + \eta^v[\phi^c\tilde{X}, \phi^c\tilde{Y}])\xi^c + \epsilon_2\eta^v(\tilde{X})(\phi^c[\xi^c, \phi^c\tilde{Y}] - [\xi^c, \tilde{Y}]) +$$

$$+ \epsilon_2\eta^v(\tilde{Y})(\phi^c[\phi^c\tilde{X}, \xi^c] - [\tilde{X}, \xi^c]) + \eta^v(\tilde{Y})\eta^v[\phi^c\tilde{X}, \xi^c]\xi^v + \eta^v(\tilde{X})\eta^v[\xi^c, \phi^c\tilde{Y}]\xi^v.$$

In the virtue of Proposition 1.6 we obtain, that $\Phi_1(X, Y) = 0$ and $\Phi_2(X, Y) = 0$ for any vector fields X, Y on TG.

Suppose now, that (f_1, ξ^c, η^v) is normal structure i.e. $\Phi_1(X, Y) = 0$. Let $X_1, .., X_n$ be a basis of the left invariant vector fields on G. Then $\eta(X_i) = \text{const}$ and $\eta^c(X_i^c) = (\eta(X_i))^c = 0$. Hence:

$$0 = \tilde{\Phi}_1(X_i^c, X_j^c) = \phi^c[X_i^c, X_j^c] - [\phi^c X_i^c, X_j^c] - [X_i^c, \phi^c X_j^c] + \phi^c[\phi^c X_i^c, \phi^c X_j^c] =$$

$$= (\phi[X_i, X_j] - [\phi X_i, X_j] - [X_i, \phi X_j] + \phi[\phi X_i, \phi X_j])^c.$$

Thus, we have:

$$\Phi(X_i, X_j) = \phi[X_i, X_j] - [\phi X_i, X_j] - [X_i, \phi X_j] + \phi[\phi X_i, \phi X_j] = 0$$

for any left invariant vector fields X_i, X_j, what means that $\Phi(X, Y) = 0$ for any vector fields X, Y on G. If (f_2, ξ^v, η^c) is the left invariant normal almost paracontact structure, then $\Phi_2(X, Y) = 0$. Since $\eta^v(X_j^v) = 0, j = 1, \ldots, n$ we have:

$$0 = \tilde{\Phi}_2(X_i^v, X_j^v) = \phi^c[X_i^v, X_j^v] - [\phi^c X_i^v, X_j^v] - [X_i^v, \phi^c X_j^v] + \phi^c[\phi^c X_i, \phi^c X_j] =$$

$$= (\phi[X_i, X_j] - [\phi X_i, X_j] - [X_i, \phi X_j] + \phi[\phi X_i, \phi X_j])^v .$$

Hence

$$\phi[X_i, X_j] - [\phi X_i, X_j] - [X_i, \phi X_j] + \phi[\phi X_i, \phi X_j] = 0 .$$

Or $\Phi(X, Y) = 0$ for any vector fields X, Y on G.

We obtain the following:

Theorem 1.8. *The following conditions are equivalent:*

(i) (ϕ, ξ, η) *is a left invariant normal almost paracontact structure on G.*

(ii) (f_1, ξ^c, η^v) *is a left invariant normal almost paracontact structure on TG.*

(iii) (f_2, ξ^v, η^c) *is a left invariant normal almost paracontact structure on TG.*

2. Now we shall consider lifts of left invariant weak-normal almost paracontact structures from G to TG. We recall, that the weak-normality of an almost paracontact structure (ϕ, ξ, η) on G means the integrability of two almost product structures $F_1 = \phi + \eta \otimes \xi$ and $F_2 = \phi - \eta \otimes \xi$ ([2]). Suppose now, that (ϕ, ξ, η) is a left invariant weak-normal almost paracontact structure on G. On account of Theorem 1.4 we know that $F_1^c = \phi^c + \eta^c \otimes \xi^v + \eta^v \otimes \xi^c$ and $F_2^c = \phi^c - \eta^c \otimes \xi^v - \eta^v \otimes \xi^c$ are the left invariant normal almost product structures on TG. Putting $\tilde{F}_{(\epsilon_1, \epsilon_2)} = \phi^c + \epsilon_1 \eta^c \otimes \xi^v + \epsilon_2 \eta^v \otimes \xi^c$ we have:

$$\begin{aligned}
&[\tilde{F}_{(\epsilon_1, \epsilon_2)}, \tilde{F}_{(\epsilon_1, \epsilon_2)}](\tilde{X}, \tilde{Y}) = [\phi^c \tilde{X}, \phi^c \tilde{Y}] + \epsilon_1 \eta^c(\tilde{Y})[\phi^c \tilde{X}, \xi^v] + \epsilon_2 \eta^v(\tilde{Y})[\phi^c \tilde{X}, \xi^c] + \\
&+ \epsilon_1 \eta^c(\tilde{X})[\xi^v, \phi^c \tilde{Y}] + \epsilon_2 \eta^v(\tilde{X})[\xi^c, \phi^c \tilde{Y}] + [\tilde{X}, \tilde{Y}] - \Big\{ \phi^c[\phi^c \tilde{X}, \tilde{Y}] + \epsilon_1 \eta^c(\tilde{X}) \phi^c[\xi^v, \tilde{Y}] + \\
&+ \epsilon_2 \eta^v(\tilde{X}) \phi^c[\xi^c, \tilde{Y}] + \epsilon_1 \eta^c[\phi^c \tilde{X}, \tilde{Y}]\xi^v + \eta^c(\tilde{X}) \eta^c[\xi^v, \tilde{Y}]\xi^v + \epsilon_1 \epsilon_2 \eta^v(\tilde{X}) \eta^c[\xi^c, \tilde{Y}]\xi^v + \\
&+ \epsilon_2 \eta^v[\phi^c \tilde{X}, \tilde{Y}]\xi^c + \epsilon_1 \epsilon_2 \eta^c(\tilde{X}) \eta^v[\xi^v, \tilde{Y}]\xi^c + \eta^v(\tilde{X}) \eta^v[\xi^c, \tilde{Y}]\xi^c + \phi^c[\tilde{X}, \phi^c \tilde{Y}] + \\
&+ \epsilon_1 \eta^c(\tilde{Y}) \phi^c[\tilde{X}, \xi^v] + \epsilon_2 \eta^v(\tilde{Y}) \phi^c[\tilde{X}, \xi^c] + \epsilon_1 \eta^c[\tilde{X}, \phi^c \tilde{Y}]\xi^v + \eta^c(\tilde{Y}) \eta^c[\tilde{X}, \xi^v]\xi^v + \\
&+ \epsilon_1 \epsilon_2 \eta^v(\tilde{Y}) \eta^c[\tilde{X}, \xi^c]\xi^v + \epsilon_2 \eta^v[\tilde{X}, \phi^c \tilde{Y}]\xi^c + \epsilon_1 \epsilon_2 \eta^c(\tilde{Y}) \eta^v[\tilde{X}, \xi^v]\xi^c + \\
&+ \eta^v(\tilde{Y}) \eta^v[\tilde{X}, \xi^c]\xi^c \Big\} .
\end{aligned} \tag{2.1}$$

for any left invariant vector fields $\widetilde{X}$, $\widetilde{Y}$ on TG.

Since $F_1^c = \widetilde{F}_{(1,1)}$ and $F_2^c = \widetilde{F}_{(-1,-1)}$ and because of the integrability of F_1^c and F_2^c we have:

$$[\widetilde{F}_{(1,1)}, \widetilde{F}_{(1,1)}] = 0 \tag{2.2}$$

$$[\widetilde{F}_{(-1,-1)}, \widetilde{F}_{(-1,-1)}] = 0 \tag{2.3}$$

If we add (2.2) to (2.3) and next substract (2.3) from (2.2) and simultineously use (2.1) we shall obtain:

$$\begin{aligned}
&[\phi^c\widetilde{X}, \phi^c\widetilde{Y}] + [\widetilde{X}, \widetilde{Y}] - \phi^c[\phi^c\widetilde{X}, \widetilde{Y}] - \eta^v(\widetilde{X})\eta^v[\xi^c, \widetilde{Y}]\xi^c - \eta^v(\widetilde{X})\eta^c[\xi^c, \widetilde{Y}]\xi^v - \\
&-\eta^c(\widetilde{X})\eta^v[\xi^v, \widetilde{Y}]\xi^c - \eta^c(\widetilde{X})\eta^c[\xi^v, \widetilde{Y}]\xi^v - \phi^c[\widetilde{X}, \phi^c\widetilde{Y}] - \eta^v(\widetilde{Y})\eta^v[\widetilde{X}, \xi^c]\xi^c - \\
&-\eta^v(\widetilde{Y})\eta^c[\widetilde{X}, \xi^c]\xi^v - \eta^c(\widetilde{Y})\eta^v[\widetilde{X}, \xi^v]\xi^c - \eta^c(\widetilde{Y})\eta^c[\widetilde{X}, \xi^v]\xi^v = 0
\end{aligned} \tag{2.4}$$

for any left invariant vector fields $\widetilde{X}$, $\widetilde{Y}$ on TG.

$$\begin{aligned}
&\eta^v(\widetilde{Y})[\phi^c\widetilde{X}, \xi^c] + \eta^c(\widetilde{Y})[\phi^c\widetilde{X}, \xi^v] + \eta^v(\widetilde{X})[\xi^c, \phi^c\widetilde{Y}] + \eta^c(\widetilde{X})[\xi^v, \phi^c\widetilde{Y}] - \eta^v[\phi^c\widetilde{X}, \widetilde{Y}]\xi^c - \\
&-\eta^c[\phi^c\widetilde{X}, \widetilde{Y}]\xi^v - \eta^v(\widetilde{X})\phi^c[\xi^c, \widetilde{Y}] - \eta^c(\widetilde{X})\phi^c[\xi^v, \widetilde{Y}] - \eta^v[\widetilde{X}, \phi^c\widetilde{Y}] - \\
&-\eta^c[\widetilde{X}, \phi^c\widetilde{Y}]\xi^v - \eta^v(\widetilde{Y})\phi^c[\widetilde{X}, \xi^c] - \eta^c(\widetilde{Y})\phi^c[\widetilde{X}, \xi^v] = 0
\end{aligned} \tag{2.5}$$

for any left invariant vector fields $\widetilde{X}$, $\widetilde{Y}$ on TG. Let $X_1, \ldots, X_n$ be a basis of the left invariant vector fields on G. Without loss of generality of our considerations we may take $X_1 = \xi$ and $\eta(X_i) = 0$ for $i > 1$. If we take $\widetilde{X} = X_i^c$ and $\widetilde{Y} = X_j^c$ in (2.4) and 2.5) we get:

$$\begin{aligned}
&[\phi^c X_i^c, \phi^c X_j^c] + [X_i^c, X_j^c] - \phi^c[\phi^c X_i^c, X_j^c] - \eta^v(X_i^c)\eta^v[\xi^c, X_j^c]\xi^c - \\
&-\phi^c[X_i^c, \phi^c X_j^c] - \eta^v(X_j^c)\eta^v[X_i^c, \xi^c]\xi^c,
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
&\eta^v(X_j^c)[\phi^c X_i^c, \xi^c] + \eta^v(X_i^c)[\xi^c, \phi^c X_j^c] - \eta^v[\phi^c X_i^c, X_j^c]\xi^c - \\
&-\eta^v(X_i^c)\phi^c[\xi^c, X_j^c] - \eta^v[X_i^c, \phi^c X_j^c]\xi^c - \eta^v(X_j^c)\phi^c[X_i^c, \xi^c] = 0\,.
\end{aligned} \tag{2.7}$$

Similarly, putting $\widetilde{X} = \xi^v$, $\widetilde{Y} = X_i^c$ $(i > 1)$ and next $\widetilde{X} = \xi^c$, $\widetilde{Y} = X_i^v$ $(i > 1)$ and $\widetilde{X}_i = X_i^v$, $\widetilde{Y} = X_j^c$ $(i, j > 1)$ in (2.4) and (2.5) we obtain:

$$[\xi^v, X_i^c] - \eta^c[\xi^v, X_i^c]\xi^v - \phi^c[\xi^v, \phi^c X_i^c] = 0, \tag{2.8}$$

$$[\xi^v, \phi^c X_i^c] - \phi^c[\xi^v, X_i^c] - \eta^c[\xi^v, \phi^c X_i^c]\xi^v = 0 \tag{2.9}$$

$$[\xi^c, X_i^v] - \eta^c [\xi^c, X_i^v]\xi^v - \phi^c [\xi^c, \phi^c X_i^v] = 0\,, \tag{2.10}$$

$$[\xi^c, \phi^c X_i^v] - \phi^c [\xi^c, X_i^v] - \eta^c [\xi^c, \phi^c X_i^v]\xi^v = 0, \tag{2.11}$$

$$[\phi^c X_i^v, \phi^c X_j^c] - [X_i^v, X_j^c] - \phi^c [\phi^c X_i^v, X_j^c] - \phi^c [X_i^v, \phi^c X_j^c] = 0, \tag{2.12}$$

$$-\eta^c [\phi^c X_i^v, X_j^c]\xi^v - \eta^c [X_i^v, \phi^c X_j^c]\xi^v = 0. \tag{2.13}$$

Now we shall investigate the problem when the almost product structure $\widetilde{F}_{(\epsilon_1,\, \epsilon_2)}$ is integrable, where $\epsilon_1 \cdot \epsilon_2 = -1$. We have:

$$[\widetilde{F}_{(\epsilon_1,\, \epsilon_2)}, \widetilde{F}_{(\epsilon_1,\, \epsilon_2)}](X_i^c, X_j^c) = [\phi^c X_i^c, \phi^c X_j^c] + \epsilon_2 \eta^v (X_j^c)[\phi^c X_j^c, \xi^c] +$$

$$+ \epsilon_2 \eta^v (X_i^c)[\xi^c, \phi^c X_j^c] + [X_i^c, X_j^c] - \Big\{\phi^c [\phi^c X_i^c, X_j^c] + \epsilon_2 \eta^v (X_i^c)\phi^c [\xi^c, X_j^c] +$$

$$+ \epsilon_2 \eta^v [\phi^c X_i^c, X_j^c]\xi^c + \eta^v (X_i^c)\eta^v [\xi^c, X_j^c]\xi^c + \phi^c [X_i^c, \phi^c X_j^c] + \epsilon_2 \eta^v (X_j^c)\phi [X_i^c, \xi^c] +$$

$$+ \epsilon_2 \eta^v [X_i^c, \phi^c X_j^c]\xi^c + \eta^v (X_j^c)\eta^v [X_i^c, \xi^c]\xi^c\Big\}, \quad i, j = 1, .., n.$$

In virtue of (2.6) and (2.7) we obtain:

$$[\widetilde{F}_{(\epsilon_1,\, \epsilon_2)}, \widetilde{F}_{(\epsilon_1,\, \epsilon_2)}](X_i^c, X_j^c) = 0, \quad i, j = 1, .., n. \tag{2.14}$$

Taking into account the formulas (1.1) – (1.3) we have:

$$[\widetilde{F}_{(\epsilon_1,\, \epsilon_2)}, \widetilde{F}_{(\epsilon_1,\, \epsilon_2)}](X_i^v, X_j^v) = 0, \quad i, j = 1, .., n. \tag{2.15}$$

Moreover for $i, j > 1$, we have:

$$[\widetilde{F}_{(\epsilon_1,\, \epsilon_2)}, \widetilde{F}_{(\epsilon_1,\, \epsilon_2)}](X_i^v, X_j^c) = [\phi^c X_i^v, \phi^c X_j^c] + [X_i^v, X_j^c] - \phi^c [\phi^c X_i^v, X_j^c] -$$

$$- \epsilon_1 \eta^c [\phi^c X_i^v, X_j^c]\xi^v - \phi^c [X_i^v, \phi^c X_j^c] - \epsilon_1 \eta^c [X_i^v, \phi^c X_j^c]\xi^v.$$

Because of (2.12) and (2.13) we have:

$$[\widetilde{F}_{(\epsilon_1,\, \epsilon_2)}, \widetilde{F}_{(\epsilon_1,\, \epsilon_2)}](X_i^v, X_j^c) = 0 \tag{2.16}$$

for $i, j > 1$. Putting $\widetilde{X} = \xi^v$, $\widetilde{Y} = X_i^c$ in (2.1) we obtain:

$$[\widetilde{F}_{(\epsilon_1,\, \epsilon_2)}, \widetilde{F}_{(\epsilon_1,\, \epsilon_2)}](\xi^v, X_i^c) = \epsilon_1 [\xi^v, \phi^c X_i^c] + [\xi^v, X_i^c] - \epsilon_1 \phi^c [\xi^v, X_i^c] -$$

$$- \eta^c [\xi^v, X_i^c]\xi^v - \phi^c [\xi^v, \phi^c X_i^c] - \epsilon_1 \eta^c [\xi^v, \phi^c X_i^c]\xi^v.$$

On account of (2.8) and (2.9) we have:

$$[\widetilde{F}_{(\epsilon_1,\,\epsilon_2)}, \widetilde{F}_{(\epsilon_1,\,\epsilon_2)}](\xi^v, X_i^c) = 0. \tag{2.17}$$

Putting $\widetilde{X} = \xi^c$, $\widetilde{Y} = X_i^v$ in (2.1) we obtain:

$$[\widetilde{F}_{(\epsilon_1,\,\epsilon_2)}, \widetilde{F}_{(\epsilon_1,\,\epsilon_2)}](\xi^c, X_i^v) = \epsilon_2[\xi^c, \phi^c X_i^v] + [\xi^c, X_i^v] - \epsilon_2\phi^c[\xi^c, X_i^v] +$$
$$+ \eta^c[\xi^c, X_i^v]\xi^v - \phi^c[\xi^c, \phi^c X_i^v] - \epsilon_1\eta^c[\xi^c, \phi^c X_i^v]\xi^v = \epsilon_2[\xi^c, \phi^c X_i^v] + [\xi^c, X_i^v] -$$
$$- \xi_2\phi^c[\xi^c, X_i^v] + \eta^c[\xi^c, X_i^v]\xi^v - \phi^c[\xi^c, \phi^c X_i^v] + \epsilon_2\eta^c[\xi^c, \phi^c X_i^v]\xi^v = \epsilon_2\Big\{[\xi^c, \phi^c X_i^v] -$$
$$- \phi^c[\xi^c, X_i^v] + \eta^c[\xi^c, \phi^c X_i^v]\xi^v\Big\} + [\xi^c, X_i^v] + \eta^c[\xi^c, X_i^v]\xi^v - \phi^c[\xi^c, \phi^c X_i^v].$$

In virtue of 2.10) and (2.11) we have:

$$\begin{aligned}[\widetilde{F}_{(\epsilon_1,\,\epsilon_2)}, \widetilde{F}_{(\epsilon_1,\,\epsilon_2)}](\xi^c, X_i^v) &= 2 \quad \epsilon_2\eta^c[\xi^c, \phi^c X_i^v]\xi^v + \eta^c[\xi^c, X_i^v]\xi^v \quad = \\ &= 2\Big\{\epsilon_2\eta[\xi, \phi X_i]\xi + \eta[\xi, X_i]\xi\Big\}^v.\end{aligned} \tag{2.18}$$

Thus, we have:

Theorem 2.1. *Suppose that* $[\widetilde{F}_{(1,\,1)}, \widetilde{F}_{(1,\,1)}] = 0$ *and* $[\widetilde{F}_{(-1,-1)}, \widetilde{F}_{(-1,-1)}] = 0$. *Then* $[\widetilde{F}_{(\epsilon_1,\,\epsilon_2)}, \widetilde{F}_{(\epsilon_1,\,\epsilon_2)}] = 0$, *where* $\epsilon_1 \cdot \epsilon_2 = -1$, *if and only if* $\epsilon_2\eta[\xi, \phi X] + \eta[\xi, X] = 0$ *for any left invariant vector field X on G such that* $\eta(X) = 0$.

Now, let us consider the almost paracontact structure $(\phi^c + \epsilon_1\eta^c \otimes \xi^v, \xi^c, \eta^v)$. The weak-normality of this structure means the integrability of the almost product structures $\widetilde{F}_{(\epsilon_1,\,1)}$ and $\widetilde{F}_{(\epsilon_1,\,-1)}$. $\epsilon_1 = \pm 1$. Suppose that $[\widetilde{F}_{(1,\,1)}, \widetilde{F}_{(1,\,1)}] = 0$ and $[\widetilde{F}_{(-1,\,-1)}, \widetilde{F}_{(-1,\,-1)}] = 0$. If we take $\epsilon_1 = 1$ then from Theorem 2.1 we have: $[\widetilde{F}_{(1,\,-1)}, \widetilde{F}_{(1,\,-1)}] = 0$ if and only if $\eta[\xi, X] - \eta[\xi, \phi X] = 0$ for any left invariant vector field X on G such that $\eta(X) = 0$. If $\epsilon_1 = -1$ then because of the same reasons we have: $[\widetilde{F}_{(-1,\,1)}, \widetilde{F}_{(-1,\,1)}] = 0$ if and only if $\eta[\xi, X] + \eta[\xi, \phi X] = 0$ for any left invariant vector field X on G such that $\eta(X) = 0$. We have proved the following:

Theorem 2.2. *Suppose that* (ϕ, ξ, η) *is a weak-normal left invariant almost paracontact structure on G. Then* $(\phi^c + \xi\eta^c \otimes \xi^v, \xi^c, \eta^v)$ *is a weak-normal left invariant almost paracontact structure on TG if and only if*

$$\eta[\xi, X] - \epsilon\eta[\xi, \phi X] = 0 \tag{2.19}$$

for any left invariant vector field X on G such that $\eta(X) = 0$.

Remark 1. If (ϕ, ξ, η) is normal, then the conditions (1.11) – (1.13) are satisfied and therefore the condition (2.19) is also satisfied.

Remark 2. We can give similar theorem for the structure: $(\phi^c + \epsilon\eta^v \otimes \xi^c, \xi^v, \eta^c)$.

Now, suppose that $(\phi^c + \epsilon\eta^c \otimes \xi^v, \xi^c, \eta^v)$ is a weak-normal left invariant almost paracontact structure on TG, what means that:

$$[\widetilde{F}_{(\epsilon,1)}, \widetilde{F}_{(\epsilon,1)}] = 0 \text{ and } [\widetilde{F}_{(\epsilon,-1)}, \widetilde{F}_{(\epsilon,-1)}] = 0 . \tag{2.20}$$

On account of (2.1) we have:

$$\begin{aligned}
&[\widetilde{F}_{(\epsilon,1)}, \widetilde{F}_{(\epsilon,1)}](\widetilde{X}, \widetilde{Y}) = [\phi^c\widetilde{X}, \phi^c\widetilde{Y}] + \epsilon\eta^c(\widetilde{Y})[\phi^c\widetilde{X}, \xi^v] + \eta^v(\widetilde{Y})[\phi^c\widetilde{X}, \xi^c] + \\
&+ \epsilon\eta^c(\widetilde{X})[\xi^v, \phi^c\widetilde{Y}] + \eta^v(\widetilde{X})[\xi^c, \phi^c\widetilde{Y}] + [\widetilde{X}, \widetilde{Y}] - \Big\{\phi^c[\phi^c\widetilde{X}, \widetilde{Y}] + \epsilon\eta^c(\widetilde{X})\phi^c[\xi^v, \widetilde{Y}] + \\
&+ \eta^v(\widetilde{X})\phi^c[\xi^c, \widetilde{Y}] + \epsilon\eta^c[\phi^c\widetilde{X}, \widetilde{Y}]\xi^v + \eta^c(\widetilde{X})\eta^c[\xi^v, \widetilde{Y}]\xi^v + \epsilon\eta^v(\widetilde{X})\eta^c[\xi^c, \widetilde{Y}]\xi^v + \\
&+ \eta^v[\phi^c\widetilde{X}, \widetilde{Y}]\xi^c + \epsilon\eta^c(\widetilde{X})\eta^v[\xi^v, \widetilde{Y}]\xi^c + \eta^v(\widetilde{X})\eta^v[\xi^c, \widetilde{Y}]\xi^c + \phi^c[\widetilde{X}, \phi^c\widetilde{Y}] + \\
&+ \epsilon\eta^c(\widetilde{Y})\phi^c[\widetilde{X}, \xi^v] + \eta^v(\widetilde{Y})\phi^c[\widetilde{X}, \xi^c] + \epsilon\eta^c[\widetilde{X}, \phi^c\widetilde{Y}]\xi^v + \eta^c(\widetilde{Y})\eta^c[\widetilde{X}, \xi^v]\xi^v + \\
&+ \epsilon\eta^v(\widetilde{Y})\eta^c[\widetilde{X}, \xi^c]\xi^v + \eta^v[\widetilde{X}, \phi^c\widetilde{Y}]\xi^c + \epsilon\eta^c(\widetilde{Y})\eta^v[\widetilde{X}, \xi^v]\xi^c + \eta^v(\widetilde{Y})\eta^v[\widetilde{X}, \xi^c]\xi^c\Big\} = 0
\end{aligned} \tag{2.21}$$

for any left invariant vector fields $\widetilde{X}, \widetilde{Y}$ on TG.

$$\begin{aligned}
&[\widetilde{F}_{(\epsilon,-1)}, \widetilde{F}_{(\epsilon,-1)}](\widetilde{X}, \widetilde{Y}) = [\phi^c\widetilde{X}, \phi^c\widetilde{Y}] + \epsilon\eta^c(\widetilde{Y})[\phi^c\widetilde{X}, \xi^v] - \eta^v(\widetilde{Y})[\phi^c\widetilde{X}, \xi^c] + \\
&+ \epsilon\eta^c(\widetilde{X})[\xi^v, \phi^c\widetilde{Y}] - \eta^v(\widetilde{X})[\xi^c, \phi^c\widetilde{Y}] + [\widetilde{X}, \widetilde{Y}] - \Big\{\phi^c[\phi^c\widetilde{X}, \widetilde{Y}] + \epsilon\eta^c(\widetilde{X})\phi^c[\xi^v, \widetilde{Y}] - \\
&- \eta^v(\widetilde{X})\phi^c[\xi^c, \widetilde{Y}] + \epsilon\eta^c[\phi^c\widetilde{X}, \widetilde{Y}]\xi^v + \eta^c(\widetilde{X})\eta^c[\xi^v, \widetilde{Y}]\xi^v - \epsilon\eta^v(\widetilde{X})\eta^c[\xi^c, \widetilde{Y}]\xi^v - \\
&- \eta^v[\phi^c\widetilde{X}, \widetilde{Y}]\xi^c - \epsilon\eta^c(\widetilde{X})\eta^v[\xi^v, \widetilde{Y}]\xi^c + \eta^v(\widetilde{X})\eta^v[\xi^v, \widetilde{Y}]\xi^c + \phi^c[\widetilde{X}, \phi^c\widetilde{Y}] + \\
&+ \epsilon\eta^c(\widetilde{Y})\phi^c[\widetilde{X}, \xi^v] - \eta^v(\widetilde{Y})\phi^c[\widetilde{X}, \xi^c] + \epsilon\eta^c[\widetilde{X}, \phi^c\widetilde{Y}]\xi^v + \eta^c(\widetilde{Y})\eta^c[\widetilde{X}, \xi^v]\xi^v + \\
&+ \epsilon\eta^v(\widetilde{Y})\eta^c[\widetilde{X}, \xi^c]\xi^v - \eta^v[\widetilde{X}, \phi^c\widetilde{Y}]\xi^c - \epsilon\eta^c(\widetilde{Y})\eta^v[\widetilde{X}, \xi^v]\xi^c + \eta^v(\widetilde{Y})\eta^v[\widetilde{X}, \xi^c]\xi^c\Big\} = 0
\end{aligned} \tag{2.22}$$

for any left invariant vector fields $\widetilde{X}, \widetilde{Y}$ on TG. After adding (2.21) and (2.22) we have:

$$\begin{aligned}
&[\phi^c\widetilde{X}, \phi^c\widetilde{Y}] + \epsilon\eta^c(\widetilde{Y})[\phi^c\widetilde{X}, \xi^v] + \epsilon\eta^c(\widetilde{X})[\xi^v, \phi^c\widetilde{Y}] + [\widetilde{X}, \widetilde{Y}] - \Big\{\phi^c[\phi^c\widetilde{X}, \widetilde{Y}] + \\
&+ \epsilon\eta^c(\widetilde{X})\phi^c[\xi^v, \widetilde{Y}] + \epsilon\eta^c[\phi^c\widetilde{X}, \widetilde{Y}]\xi^v + \eta^c(\widetilde{X})\eta^c[\xi^v, \widetilde{Y}]\xi^v + \eta^v(\widetilde{X})\eta^v[\xi^c, \widetilde{Y}]\xi^c + \\
&+ \phi^c[\widetilde{X}, \phi^c\widetilde{Y}] + \epsilon\eta^c(\widetilde{Y})\phi^c[\widetilde{X}, \xi^v] + \epsilon\eta^c[\widetilde{X}, \phi^c\widetilde{Y}]\xi^v + \eta^c(\widetilde{Y})\eta^c[\widetilde{X}, \xi^v]\xi^v + \\
&+ \eta^v(\widetilde{Y})\eta^v[\widetilde{X}, \xi^c]\xi^c\Big\} = 0
\end{aligned} \tag{2.23}$$

for any left invariant vector fields $\widetilde{X}, \widetilde{Y}$ on TG. Substracting (2.21) from (2.22) gives:

$$\begin{aligned}
&\eta^v(\tilde{Y})[\phi^c\tilde{X}, \xi^c] + \eta^v(\tilde{X})[\xi^c, \phi^c\tilde{Y}] - \Big\{\eta^v(\tilde{X})\phi^c[\xi^c, \tilde{Y}] + \epsilon\eta^v(\tilde{X})\eta^c[\xi^c, \tilde{Y}]\xi^v + \\
&+ \eta^v[\phi^c\tilde{X}, \tilde{Y}]\xi^c + \epsilon\eta^c(\tilde{X})\eta^v[\xi^v, \tilde{Y}]\xi^c + \eta^v(\tilde{Y})\phi^c[\tilde{X}, \xi^c] + \epsilon\eta^v(\tilde{Y})\eta^c[\tilde{X}, \xi^c]\xi^v + \\
&+ \eta^v[\tilde{X}, \phi^c\tilde{Y}]\xi^c + \epsilon\eta^c(\tilde{Y})\eta^v[\tilde{X}, \xi^v]\xi^c\Big\} = 0
\end{aligned} \tag{2.24}$$

for all left invariant vector fields $\tilde{X}$, $\tilde{Y}$ on TG. From (2.1) we have:

$$\begin{aligned}
&[\tilde{F}_{(-\epsilon,-\epsilon)}, \tilde{F}_{(-\epsilon,\epsilon)}](\tilde{X}, \tilde{Y}) = [\phi^c\tilde{X}, \phi^c\tilde{Y}] - \epsilon\eta^c(\tilde{Y})[\phi^c\tilde{X}, \xi^v] - \epsilon\eta^v(\tilde{Y})[\phi^c\tilde{X}, \xi^c] - \\
&- \epsilon\eta^c(\tilde{X})[\xi^v, \phi^c\tilde{Y}] - \epsilon\eta^v(\tilde{X})[\xi^c, \phi^c\tilde{Y}] + [\tilde{X}, \tilde{Y}] - \Big\{\phi^c[\phi^c\tilde{X}, \tilde{Y}] - \epsilon\eta^c(\tilde{X})\phi^c[\xi^v, \tilde{Y}] - \\
&- \epsilon\eta^v(\tilde{X})\phi^c[\xi^c, \tilde{Y}] - \epsilon\eta^c[\phi^c\tilde{X}, \tilde{Y}]\xi^v + \eta^c(\tilde{X})\eta^c[\xi^v, \tilde{Y}]\xi^v + \eta^v(\tilde{X})\eta^c[\xi^c, \tilde{Y}]\xi^v - \\
&- \epsilon\eta^v[\phi^c\tilde{X}, \tilde{Y}]\xi^c + \eta^c(\tilde{X})\eta^v[\xi^v, \tilde{Y}]\xi^c + \eta^v(\tilde{X})\eta^v[\xi^c, \tilde{Y}]\xi^c + \phi^c[\tilde{X}, \phi^c\tilde{Y}] - \\
&- \epsilon\eta^c(\tilde{Y})\phi^c[\tilde{X}, \xi^v] - \epsilon\eta^v(\tilde{Y})\phi^c[\tilde{X}, \xi^c] - \epsilon\eta^c[\tilde{X}, \phi^c\tilde{Y}]\xi^v + \eta^c(\tilde{Y})\eta^c[\tilde{X}, \xi^v]\xi^v + \\
&+ \eta^v(\tilde{Y})\eta^c[\tilde{X}, \xi^c]\xi^v - \epsilon\eta^v[\tilde{X}, \phi^c\tilde{Y}]\xi^c + \eta^c(\tilde{Y})\eta^v[\tilde{X}, \xi^v]\xi^c + \eta^v(\tilde{Y})\eta^v[\tilde{X}, \xi^c]\xi^c\Big\}.
\end{aligned}$$

After having used (2.23) and (2.24), we have:

$$\begin{aligned}
&[\tilde{F}_{(-\epsilon,-\epsilon)}, \tilde{F}_{(-\epsilon,-\epsilon)}](\tilde{X}, \tilde{Y}) = 2\,(-\epsilon\eta^c(\tilde{Y})[\phi^c\tilde{X}, \xi^v] - \\
&- \epsilon\eta^c(\tilde{X})[\xi^v, \phi^c\tilde{Y}] - \Big\{-\epsilon\eta^c(\tilde{X})\phi^c[\xi^v, \tilde{Y}] - \epsilon\eta^c[\phi^c\tilde{X}, \tilde{Y}]\xi^v + \eta^v(\tilde{X})\eta^c[\xi^c, \tilde{Y}]\xi^v + \\
&+ \eta^c(\tilde{X})\eta^v[\xi^v, \tilde{Y}]\xi^c - \epsilon\eta^c(\tilde{Y})\phi^c[\tilde{X}, \xi^v] - \epsilon\eta^c[\tilde{X}, \phi^c\tilde{Y}]\xi^v + \eta^v(\tilde{Y})\eta^c[\tilde{X}, \xi^c]\xi^v + \\
&+ \eta^c(\tilde{Y})\eta^v[\tilde{X}, \xi^v]\xi^c\Big\}\,).
\end{aligned} \tag{2.25}$$

Let $X_1, \ldots, X_n$ be a basis of left invariant vector fields on G such that $X_1 = \xi$ and $\eta(X_i) = 0$ for $i > 1$. In virtue of Theorem 1.1 the vector fields ξ^c, ξ^v, $X_2^c, \ldots, X_n^c$, $X_2^v, \ldots, X_n^v$ constitute a basis of left invariant vector fields on TG. Now, for $\tilde{X} = X_i^c$, $\tilde{Y} = X_j^c$ and next $\tilde{X} = X_i^v$, $\tilde{Y} = X_j^v$ in (2.25) we have:

$$[\tilde{F}_{(-\epsilon,-\epsilon)}, \tilde{F}_{(-\epsilon,-\epsilon)}](X_i^c, X_j^c) = 0, \tag{2.26}$$

$$[\tilde{F}_{(-\epsilon,-\epsilon)}, \tilde{F}_{(-\epsilon,-\epsilon)}](X_i^v, X_j^v) = 0. \tag{2.27}$$

Putting $\tilde{X} = X_i^c$, $\tilde{Y} = X_j^v$, $i, j > 1$ in (2.25) we have:

$$\begin{aligned}
&[\tilde{F}_{(-\epsilon,-\epsilon)}, \tilde{F}_{(-\epsilon,-\epsilon)}](X_i^c, X_j^v) = \\
&= -2\epsilon(\eta^c[\phi^c X_i^c, X_j^v] + \eta^c[X_i^c, \phi^c X_j^v])\xi^v.
\end{aligned} \tag{2.28}$$

Taking $\widetilde{X} = \xi^v$, $\widetilde{Y} = X_i^c$, $i > 1$ in (2.25), we get:

$$\begin{aligned}&[\widetilde{F}_{(-\epsilon,-\epsilon)}, \widetilde{F}_{(-\epsilon,-\epsilon)}](\xi^v, X_i^c) = \\ &= 2\,(-\epsilon\,[\xi^v, \phi^c X_i^c] - \left\{-\epsilon\phi^c[\xi^v, X_i^c] - \epsilon\eta^c[\xi^v, \phi^c X_i^c]\xi^v\right\}) = \\ &= 2\epsilon(\phi^c[\xi^v, X_i^c] + \eta^c[\xi^v, \phi^c X_i^c]\xi^v - [\xi^v, \phi^c X_i^c]).\end{aligned} \tag{2.29}$$

For $\widetilde{X} = \xi^c$, $\widetilde{Y} = X_i^v$, $i > 1$ in (2.25) we have:

$$\begin{aligned}&[\widetilde{F}_{(-\epsilon,-\epsilon)}, \widetilde{F}_{(-\epsilon,-\epsilon)}](\xi^c, X_i^v) = \\ &= -2\,\left\{\eta^c[\xi^c, X_i^v]\xi^v - \epsilon\eta^c[\xi^c, \phi^c X_i^v]\xi^v\right\}.\end{aligned} \tag{2.30}$$

Taking $\widetilde{X} = X_i^c$, $\widetilde{Y} = X_j^c$, $i, j > 1$ in (2.23) we have:

$$[\phi^c X_i^c, \phi^c X_j^c] + [X_i^c, X_j^c] - \phi^c[\phi^c X_i^c, X_j^c] - \phi^c[X_i^c, \phi^c X_j^c] = 0$$

and because of (1.1), (1.2), (1.3) we have:

$$[\phi X_i, \phi X_j] + [X_i, X_j] - \phi[\phi X_i, X_j] - \phi[X_i, \phi X_j] = 0\,. \tag{2.31}$$

Putting $X = X_i^c$, $Y = X_j^v$ in (2.23) we obtain:

$$\begin{aligned}&[\phi^c X_i^c, \phi^c X_j^v] + [X_i^c, X_j^v] - \phi^c[\phi^c X_i^c, X_j^v] - \phi^c[X_i^c, \phi^c X_j^v] - \epsilon\left\{\eta^c[\phi^c X_i^c, X_j^v] + \right. \\ &\left. + \eta^c[X_i^c, \phi^c X_j^v]\right\}\xi^v = 0\end{aligned}$$

and again because of (1.1), (1.2), (1.3) and (2.31) we have:

$$\begin{aligned}&[\phi^c X_i^c, \phi^c X_j^v] + [X_i^c, X_j^v] - \phi^c[\phi^c X_i^c, X_j^v] - \phi^c[X_i^c, \phi^c X_j^v] = \\ &= \left\{[\phi X_i, \phi X_j] + [X_i, X_j] - \phi[\phi X_i, X_j] - \phi[X_i, \phi X_j]\right\}^v = 0\end{aligned}$$

hence:

$$\eta^c[\phi^c X_i^c, X_j^v] + \eta^c[X_i^c, \phi^c X_j^v] = 0\,. \tag{2.32}$$

If we take $X = \xi^v$, $Y = X_i^c$, $i > 1$ in (2.23) we obtain:

$$\begin{aligned}&\epsilon[\xi^v, \phi^c X_i^c] + [\xi^v, X_i^c] - \epsilon\phi^c[\xi^v, X_i^c] - \\ &- \eta^c[\xi^v, X_i^c]\xi^v - \phi^c[\xi^v, \phi^c X_i^c] - \epsilon\eta^c[\xi^v, \phi^c X_i^c] = 0.\end{aligned} \tag{2.33}$$

Inserting $\phi^c X_i^c$ instead of X_i^c in (2.33) we have:

$$\begin{aligned}&\epsilon[\xi^v, X_i^c] + [\xi^v, \phi^c X_i^c] - \epsilon\phi^c[\xi^v, \phi^c X_i^c] - \\ &-\eta^c[\xi^v, \phi^c X_i^c]\xi^v - \phi^c[\xi^v, X_i^c] - \epsilon\eta^c[\xi^v, X_i^c] = 0\end{aligned} \tag{2.34}$$

From (2.33) and (2.34) we obtain:

$$\phi^c[\xi^v, X_i^c] + \eta^c[\xi^v, \phi^c X_i^c] - [\xi^v, \phi^c X_i^c] = 0\,. \tag{2.35}$$

Putting $X = \xi^c$, $Y = X_i^v$ in (2.23) we obtain:

$$[\xi^c, X_i^v] - \phi^c[\xi^c, \phi^c X_i^v] - \epsilon\eta^c[\xi^c, \phi^c X_i^v]\xi^v = 0.$$

Hence:

$$[\xi, X_i] - \phi[\xi, \phi X_i] - \epsilon\eta[\xi, \phi X_i]\xi = 0. \tag{2.36}$$

From (2.35) if follows:

$$\phi[\xi, X_i] + \eta[\xi, \phi X_i]\xi - [\xi, \phi X_i] = 0$$

or

$$\eta[\xi, X_i]\xi = [\xi, X_i] - \phi[\xi, \phi X_i]\,. \tag{2.37}$$

From (2.36) and (2.37) it follows that:

$$\eta[\xi, X_i]\xi - \epsilon\eta[\xi, \phi X_i]\xi = 0. \tag{2.38}$$

Combining (2.26) – (2.30) with (2.32), (2.35), (2.38) we obtain:

$$[\tilde{F}_{(-\epsilon,-\epsilon)}, \tilde{F}_{(-\epsilon,-\epsilon)}](\tilde{X}, \tilde{Y}) = 0 \tag{2.39}$$

for any left invariant vector fields X, Y on TG.

Hence we have:

Theorem 2.3. *Suppose that* $[\tilde{F}_{(\epsilon,1)}, \tilde{F}_{(\epsilon,1)}] = 0$ *and* $[\tilde{F}_{(\epsilon,-1)}, \tilde{F}_{(\epsilon,-1)}] = 0$. *Then* $[\tilde{F}_{(-\epsilon,-\epsilon)}, \tilde{F}_{(-\epsilon,-\epsilon)}] = 0$.

Hence we may state:

Theorem 2.4. *Suppose that* $(\phi^c + \epsilon\eta^v \otimes \xi^c, \xi^v, \eta^c)$ *is a weak-normal left invariant almost paracontact structure on TG. Then* (ϕ, ξ, η) *is a weak-normal almost paracontact left invariant structure on G.*

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STRESZCZENIE

W tej pracy, która jest kontynuacją pracy [2], rozpatrujemy podniesienia lewo-niezmienniczych prawie para-kontaktowych normalnych lub słabo-normalnych struktur na grupie Lie'go G do lewo-niezmienniczych prawie para-kontaktowych normalnych lub słabo-normalnych struktur na TG mającej naturalną strukturę grupy Lie'go.

РЕЗЮМЕ

В этой работе занимаемся поднятием лево-инвариантных почти параконтактных нормальных и слабо-нормальных структур на группе Ли.