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Instytut Matematyki
Uniwersytet Marii Curie-Skłodowskiej

Andrzej BUCKI, Ryszard HOŁUBOWICZ, Andrzej MIERNOWSKI

On Integrability of Almost Paracontact Structures

O całkowalności prawie para-kontaktowych struktur

Об интегрируемости почти параконтактных структур

0. Introduction. The normality of an almost contact structure Ω on a manifold M denotes, as one knows, the integrability of some almost complex structure on $M \times R$. In [3] A. Morimoto has defined the integrability of the structure Ω by requiring some almost complex structure on $M \times M$ to be integrable, and later he has proved that the normality is equivalent to the integrability.

In this paper we investigate conditions of integrability of almost paracontact structures being defined by J. Sato [4]. In the first section of the paper we introduce, analogously to A. Morimoto, the notion of the integrability of an almost paracontact structure Σ on M and in the Theorem 1.5 we show that this is equivalent to the normality.

In the section 2 we define the idea of the weak-normality of an almost paracontact structure Σ on M requiring some almost product structures $F_1(\Sigma)$, $F_2(\Sigma)$ on M to be integrable, which does not possess an equivalent in almost contact structures on account of the evenness of a dimension of a manifold. In the Theorem 2.1 we show that the weak-normality is a consequence of the normality and in the Theorem 2.4 we answer the question for what structures both types of normality are equivalent.

In the last section we present two examples that illustrate the weak-normality of almost paracontact structures.

In this paper manifolds, vector fields, real valued functions and differential forms on a manifold are diffrentiable of the class C^∞.

1. Normal Almost Paracontact Structures.

Definition 1.1 [4]. Let M be an n-dimensional manifold. If there exists a $(1, 1)$-tensor field ϕ, a vector field ξ and a 1-form η which satisfy the conditions:

$$\phi^2 = Id - \eta \otimes \xi\,, \tag{1.1}$$

$$\eta(\xi) = 1\,, \tag{1.2}$$

then we say that M has an almost paracontact structure and M is an almost paracontact manifold.

You can prove that the conditions (1.1) and (1.2) imply the following:

$$\phi(\xi) = 0, \tag{1.3}$$

$$\eta \circ \phi = 0, \tag{1.4}$$

$$\operatorname{rank} \phi = n-1. \tag{1.5}$$

Definition 1.2 [4]. An almost paracontact structure (ϕ, ξ, η) on M is said to be normal if the following condition is satisfied:

$$[\phi, \phi] - 2d\eta \otimes \xi = 0 \tag{1.6}$$

where $[\phi, \phi]$ is the Nijenhuis tensor for ϕ.

It is well known that normal almost paracontact structures may be defined in the following way:

Let us consider the manifold $M \times R$. Having used the almost paracontact structure (ϕ, ξ, η) on M we define an almost product structure F on $M \times R$ as follows [4]:

$$F\left(X, f\frac{d}{dt}\right) = \left(\phi(X) + f\xi, \eta(X)\frac{d}{dt}\right). \tag{1.7}$$

We know the following:

Theorem 1.1 [4]. *An almost paracontact structure (ϕ, ξ, η) on M is normal if and only if the induced almost product structure F on $M \times R$ is integrable.*

In [3] A. Morimoto has dealt with some almost complex structure on a product manifold with almost contact structures and has investigated its integrability. Similar considerations may be conducted in the case of manifolds with almost paracontact structures.

Let M and $\overline{M}$ be manifolds and $\Sigma = (\phi, \xi, \eta)$ and $\overline{\Sigma} = (\overline{\phi}, \overline{\xi}, \overline{\eta})$ be almost paracontact structures on M and $\overline{M}$ respectively. A vector field X on M will be identified with a vector field $\widetilde{X}$ on $M \times \overline{M}$ as follows: $\widetilde{X}_{(p, \overline{p})} = X_p + 0_{\overline{p}}$ for $(p, \overline{p}) \in M \times \overline{M}$, where $0_{\overline{p}}$ denotes the zero vector of $\overline{M}$ at $\overline{p}$. Similarly, we identify $\overline{X}$ on $\overline{M}$ with $\widetilde{X}$ on $M \times \overline{M}$ such as: $\widetilde{X}_{(p, p)} = 0_p + X_p$. Let $X + X \in T(M \times M)$ and put:

$$F(X + X) = \phi(X) + \eta(X)\xi + \phi(X) + \eta(X)\xi. \tag{1.8}$$

It is easy to see that $F^2 = Id_{M \times M}$, so F is an almost product structure on $M \times M$.

Remark. Observe that while $M = R$, $\phi = 0$, $\xi = d/dt$, $\eta = dt$, the definition (1.8) becomes (1.7).

We shall show that an almost paracontact structure is integrable if and only if it is normal one. To this end we shall use the following definitions and lemmas.

Definition 1.3 Let $\Sigma = (\phi, \xi, \eta)$ be an almost paracontact structure on M. We define the following tensor field ψ of the type (1, 2) and a differential 2-form θ on M:

$$\psi(X, Y) = \phi\,[X, Y] - [\phi X, Y] - [X, \phi Y] + \\ + \phi\,[\phi X, \phi Y] + \{\phi(X)\,(\eta(Y)) - \phi(Y)\,(\eta(X))\}\cdot \xi\,, \tag{1.9}$$

$$\theta(X, Y) = \eta\,[X, Y] - X(\eta(Y)) + Y(\eta(X)) + \eta\,[\phi X, \phi Y]\,. \tag{1.10}$$

Now we prove the following:

Lemma 1.2 *Let* $\Sigma = (\phi, \xi, \eta)$ *and* $\Sigma = (\phi, \xi, \eta)$ *be almost paracontact structures on* M *and* M *respectively. Then the induced by* (1.8) *almost product structure* F *on* $M \times M$ *is integrable if and only if the following conditions are satisfied:*

$$\psi = 0\,, \tag{1.11}$$

$$\psi = 0\,, \tag{1.12}$$

where ψ *is the tensor field of the type* (1, 2) *on* M *corresponding to* Σ *defined by* (1.9).

Proof. The integrability condition of the induced almost product structure F on $M \times M$ is as follows:

$$F\,[X + X, Y + Y] + F\,[F(X + X), F(Y + Y)] = \\ = [F(X + X), Y + Y] + [X + X, F(Y + Y)]\,. \tag{1.13}$$

The first term of the left hand side of (1.13) (using (1.8)) is equaled to:

$$F\,[X + X, Y + Y] = F(\,[X, Y] + [X, Y]) = \\ = \phi\,[X, Y] + \eta([X, Y])\,\xi + \phi\,[X, Y] + \eta\,[X, Y]\,\xi\,. \tag{1.14}$$

The second term of the left hand side of (1.13) is of the form:

$$F\,[F(X + X), F(Y + Y)] = F\,[\phi(X) + \eta(X)\,\xi + \phi(X) + \eta(X)\,\xi, \phi(Y) + \eta(Y)\,\xi + \phi(Y) + \\ + \eta(Y)\,\xi] = F([\phi(X) + \eta(X)\,\xi, \phi(Y) + \eta(Y)\,\xi] + [\phi(X) + \eta(X)\,\xi, \phi(Y) + \eta(Y)\,\xi]) + \\ + F([\phi(X) + \eta(X)\,\xi, \phi(Y) + \eta(Y)\,\xi] + [\phi(X) + \eta(X)\,\xi, \phi(Y) + \eta(Y)\,\xi]) = \\ = \phi\,[\phi(X) + \eta(X)\,\xi, \phi(Y) + \eta(Y)\,\xi] + \eta\,[\phi(X) + \eta(X)\,\xi, \phi(Y) + \eta(Y)\,\xi]\,\xi + \\ + \phi\,[\phi(X) + \eta(X)\,\xi, \phi(Y) + \eta(Y)\,\xi] + \eta\,[\phi(X) + \eta(X)\,\xi, \phi(Y) + \eta(Y)\,\xi]\,\xi + \\ + F(\,[\phi(X), \phi(Y)] + [\eta(X)\,\xi, \phi(Y)] + [\eta(X)\,\xi, \eta(Y)\,\xi] + [\phi(X), \eta(Y)\,\xi] + \\ + [\phi(X), \phi(Y)] + [\phi(X), \eta(Y)\,\xi] + [\eta(X)\,\xi, \phi(Y)] + [\eta(X)\,\xi, \eta(Y)\,\xi]) = \\ = \phi\,[\phi(X) + \eta(X)\,\xi, \phi(Y) + \eta(Y)\,\xi] + \eta\,[\phi(X) + \eta(X)\,\xi, \phi(Y) + \eta(Y)\,\xi]\,\xi + \phi\,[\phi(X) + \\ + \eta(X)\,\xi, \phi(Y) + \eta(Y)\,\xi] + \eta\,[\phi(X) + \eta(X)\,\xi, \phi(Y) + \eta(Y)\,\xi]\,\xi + F(-\,\phi(Y)\,(\eta(X)\,\xi + \\ + \eta(X)\,\xi\,(\eta(Y))\,\xi - \eta(Y)\,\xi\,(\eta(X))\,\xi + \phi(X)\,(\eta(Y))\,\xi + \phi(X)\,(\eta(Y))\,\xi - \phi(Y)\,(\eta(X))\,\xi +$$

$$+\eta(X)\,\xi\,(\eta(Y))\,\xi - \eta(Y)\,\xi\,(\eta(X))\,\xi) = \phi\,[\phi(X) + \eta(X)\,\xi, \phi(Y) + \eta(Y)\,\xi] + \eta\,[\phi(X) + \eta(X)\,\xi, \phi(Y) + \eta(Y)\,\xi]\,\xi + \phi\,[\phi(X) + \eta(X)\,\xi, \phi(Y) + \eta(Y)\,\xi] + \eta\,[\phi(X) + \eta(X)\,\xi, \phi(Y) + \eta(Y)\,\xi]\,\xi + F(\{-\phi(Y)\,(\eta(X)) - \eta(Y)\,\xi\,(\eta(X)) + \phi(X)\,(\eta(Y)) + \eta(X)\,\xi\,(\eta(Y))\}\,\xi + \{\eta(X)\,\xi\,(\eta(Y)) + \phi(X)\,(\eta(Y)) - \phi(Y)\,(\eta(X)) - \eta(Y)\,\xi\,(\eta(X))\}\,\xi).$$

If we put:

$$f(X, X, Y, Y) = \eta(X)\,\xi\,(\eta(Y)) + \phi(X)\,(\eta(Y)) - \phi(Y)\,(\eta(X)) - \eta(Y)\,\xi\,(\eta(X))$$

$$f(X, X, Y, Y) = -\phi(Y)\,(\eta(X)) - \eta(Y)\,\xi\,(\eta(X)) + \phi(X)\,(\eta(Y)) + \eta(X)\,\xi\,(\eta(Y))$$

and after having used (1.2), (1.3) for Σ and Σ the second term of the left hand side of (1.13) is expressed as follows:

$$F\,[F(X+X), F(Y+Y)] = \phi\,[\phi(X) + \eta(X)\,\xi, \phi(Y) + \eta(Y)\,\xi] + \eta\,[\phi(X) + \eta(X)\,\xi, \phi(Y) + \eta(Y)\,\xi]\,\xi + \phi\,[\phi(X) + \eta(X)\,\xi, \phi(Y) + \eta(Y)\,\xi] + \eta\,[\phi(X) + \eta(X)\,\xi, \phi(Y) + \eta(Y)\,\xi]\,\xi + f(X, X, Y, Y)\,\xi + f(X, X, Y, Y)\,\xi\,. \tag{1.15}$$

The first term of the right hand side of (1.13) is:

$$[F(X+X), Y+Y] = [\phi(X) + \eta(X)\,\xi + \phi(X) + \eta(X)\,\xi\,, Y+Y] = [\phi(X) + \eta(X)\,\xi, Y] + [\phi(X) + \eta(X)\,\xi, Y] + [\phi(X) + \eta(X)\,\xi, Y] + [\phi(X) + \eta(X)\,\xi, Y] = [\phi(X) + \eta(X)\,\xi, Y] + [\phi(X) + \eta(X)\,\xi, Y] - Y(\eta(X))\,\xi - Y(\eta(X))\,\xi\,. \tag{1.16}$$

The second term of the right hand side of (1.13) equals to:

$$[X+X, F(Y+Y)] = [X+X, \phi(Y) + \eta(Y)\,\xi + \phi(Y) + \eta(Y)\,\xi] = [X, \phi(Y) + \eta(Y)\,\xi] + [X, \phi(Y) + \eta(Y)\,\xi] + [X, \phi(Y) + \eta(Y)\,\xi] + [X, \phi(Y) + \eta(Y)\,\xi] = [X, \phi(Y) + \eta(Y)\,\xi] + [X, \phi(Y) + \eta(Y)\,\xi] + X(\eta(Y))\,\xi + X(\eta(Y))\,\xi\,. \tag{1.17}$$

Using (1.14), (1.15), (1.16) and (1.17) the condition (1.13) is equivalent to the following two relations:

$$\phi\,[X, Y] + \eta\,[X, Y]\,\xi = \tag{1.18}$$

$$= -\phi\,[\phi(X) + \eta(X)\,\xi, \phi(Y) + \eta(Y)\,\xi] - \eta\,[\phi(X) + \eta(X)\,\xi, \phi(Y) + \eta(Y)\,\xi]\,\xi + [\phi(X) + \eta(X)\,\xi, Y] + [X, \phi(Y) + \eta(Y)\,\xi] + X(\eta(Y))\,\xi - Y(\eta(X))\,\xi - f(X, X, Y, Y)\,\xi\,,$$

$$\phi\,[X, Y] + \eta\,[X, Y]\,\xi = \tag{1.19}$$

$$= -\phi\,[\phi(X) + \eta(X)\,\xi, \phi(Y) + \eta(Y)\,\xi] - \eta\,[\phi(X) + \eta(X)\,\xi, \phi(Y) + \eta(Y)\,\xi]\,\xi +$$

$$+ [\phi(X) + \eta(X)\,\xi, Y] + [X, \phi(Y) + \eta(Y)\,\xi] + X(\eta(Y))\,\xi - Y(\eta(X))\,\xi - f(X, X, Y, Y)\cdot\xi .$$

Now, putting $X = Y = 0$ in (1.18) and (1.19) we obtain:

$$\psi(X, Y) = 0\,, \tag{1.20}$$

$$\theta(X, Y) = 0. \tag{1.21}$$

Putting $X = Y = 0$ in (1.18) and (1.19) we get:

$$\theta(X, Y) = 0\,, \tag{1.22}$$

$$\psi(X, Y) = 0. \tag{1.23}$$

Putting $X = Y = 0$ in (1.18) and (1.19) we have:

$$[X, \eta(Y)\,\xi] - \phi\,[\phi X, \eta(Y)\,\xi] - \eta([\eta(X)\,\xi, \phi(Y)])\,\xi + \eta(Y)\,\xi\,(\eta(X))\,\xi = 0\,, \tag{1.24}$$

$$[\eta(X)\,\xi, Y] - \phi\,[\eta(X)\,\xi, \phi(Y)] - \eta([\phi(X), \eta(Y)\,\xi])\,\xi - \eta(X)\,\xi\,(\eta(Y))\,\xi = 0\,. \tag{1.25}$$

Putting $X = Y = 0$ in (1.18) and (1.19) we have:

$$[\eta(X)\,\xi, Y] - \phi\,[\eta(X)\,\xi, \phi(Y)] - \eta([\phi(X), \eta(Y)\,\xi])\,\xi - \eta(X)\,\xi\,(\eta(Y))\,\xi = 0\,, \tag{1.26}$$

$$[X, \eta(Y)\,\xi\,] - \phi\,[\phi X, \eta(Y)\,\xi] - \eta([\eta(X)\,\xi, \phi(Y)])\,\xi + \eta(Y)\,\xi\,(\eta(X))\,\xi = 0\,. \tag{1.27}$$

These eight conditions are equivalent to (1.18) and (1.19).

Now we shall show that the conditions (1.21), (1.22) and (1.24) through (1.27) are implied by (1.20) and (1.23). First, it is easy to see that (1.24) and (1.26) are equivalent and also (1.25) and (1.27) are equivalent. Now we show that (1.24) is implied by (1.20) and (1.23). To this end, first we observe that:

$$\phi([X, \xi]) = [\phi(X), \xi]\,. \tag{1.28}$$

In fact, putting $Y = \xi$ in (1.20) and by virtue of (1.2) and (1.3) we get (1.28). Now the left hand side of (1.24) is equals to:

$$\eta(Y)\,[X, \xi] - \eta(Y)\,\phi\,[\phi X, \xi] - \eta(X)\,\eta\,[\xi, \phi(Y)]\,\xi + \eta(Y)\,\xi\,(\eta(X))\,\xi =$$

$$= \eta(Y)\quad [X, \xi] - [\phi^2 X, \xi] + \xi(\eta(X))\,\xi\quad - \eta(X)\,\eta\,(\phi([\xi, Y])) =$$

$$= \eta(Y)\Big\{[X, \xi] - [X, \xi] + \eta(X)\,[\xi, \xi] - \xi(\eta(X))\,\xi + \xi(\eta(X))\,\xi\Big\} = 0\,.$$

In the same way we show that (1.27) is implied by (1.20) and (1.23). Now it is sufficient to prove that (1.20) implies (1.21), since (1.22) is implied by (1.23) in the same way. Operating ϕ on both sides of (1.20) we have:

$$\phi^2[X, Y] = \phi[\phi X, Y] + \phi[X, \phi Y] - \phi^2[\phi X, \phi Y]. \tag{1.29}$$

Using (1.20) we can calculate the first term of the right hand side of (1.29) substituting $\phi(X)$ instead of X.

$$\phi[\phi X, Y] = [\phi^2 X, Y] + [\phi X, \phi Y] - \phi[\phi^2 X, \phi Y] - \Big\{(\phi^2 X)(\eta(Y)) - \phi(Y)(\eta(\phi(X))\Big\}\xi =$$
$$= [X - \eta(X)\xi, Y] + [\phi X, \phi Y] - \phi[X - \eta(X)\xi, \phi Y] - (\phi^2 X)(\eta(Y))\xi.$$

Inserting this into (1.29) we obtain:

$$[X, Y] - \eta[X, Y]\xi = [X - \eta(X)\xi, Y] + [\phi X, \phi Y] - \phi[X - \eta(X)\xi, \phi Y] -$$
$$- (\phi^2(X))(\eta(Y))\xi + \phi[X, \phi Y] - [\phi X, \phi Y] + \eta[\phi X, \phi Y]\xi.$$

Hence we have:

$$\begin{aligned}
-\eta[X, Y]\xi &= -[\eta(X)\xi, Y] - \phi[-\eta(X)\xi, \phi Y] - (X - \eta(X)\xi)(\eta(Y))\xi + \eta[\phi X, \phi Y]\xi = \\
&= -\eta(X)[\xi, Y] + Y(\eta(X))\xi - \phi\Big\{-\eta(X)[\xi, \phi Y] + \phi(Y)(\eta(X))\xi\Big\} - \\
&\quad - X(\eta(Y))\xi + \eta(X)\xi(\eta(Y))\xi + \eta[\phi X, \phi Y]\xi = \\
&= -\eta(X)[\xi, Y] + Y(\eta(X))\xi + \eta(X)\phi[\xi, \phi Y] - \phi(Y)(\eta(X))\phi(\xi) - \\
&\quad - X(\eta(Y))\xi + \eta(X)\xi(\eta(Y))\xi + \eta[\phi X, \phi Y]\xi = \\
&= -\eta(X)[\xi, Y] + Y(\eta(X))\xi + \eta(X)[\xi, \phi^2 Y] - \\
&\quad - X(\eta(Y))\xi + \eta(X)\xi(\eta(Y))\xi + \eta([\phi X, \phi Y])\xi = \\
&= -\eta(X)[\xi, Y] + Y(\eta(X))\xi + \eta(X)[\xi, Y] - \eta(X)\eta(Y)[\xi, \xi] - \\
&\quad - \eta(X)\xi(\eta(Y))\xi - X(\eta(Y))\xi + \eta(X)\xi(\eta(Y))\xi + \eta([\phi X, \phi Y])\xi = \\
&= -X(\eta(Y))\xi + Y(\eta(X))\xi + \eta([\phi X, \phi Y])\xi = \\
&= -\Big\{X(\eta(Y)) - Y(\eta(Y)) + \eta[\phi X, \phi Y]\Big\}\xi
\end{aligned}$$

or $\theta(X, Y) = 0$. Thus we have proved that (1.20) implies (1.21) what completes the proof of the Lemma 1.2.

In the case $M = M$ and $\Sigma = \Sigma$ we have:

Definition 1.4. An almost paracontact structure Σ on a manifold M is said to be integrable if and only if the product structure F given by (1.8) on $M \times M$ is integrable.

We have the following:

Theorem 1.3. *Let $\Sigma = (\phi, \xi, \eta)$ be an almost paracontact structure on M. Then Σ is integrable if and only if the following condition is satisfied: $\psi = 0$.*

Combining Lemma 1.2. and the Theorem 1.3. we get:

Theorem 1.4. *Let Σ and $\bar{\Sigma}$ be almost paracontact structures on M and $\bar{M}$ respectively. Then the induced by Σ and $\bar{\Sigma}$ almost product structure on $M \times \bar{M}$ is integrable if and only if Σ and $\bar{\Sigma}$ are both integrable.*

If in the Theorem 1.4. we take $M = R$ and $\Sigma = (0, d/dt, dt)$ then we shall get:

Theorem 1.5. *An almost paracontact structure Σ on M is integrable if and only if Σ is normal.*

In particular, we have:

Corollary 1.6. *An almost paracontact structure Σ on M is normal if and only if condition (1.20) is satisfied.*

2. Weak-normal Almost Paracontact Structures. Assume that $\Sigma = (\phi, \xi, \eta)$ is an almost paracontact structure on a manifold M. We define (1, 1)-tensors on M:

$$F_1 = \phi - \eta \otimes \xi \, , \tag{2.1}$$

$$F_2 = \phi + \eta \otimes \xi \, . \tag{2.2}$$

It is easy to verify that $F_1^2 = F_2^2 = Id_M$.

Definition 2.1. An almost paracontact structure Σ on M is said to be weak-normal if the almost product structures F_1 and F_2 defined by (2.1) and (2.2) are both integrable.

Now we prove:

Theorem 2.1. *If an almost paracontact structure Σ on M is normal then Σ is weak-normal.*

Proof. Let $[F_1 \, F_1]$ denote the Nijenhuis tensor field for the structure F_1. It is clear that $[F_1 \, F_1](X, Y) = 0$ if and only if

$$F_1 \, [F_1 \, F_1](X, Y) = 0 \, .$$

Now we calculate:

$$F_1 \, [F_1 \, F_1](X, Y) = F_1 \, [X, Y] + F_1 \, [F_1 X, F_1 Y] - [F_1 X, Y] - [X, F_1 Y] = \phi \, [\phi X, \phi Y] -$$
$$- \eta \, [\phi X, \phi Y] \, \xi - \eta(Y) \phi \, [\phi X, \xi] + \eta(Y) \, \eta \, [\phi X, \xi] \, \xi + \phi(X) \, (\eta(Y)) \, \xi - \eta(X) \, \phi \, [\xi, \phi Y] +$$
$$+ \eta(X) \, \eta \, [\xi, \phi Y] \, \xi - \phi(Y) \, (\eta(X)) \, \xi - \eta(X) \, \xi \, (\eta(Y)) \, \xi + \eta(Y) \, \xi \, (\eta(X)) \, \xi + \phi \, [X, Y] -$$
$$- \eta([X, Y]) \, \xi - [\phi X, Y] + \eta(X) \, [\xi, Y] - Y(\eta(X)) \, \xi - [X, \phi Y] + \eta(Y) \, [X, \xi] +$$
$$+ X(\eta(Y)) \, \xi = \psi(X, Y) - \theta(X, Y) \, \xi - \eta(Y) \quad \psi(\phi X, \xi) - \theta(\phi X, \xi) \, \xi \quad - \eta(X) \quad \psi(\xi, \phi Y) -$$
$$- \theta(\xi, \phi Y) \, \xi \quad .$$

Thus the integrability condition $F_1\,[F_1\,F_1](X, Y) = 0$ of the structure F_1 is equivalent to:

$$\psi(X,Y) - \theta(X,Y)\,\xi - \eta(Y) \quad \psi(\phi X,\xi) - \theta(\phi X,\xi)\,\xi \quad - \tag{2.3}$$
$$-\eta(X) \quad \psi(\xi,\phi Y) - \theta(\xi,\phi Y)\,\xi \quad = 0\,.$$

In the similar way we have:

$$F_2\,[F_2\,F_2]\,(X,Y) = \psi(X,Y) + \theta(X,Y)\,\xi + \eta(Y) \quad \psi(\phi X,\xi) - \theta(\phi X,\xi)\,\xi \quad +$$
$$+\,\eta(X) \quad \psi(\xi,\phi Y) - \theta(\xi,\phi Y)\,\xi$$

or the integrability condition of the structure F_2 is equivalent to the following one:

$$\psi(X,Y) + \theta(X,Y)\,\xi + \eta(Y) \quad \psi(\phi X,\xi) - \theta(\phi X,\xi)\,\xi \quad + \tag{2.4}$$
$$+\,\eta(X) \quad \psi(\xi,\phi Y) - \theta(\xi,\phi Y)\,\xi \quad = 0\,.$$

By virtue of the Corollary 1.6 we come to the end of the proof.

Observe that adding (2.3) and (2.4) we get:

$$\psi(X,Y) + \eta(Y)\,\theta(\phi X,\xi)\,\xi + \eta(X)\,\theta(\xi,\phi Y)\,\xi = 0\,. \tag{2.5}$$

Substracting (2.4) from (2.3) gives us:

$$\theta(X,Y)\,\xi + \eta(Y)\,\psi(\phi X,\xi) + \eta(X)\,\psi(\xi,\phi Y) = 0\,. \tag{2.6}$$

It is clear that the conditions (2.3) and (2.4) are equivalent to (2.5) and (2.6). Inserting $\phi(X)$ and ξ instead of X and Y into (1.9) and (1.10) we obtain:

$$\psi(\phi X,\xi) = -\theta(X,\xi)\,\xi - (\phi\circ\psi)(X,\xi) \tag{2.7a}$$

$$\theta(\phi X,\xi)\,\xi = -\psi(X,\xi) - (\phi\circ\psi)(\phi X,\xi) \tag{2.7b}$$

Now (2.5) and (2.6) may be written as follows:

$$\psi(X,Y) - \eta(Y)\,\psi(X,\xi) - \eta(X)\,\psi(\xi,Y) - \eta(Y)\,(\phi\circ\psi)(\phi X,\xi) - \tag{2.8a}$$
$$-\,\eta(X)\,(\phi\circ\psi)(\xi,\phi Y) = 0$$

$$[\theta(X,Y) - \eta(Y)\,\theta(X,\xi) - \eta(X)\,\theta(\xi,Y)]\,\xi - \eta(Y)\,(\phi\circ\psi)(X,\xi) - \tag{2.8b}$$
$$-\,\eta(X)\,(\phi\circ\psi)(\xi,Y) = 0\,.$$

Now, inserting $\phi(X)$ and ξ instead of X and Y into (2.5) and (2.6) we get:

$$\psi(\phi X, \xi) = -\theta(X, \xi)\,\xi \tag{2.9a}$$

$$\theta(\phi X, \xi)\,\xi = -\psi(X, \xi) \tag{2.9b}$$

From (2.7a), (2.7b), (2.9a) and (2.9b) we have:

$$(\phi \circ \psi)(X, \xi) = 0$$

Hence (2.8a) and (2.8b) become:

$$\psi(X, Y) - \eta(Y)\,\psi(X, \xi) - \eta(X)\,\psi(\xi, Y) = 0 \tag{2.10a}$$

$$\theta(X, Y) - \eta(Y)\,\theta(X, \xi) - \eta(X)\,\theta(\xi, Y) = 0 \tag{2.10b}$$

We have the following:

Lemma 2.2. *The condition* (2.10a) *implies* (2.10b).

Proof. By operating ϕ on (2.10a), we have:

$$\begin{aligned}\phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[\phi X, \phi Y] + \eta(X)\,\phi^2[\xi, Y] + \eta(X)\,\phi[\xi, \phi Y] - \\ -\eta(Y)\,\phi^2[X, \xi] + \eta(Y)\,\phi[\phi X, \xi] = 0\,.\end{aligned} \tag{2.11}$$

Inserting $\phi(X)$ instead of X into (2.10a) we have:

$$\begin{aligned}\phi[\phi X, Y] - [\phi^2 X, Y] - [\phi X, \phi Y] + \phi[\phi^2 X, \phi Y] + \phi^2(X)(\eta(Y))\,\xi - \eta(Y)\,\phi[\phi X, \xi] + \\ + \eta(Y)[X, \xi] + \eta(Y)\,\xi(\eta(X))\,\xi = 0\,.\end{aligned}$$

Hence

$$\begin{aligned}\phi[\phi X, Y] = [X, Y] - [\eta(X)\xi, Y] + [\phi X, \phi Y] - \phi[X, \phi Y] + \phi[\eta(X)\xi, \phi Y] - X(\eta(Y))\,\xi + \\ + \eta(X)\,\xi(\eta(Y))\,\xi + \eta(Y)\,\phi[\phi X, \xi] - \eta(Y)[X, \xi] - \eta(Y)\,\xi(\eta(X))\,\xi\,.\end{aligned} \tag{2.12}$$

Inserting (2.12) into (2.11) we obtain:

$$\begin{aligned}&[X, Y] - \eta[X, Y]\,\xi - [X, Y] + [\eta(X)\xi, Y] - [\phi X, \phi Y] + \phi[X, \phi Y] - \phi[\eta(X)\xi, \phi Y] + \\ &+ X(\eta(Y))\,\xi - \eta(X)\,\xi(\eta(Y))\,\xi - \phi[X, \phi Y] + [\phi X, Y] - \eta[\phi X, \phi Y]\,\xi - \eta(X)[\xi, Y] + \\ &+ \eta(X)\,\eta[\xi, Y]\,\xi + \eta(X)\,\phi[\xi, \phi Y] - \eta(Y)[X, \xi] + \eta(Y)\,\eta[X, \xi]\,\xi + \eta(Y)\,\phi[\phi X, \xi] - \\ &- \eta(Y)\,\phi[\phi X, \xi] + \eta(Y)[X, \xi] + \eta(Y)\,\xi(\eta(X))\,\xi = 0\end{aligned}$$

or

$$\eta [X, Y] \xi + Y(\eta(X)) \xi - X(\eta(Y)) \xi + \eta(X) \xi (\eta(Y)) \xi + \eta [\phi X, \phi Y] \xi - \eta(X) \eta [\xi, Y] \xi -$$

$$- \eta(Y) \eta [X, \xi] \xi - \eta(Y) \xi (\eta(X)) \xi = 0 .$$

Hence

$$\theta(X, Y) - \eta(Y) \theta (X, \xi) - \eta(X) \theta (\xi, Y) = 0.$$

Now we prove the following:

Theorem 2.3. *An almost paracontact structure $\Sigma = (\phi, \xi, \eta)$ on M is weak-normal if and only if the following conditionsare satisfied:*

$$\psi(\phi(X), \phi(Y)) = 0 \qquad (2.13a)$$

$$(\phi \circ \psi)(X, \xi) = 0 \qquad (2.13b)$$

for any vector fields X, Y

Proof. If F_1 and F_2 are both integrable, then (2.13a) and (2.13b) are clear. Now, assume that (2.13a) is satisfied. We insert $\phi(X)$ and $\phi(Y)$ instead of X and Y respectively into (2.13a) Thus we have:

$$0 = \psi(\phi^2 X, \phi^2 Y) = \psi(X - \eta(X) \xi, Y - \eta(Y) \xi) =$$

$$= \psi(X, Y) - \eta(Y) \psi (X, \xi) - \eta(X) \psi (\xi, Y)$$

or the condition (2.10a), which combining with the Lemma 2.2 and the condition (2.13b) give us the integrability of both structures F_1 and F_2.

Moreover we can state:

Theorem 2.4. *A weak-normal almost paracontact structure $\Sigma = (\phi, \xi, \eta)$ on M is normal if and only if the following condition is satisfied:*

$$\mathcal{L}_\xi \eta = 0 \qquad (2.14)$$

where $\mathcal{L}_\xi$ denotes the Lie derivative with respect to the field ξ.

Proof. Observe that:

$$(\mathcal{L}_\xi \eta) (X) = \xi(\eta(X)) - \eta [\xi, X] = \xi(\eta(X)) + \eta [X, \xi] = \theta (X, \xi)$$

or

$$\mathcal{L}_\xi \eta = \theta(-, \xi) . \qquad (2.15)$$

It is clear that if Σ is normal then $\mathcal{L}_\xi\, \eta = 0$. Now, conversely, if (2.14) then $\theta(-, \xi) = 0$ and by virtue of (2.5) we have:

$$\psi(X, Y) = -\eta(Y)\,\theta(\phi X, \xi)\,\xi - \eta(X)\,\theta(\xi, \phi Y) = 0$$

and this means that Σ is normal.

3. Examples.

Example 3.1. Let ω be a contact form on an odd-dimensional manifold M. It is known [2], that then there exists a vector field ξ on M such that: $\omega(\xi) = 1$, $d\omega(X, \xi) = 0$, for any vector field on M. Putting $\phi = Id_M - \omega \otimes \xi$, we obtain an almost paracontact structure (ϕ, ξ, ω) on M. This structure is not weak-normal.

Example 3.2. Let (M, g) be a Riemannian manifold that admits unit vector field ξ. Then putting $\eta = g(-, \xi)$ and $\phi = Id_M - \eta \otimes \xi$ we obtain an almost paracontact structure on M, so-called an almost paracontact ξ-structure on M. Such structures with additional conditions on ξ were investigated by T. Adati and T. Miyazawa in [1]. It turns out that in the case of ξ-structures, the normality and the weak-normality have the natural geometrical interpretation being illustrated in the following:

Theorem 3.1. *Let $\Sigma = (\phi, \xi, \eta)$ be a ξ-structure on a Riemannian manifold (M, g). Then:*

(i) *Σ is normal if and only if $d\eta = 0$,*

(ii) *Σ is weak-normal if and only if $d\eta = \eta \wedge \alpha$, where α is some 1-form on M.*

Proof. We have:

$$\psi(X, Y) = [X, Y] - \eta\,[X, Y]\,\xi - [X - \eta(X)\,\xi,\ Y] - [X,\ Y - \eta(Y)\,\xi] +$$

$$+ [X - \eta(X)\,\xi,\ Y - \eta(Y)\,\xi] - \eta\,[X - \eta(X)\,\xi,\ Y - \eta(Y)\,\xi]\,\xi + (X - \eta(X)\,\xi)(\eta(Y))\,\xi -$$

$$-(Y - \eta(Y)\,\xi)(\eta(X))\,\xi = [X, Y] - \eta\,[X, Y]\,\xi - [X, Y] + \eta(X)\,[\xi, Y] - Y(\eta(X))\,\xi -$$

$$-[X, Y] + \eta(Y)\,[X, \xi] + X(\eta(Y))\,\xi + [X, Y] - \eta(Y)\,[X, \xi] - X(\eta(Y))\,\xi - \eta(X)\,[\xi, Y] +$$

$$+ Y(\eta(X))\,\xi + \eta(X)\,\xi\,(\eta(Y))\,\xi - \eta(Y)\,\xi\,(\eta(X))\,\xi - \eta\,[X, Y]\,\xi + \eta(Y)\,\eta\,[X, \xi]\,\xi +$$

$$+ X(\eta(Y))\,\xi + \eta(X)\,\eta\,[X, \xi]\,\xi - Y(\eta(X))\,\xi - \eta(X)\,\xi\,(\eta(Y))\,\xi + \eta(Y)\,\xi\,(\eta(X))\,\xi +$$

$$+ X(\eta(Y))\,\xi - \eta(X)\,\xi\,(\eta(Y))\,\xi - Y(\eta(X))\,\xi + \eta(Y)\,\xi\,(\eta(X))\,\xi = -2\,\eta\,[X, Y]\,\xi +$$

$$+ 2\,X(\eta(Y))\,\xi - 2\,Y(\eta(X))\,\xi - \eta(Y)\ \ -\xi(\eta(X)) - \eta\,[X, \xi]\ \ \xi - \eta(X)\ \ \xi(\eta(Y)) -$$

$$-\eta\,[\xi, Y]\ \ \xi = 2(2\,d\eta(X, Y) - \eta(Y)\,d\eta(X, \xi) - \eta(X)\,d\eta(\xi, Y))\,\xi$$

and this means:

$$\psi(X, Y) = 2(2d\eta(X, Y) - \eta(Y)\,d\eta(X, \xi) - \eta(X)\,d\eta(\xi, Y))\,\xi\,. \tag{3.1}$$

Similarly:

$$\psi(\phi X, \phi Y) = 4(d\eta(X, Y) - \eta(Y)\, d\eta(X, \xi) - \eta(X)\, d\eta(\xi, Y))\, \xi\,. \tag{3.2}$$

and

$$(\phi \circ \psi)(X, \xi) = 0\,.$$

Now we shall show (i). Suppose that being under considerations an almost paracontact ξ-structure is normal. Then by virtue of the Corollary 1.6 $\psi(X, Y) = 0$ and also $\theta(X, Y) = 0$. Thus we have:

$$\begin{aligned} 0 = \theta(X, \xi) = \eta\,[X, \xi] + \xi(\eta(X)) = -2\, d\eta(X, \xi)\,, \\ 0 = \theta(\xi, Y) = \eta\,[\xi, Y] - \xi(\eta(Y)) = -2\, d\eta(\xi, Y)\,. \end{aligned} \tag{3.3}$$

Inserting (3.3) into (3.1) and remembering that $\psi = 0$ we come to the conclusion that $d\eta = 0$. Conversely, if η is closed form on M, then by virtue of (3.1) $\psi = 0$, which means that a ξ-structure is normal.

Now we shall prove (ii). Suppose that an almost paracontact ξ-structure is weak-normal. By (3.2) we have:

$$d\eta(X, Y) - \eta(Y)\, d\eta\,(X, \xi) - \eta(X)\, d\eta\,(\xi, Y) = 0\,. \tag{3.4}$$

Hence

$$d\eta(X, Y) = 0 \ \text{ for } X, Y \text{ such that } \ \eta(X) = \eta(Y) = 0\,. \tag{3.5}$$

Observe that (3.5) is equivalent to (3.4).

In fact, let us take two vector fields X and Y on M. Then we have:

$$\eta(X - \eta(X)\, \xi) = \eta(Y - \eta(Y)\, \xi) = 0$$

and by virtue of (3.5)

$$0 = d\eta(X - \eta(X)\, \xi,\, Y - \eta(Y)\, \xi) = d\eta(X, Y) - \eta(X)\, d\eta(\xi, Y) - \eta(Y)\, d\eta(X, \xi)$$

and hence (3.4) holds. On the other hand (3.5) is equivalent to:

$$\eta\,[X, Y] = 0 \ \text{ for } X, Y \text{ such that } \ \eta(X) = \eta(Y) = 0 \tag{3.6}$$

which, on account of the Frobenius Theorem, is equivalent to the complete integrability of the distribution that is generated by the form η, and hence, in fact, $d\eta = \eta \wedge \alpha$ for some 1-form α on M.

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STRESZCZENIE

W pierwszym rozdziale tej pracy wykazujemy między innymi:

Twierdzenie. *Niech* $\Sigma = (\phi, \xi, \eta)$ *będzie prawie para-kontaktową strukturą na M. Wtedy* Σ *jest całkowalna, wtedy i tylko wtedy gdy* $\psi = 0$, *gdzie:*

$$\psi(X, Y) = \phi([X, Y]) - [\phi X, Y] - [X, \phi Y] + \phi[\phi X, \phi Y] + \phi(X)\,\eta(Y) - \phi(Y)\,\eta(X) \quad \xi$$

oraz podstawowy wynik tego rozdziału sformułowany następująco:

Twierdzenie. *Prawie para-kontaktowa struktura* Σ *na M jest całkowalna wtedy i tylko wtedy gdy jest normalna.*

W rozdziale drugim wprowadzamy pojęcie słabej normalności prawie para-kontaktowej struktury Σ na M.

W ostatniej części pracy podajemy przykład prawie para-kontaktowej struktury, która nie jest słabo-normalna oraz geometryczne interpretacje normalności i słabej normalności.

РЕЗЮМЕ

В первой части этой работы доказано в частности:

Теорема. Пусть $\Sigma = (\phi, \xi, \eta)$ будет почти паракoнтактной структурой на M. Тогда Σ интегрируемая тогда и только тогда, когда $\psi = 0$, где

$$\psi(X, Y) = \phi([X, Y]) - [\phi X, Y] - [X, \phi Y] + \phi[\phi X, \phi Y] + \phi(X)\,\eta(Y) - \phi(Y)\,\eta(X) \quad \xi$$

и фундаментальный результат этой части.

Теорема. Почти паракoнтактная структура на M интегрируемая тогда и только тогда, когда она нормальная.

Во второй части введено понятие слабо-нормальности почти паракoнтактной структуры на M.

В последней части этой работы даются примеры почти паракoнтактной структуры, которая не слабо-нормальная, также дается геометрическую интерпретацию нормальности и слабо-нормальности.