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On the Functional $zf'(z)/f(z)$ over Functions with Positive Real Part

Abstract. Let $|z| < 1$. We investigate the range of the functional $f \mapsto zf'(z)/f(z)$ when f varies over the class $\mathbf{P}$ of all analytic functions with positive real part on the unit disc, normalized by $f(0) = 1$. We give the explicit description of this range for all $|z| < 1$.

1. Introduction and basic tools. Let $\mathbf{H}(\Delta)$ be the class of all complex functions analytic on the unit disc $\Delta = \{z \in \mathbf{C}: |z| < 1\}$, and let

$$\mathbf{P} = \{f \in \mathbf{H}(\Delta): \operatorname{Re} f(z) > 0 \text{ on } \Delta, f(0) = 1\}.$$

By the Riesz-Herglotz formula we have the integral representation for the class $\mathbf{P}$:

$$\mathbf{P} = \left\{ \int_0^{2\pi} q(\cdot, t) d\nu(t) : \nu \in \mathbb{P}(0, 2\pi) \right\}, \tag{1}$$

where

$$q(z,t) = (1 + ze^{it})/(1 - ze^{it}), \ z \in \Delta, \ 0 \le t \le 2\pi, \tag{2}$$

and $\mathbb{P}(\alpha, \beta)$ means the family of all probability measures on the interval $[\alpha, \beta]$, see [3],[5].

One of the subjects in geometric function theory is the study of continuous functionals on compact classes of analytic functions. We shall be concerned with the functional

$$J_z(f) = zf'(z)/f(z), \ f \in \mathbf{P},$$

where $z \in \Delta \setminus \{0\}$ is a fixed number. Denote

$$D(z) = J_z(\mathbf{P}) = \{zf'(z)/f(z): f \in \mathbf{P}\}. \tag{3}$$

The main purpose of the present paper is to give a detailed description of $D(z)$. It appears that the set $D(z)$ is always convex, has a smooth boundary which is either an elliptic Booth's lemniscate or the union of two circular arcs and two arcs of an elliptic Booth's lemniscate. The similar results for the univalent functions, the close-to-convex functions and the typically real functions are given in [6], see also [3; Th.10.6], in [2], see also [5; v.I, p.27], [9] and in [7, 8, 11], respectively.

Proposition 1. *For all points $z \in \Delta \setminus \{0\}$ the set (3) is a compact convex Jordan domain, centrally symmetric with respect to 0. Moreover, $D(z) = D(|z|)$.*

Proof. Convexity of the set $D(z)$ follows from the fact that for any $f, g \in \mathbf{P}$ and $0 < \lambda < 1$ we have $h = f^{1-\lambda} g^{\lambda} \in \mathbf{P}$. Similarly, the equivalences $f \in \mathbf{P} \Leftrightarrow 1/f \in \mathbf{P}$ and $f \in \mathbf{P} \Leftrightarrow f \circ \tau_\alpha \in \mathbf{P}$, where $\tau_\alpha(z) = ze^{i\alpha}, \alpha \in \mathbf{R}$, imply that $D(z) = -D(z)$ and $D(e^{i\alpha}z) = D(z)$.

To find (3), we will show that all the boundary points of (3) are situated on some Jordan arcs. To do this we apply the following result due to Rusheweyh [10].

Theorem 1 (Rusheweyh). *Assume that the functions $\phi, \psi \colon [\alpha, \beta] \mapsto \mathbf{C}$ are continuous and $0 \notin \{(1-\lambda)\psi(s) + \lambda\psi(t) \colon \alpha \le s \le t \le \beta,\ 0 \le \lambda \le 1\}$. Next let*

$$J(\nu) = \left(\int_\alpha^\beta \phi(t) d\nu(t) \right) \left(\int_\alpha^\beta \psi(t) d\nu(t) \right)^{-1} \quad \text{for } \nu \in \mathbb{P}(\alpha, \beta). \tag{4}$$

Then the following sets

$$J(\mathbb{P}(\alpha, \beta)) \tag{5}$$

and

$$\{[(1-\lambda)\phi(s) + \lambda\phi(t)] / [(1-\lambda)\psi(s) + \lambda\psi(t)] \colon \alpha \le s \le t \le \beta,\ 0 \le \lambda \le 1\} \tag{6}$$

are exactly the same.

Remarks.

1. The original proof of the Rusheweyh result follows easily from the Carathéodory theorem on convex hulls and from the fact that $w \in J(\mathbb{P}(\alpha, \beta))$ if and only if

$$0 \in \left\{ \int_\alpha^\beta (\phi - w\psi) d\nu \colon \nu \in \mathbb{P}(\alpha, \beta) \right\} = \operatorname{conv}\{\phi(t) - w\psi(t) \colon \alpha \le t \le \beta\},$$

where conv A means the convex hull of A.

2. For continuous functions $\phi, \psi \colon [\alpha, \beta] \mapsto \mathbf{R}$ with $0 \notin \psi([\alpha, \beta])$ the Rusheweyh result says that $J(\mathbb{P}(\alpha, \beta)) = \{\phi(t)/\psi(t) \colon \alpha \le t \le \beta\}$.

Following Theorem 1 the identity of the sets (5)-(6) implies

Proposition 2. *Denote $|z| = r$, $r < 1$. Then $D(z) = \Phi(I)$, where*

$$\Phi(s, t, \lambda) = \frac{1}{2} \left(q(r, s) + q(r, t) - \frac{1 + q(r, t) q(r, s)}{(1-\lambda) q(r, s) + \lambda q(r, t)} \right) \tag{7}$$

and

$$I = \{(s, t, \lambda) \colon 0 \le s \le t \le 2\pi, 0 \le \lambda \le 1\}. \tag{8}$$

Thus our problem reduces to finding the image of the set (8) by the mapping (7). The following lemma, similar to Lemma 1 from [8], is a useful tool for solving this kind of problems. The symbols ∂A , int A and $\overline{A}$ will denote the topological boundary of A, the interior of A and the closure of A, respectively.

Lemma 1. *Let U be an open set such that $\overline{U}$ is compact and suppose that a mapping $\Phi\colon \overline{U} \mapsto \mathbf{C}$ is continuous on $\overline{U}$ and $\Phi(U)$ is open. Denote $\Gamma = \Phi(\partial U)$. Then*
(i) $\partial\Phi(\overline{U}) \subset \Gamma$.
(ii) *If Γ is a Jordan curve, then $\partial\Phi(\overline{U}) = \Gamma$.*
(iii) *If Γ is a Jordan domain, then $\partial\Phi(\overline{U}) = \partial\Gamma$.*

Proof.
(i) By continuity we have $\Phi(\overline{U}) \subset \overline{\Phi(U)}$, whence $\overline{\Phi(\overline{U})} = \overline{\Phi(U)}$ and $\partial\Phi(\overline{U}) \subset \partial\Phi(U)$. If $\Phi(U)$ is open and $\Phi(\overline{U})$ is closed, then we have $\partial\Phi(\overline{U}) \subset \partial\Phi(U) = \overline{\Phi(U)} \setminus \Phi(U) = \Phi(\overline{U}) \setminus \Phi(U) \subset \Phi(\partial U)$ and (i) holds.

(ii) From (i) we have $\partial\Phi(\overline{U}) \subset \Gamma$. Because $\Phi(U)$ is bounded, we can find $x \in \Phi(U)$ and $y \in \mathbf{C} \setminus \Phi(\overline{U})$. If $\partial\Phi(\overline{U}) \neq \Gamma$, then the set $\mathbf{C} \setminus \partial\Phi(\overline{U})$ is arcwise connected, thus there exists a path

$$\alpha\colon [0,1] \mapsto \mathbf{C} \setminus \partial\Phi(\overline{U}),\ \alpha(0) = x \in \Phi(U),\ \alpha(1) = y \notin \Phi(\overline{U}). \tag{9}$$

Then we have $\alpha(t_0) \in \partial\Phi(\overline{U})$, where $t_0 = \sup\{t\colon \alpha([0,t]) \subset \Phi(\overline{U})\}$, which contradicts (9).
The proof of (iii) is similar.

2. Description of $\mathbf{D}(z)$. We start with a proposition which allows us to use Lemma 1.

Proposition 3. *Denote Φ and I like in (7) and (8) and let $r_0 = c - \sqrt{c}$, $c = (\sqrt{5}+1)/2$.*
(i) *If $r = |z| \le r_0$, then Φ is an open mapping on $\operatorname{int}(I)$.*
(ii) *If $r_0 < r < 1$, then Φ is an open mapping on the set*

$$\operatorname{int} I \setminus \left(\left\{\gamma_1(s)\colon \frac{\pi}{2} + q < s < q_1\right\} \cup \left\{\gamma_2(s)\colon q < s < q_2\right\} \right), \tag{10}$$

where

$$\begin{aligned}
&\gamma_1(s) = (s, t_1(s), \lambda_1(s)) , \qquad t_1(s) = 5\pi/2 - s ,\\
&\gamma_2(s) = (s, t_2(s), \lambda_2(s)) , \qquad t_2(s) = 3\pi/2 - s ,\\
&\lambda_1(s) = \frac{(1 - 2r\sin s + r^2)\left[(1-r^2)^2 + 2r(1+r^2)\sin s\right]}{4r^2(\cos s - \sin s)[(\cos s + \sin s)(1+r^2) - 2r]},\ \lambda_2(s) = \lambda_1(-s),\\
&p = \frac{(1-r^2)^2}{2r(1+r^2)},\ q = \arcsin p,\\
&q_1 = \min\{\pi + q, 3\pi/2 - q\} \text{ and } q_2 = \min\left\{\frac{\pi}{2} + q, \pi - q\right\}
\end{aligned} \tag{11}$$

Note that $q_1 = \pi + q$, $q_2 = \pi/2 + q$ iff $\sqrt{2} - 1 \le r < 1$ and $q_1 = 3\pi/2 - q$, $q_2 = \pi - q$ iff $r_0 < r \le \sqrt{2} - 1$.

Proof. To shorten the notation, we shall use the following substitutions: $A = (1+ar)/(1-ar)$, $B = (1+br)/(1-br)$, $a = e^{is}$, $b = e^{it}$, $A' = 2r/(1-ar)^2$, $B' = 2r/(1-br)^2$, $\Phi = \Phi(s,t,\lambda) = (A + B - (1+AB)/[(1-\lambda)A + \lambda B])/2$, $M = [(1-\lambda)A + \lambda B]^2$. To determine the set on which Φ is an open mapping, we find the points $(s,t,\lambda) \in \operatorname{int}(I)$ such that the rank of the matrix

$$\begin{pmatrix} \operatorname{Re} \partial\Phi/\partial s & \operatorname{Im} \partial\Phi/\partial s \\ \operatorname{Re} \partial\Phi/\partial t & \operatorname{Im} \partial\Phi/\partial t \\ \operatorname{Re} \partial\Phi/\partial \lambda & \operatorname{Im} \partial\Phi/\partial \lambda \end{pmatrix}$$

equals 2. To do this, we solve the system of equations

$$\left\{ \begin{array}{l} \frac{\partial\overline{\Phi}}{\partial s}\frac{\partial\Phi}{\partial\lambda} = \frac{\partial\overline{\Phi}}{\partial\lambda}\frac{\partial\Phi}{\partial s} \\ \frac{\partial\overline{\Phi}}{\partial t}\frac{\partial\Phi}{\partial\lambda} = \frac{\partial\overline{\Phi}}{\partial\lambda}\frac{\partial\Phi}{\partial t} \end{array} \right. \tag{12}$$

for $0 \le s < t < 2\pi, 0 < \lambda < 1$. Because we have $\frac{\partial\Phi}{\partial s} = iaA'(1-\lambda)[1 + A^2 - \lambda(A-B)^2]/2M$, $\frac{\partial\Phi}{\partial t} = ibB'\lambda[1 + B^2 - (1-\lambda)(A-B)^2]/2M$, $\frac{\partial\Phi}{\partial\lambda} = (1+AB)(B-A)/2M$, the system (12) is equivalent to

$$\left\{ \begin{array}{l} \operatorname{Re}\left[aA'(1 + A^2 - \lambda(A-B)^2)(1 + \overline{AB})(\overline{A} - \overline{B})\right] = 0, \\ \operatorname{Re}\left[bB'(1 + B^2 - (1-\lambda)(A-B)^2)(1 + \overline{AB})(\overline{A} - \overline{B})\right] = 0. \end{array} \right. \tag{13}$$

Observe that $\overline{aA'} = 2ar/(a-r)^2$, $\overline{A} = (a+r)/(a-r)$ and that for $\overline{bB}$, $\overline{B}$ similar equalities hold. After eliminating λ from the system (13) and after some labor we deduce the equation

$$(a-b)(1 + a^2b^2)(1 - r^2)(ab + r^2)(1 + abr^2) = 0.$$

Because of $0 < r < 1 = |a| = |b|$ and $a \ne b$, this yields to $a^2b^2 = -1$. Thus for all points (s,t,λ) with $s < t < s + 2\pi$, $0 < \lambda < 1$, that are critical for Φ, we have $s + t = \pm\pi/2 \pmod{2\pi}$.

Considering only the points $(s,t,\lambda) \in I$ we get the following possibilities:

$$t_1(s) = \left\{ \begin{array}{ll} \pi/2 - s, & \text{for } s \in [0, \pi/4) \\ 5\pi/2 - s, & \text{for } s \in [\pi/2, 5\pi/4) \end{array} \right. . \tag{14}$$

and

$$t_2(s) = \left\{ \begin{array}{ll} 3\pi/2 - s, & \text{for } s \in [0, 3\pi/4) \\ 7\pi/2 - s, & \text{for } s \in [3\pi/2, 7\pi/4) \end{array} \right. . \tag{15}$$

Substituting $b = \exp(it_j(s))$, $j = 1, 2$, in the system (13) we obtain

$$\lambda_1(s) = \frac{(ia + r)(ar - i)(a + ir - ia^2r - 2ar^2 + ir^3 - ia^2r^3 + ar^4)}{2r^2(-i + a^2)(i + a^2 - 2(1+i)ar + ir^2 + a^2r^2)}$$

and

$$\lambda_2(s) = \frac{(ar+i)(r-ia)(a-ir+ia^2r-2ar^2-ir^3+ia^2r^3+ar^4)}{2r^2(i+a^2)(-i+a^2+2(-1+i)ar-ir^2+a^2r^2)},$$

respectively, which we can transform to

$$\lambda_1(s) = \frac{|r+ia|^2(p+\sin s)}{2r(\cos s-\sin s)[(\cos s+\sin s)-2r/(1+r^2)]}$$

and

$$\lambda_2(s) = \frac{|r-ia|^2(p-\sin s)}{2r(\cos s+\sin s)[(\cos s-\sin s)-2r/(1+r^2)]},$$

which aprees with (11).
We have also

$$1-\lambda_1(s) = \frac{-|a-r|^2(p+\cos s)}{2r(\cos s-\sin s)[(\cos s+\sin s)-2r/(1+r^2)]}$$

and

$$1-\lambda_2(s) = \frac{-|a-r|^2(p+\cos s)}{2r(\cos s+\sin s)[(\cos s-\sin s)-2r/(1+r^2)]}.$$

Now we shall study the inequalities

$$0<\lambda_j(s)<1,\ 0\le s<t_j(s)<2\pi,\ j=1,2,$$

in order to determine the range of the parameter s, for which the critical points $\gamma_j(s)$ belong to int$I$. Denote $L = p+\sin s$, $K = p+\cos s$, $M = (\cos s-\sin s)[(\cos s+\sin s)-2r/(1+r^2)]$. Then $0<\lambda_1(s)<1$ iff $KL<0$ and $LM>0$.
Suppose first that $L>0, K<0, M>0$. Then $\cos s<-p<\sin s$. If $p\ge 1$ we get a contradiction which means that for $p\ge 1$ there are no critical points inside the set I. If $0<p<1$, we get

$$\frac{\pi}{2}+q<s<\min\left\{\pi+q,\frac{3}{2}\pi-q\right\} = \begin{cases}\pi+q \text{ if } 0<q\le\pi/4,\\ 3\pi/2-q \text{ if } \pi/4\le q<\pi/2.\end{cases}$$

Suppose now that $L<0, K>0, M<0$. Then $\sin s<-p<\cos s$. If $p\ge 1$ we get again a contradiction. If $0<p<1$ we obtain

$$2\pi-q>s>\max\left\{\frac{3}{2}\pi-q,\pi+q\right\} = \begin{cases}3\pi/2-q \text{ if } 0<q\le\pi/4,\\ \pi+q \text{ if } \pi/4\le q<\pi/2.\end{cases}$$

According to (14), the admissible s are contained in $[0,\pi/4)\cup[\pi/2,5\pi/4)$ and the conditions: $0<\lambda_1(s)<1$, $s\in[0,\pi/4)\cup[\pi/2,5\pi/4)$ are equivalent to $\pi/2+q<s<\min\{\pi+q,3\pi/2-q\}$. Analogously, by (15) and $\lambda_2(s)=\lambda_1(-s)=\lambda_1(2\pi-s)$, the conditions: $0<\lambda_2(s)<1$, $s\in[0,3\pi/4)\cup[3\pi/2,7\pi/4)$ are equivalent to

$$q<s<\min\{\pi/2+q,\pi-q\} = \begin{cases}\pi/2+q \text{ if } 0<q\le\pi/4\\ \pi-q \text{ if } \pi/4\le q<\pi/2.\end{cases}$$

Putting $x = 2r/(1+r^2)$ we get $p = 1/x - x$, so we can easily find the ranges of r for which the inequalities $0 < p < 1$ and $0 < p \le 1/\sqrt{2}$ hold.

Now we are ready to state the main result containing the description of $D(z)$ for $|z| < 1$.

Theorem 2. *Let us denote* r_0, p, q, q_1, q_2 *like in* Proposition 3 *and let* $\Gamma_0(s) = 2re^{is}/(1 - r^2e^{2is})$, $r = |z|$.
(i) *If* $|z| \le r_0$, *then*

$$\partial D(z) = \{\Gamma_0(s) \colon 0 \le s \le 2\pi\}.$$

(ii) *If* $1 > |z| > r_0$, *then* $\partial D(z)$ *is a union of two circular arcs and two arcs of the curve* Γ_0:
$\partial D(z) = \{\Gamma_0(s) \colon -q \le s \le q$ *or* $\pi - q \le s \le \pi + q\} \cup \{\Gamma_1(s) \colon q + \pi/2 \le s \le q_1\} \cup \{\Gamma_2(s) \colon q \le s \le q_2\}$, *where*

$$(1-r^2)\Gamma_1(s) = \frac{r^4 + 1 - 6r^2 - i(1-r^4) + 2r(1+r^2)h(s)}{1 + r^2 - i(1-r^2) - 2rh(s)}$$

and

$$\Gamma_2(s) = \overline{\Gamma_1(-s)}, \;\; h(s) = \cos s + \sin s.$$

Proof.
(i) If $|z| \le r_0$, then from Proposition 3 and Lemma 1 we get the inclusion

$$\partial D(z) \subset \Gamma_0([0, 2\pi]).$$

Since $D(z)$ is a Jordan domain and Γ_0 is a Jordan curve this implies that

$$\partial D(z) = \Gamma_0([0, 2\pi]).$$

(ii) Denote the set (10) by U. Again from Proposition 3 and Lemma 1 we have the inclusion

$$\partial D(z) \subset \Phi(\partial U).$$

With the notation like in Proposition 3,

$$\Phi(\partial U) = \Gamma_0([0, 2\pi]) \cup \{\Phi(\gamma_1(s)) \colon q + \pi/2 < s < q_1\} \cup \{\Phi(\gamma_2(s)) \colon q < s < q_2\}.$$

Direct calculations give the identities

$$\Phi(\gamma_1(s)) = \Gamma_1(s) = \frac{1}{1-r^2} \frac{r^4 + 1 - 6r^2 - i(1-r^4) + 2r(1+r^2)(\cos s + \sin s)}{1 + r^2 - i(1-r^2) - 2r(\cos s + \sin s)}$$

and

$$\Phi(\gamma_2(s)) = \Gamma_2(s) = \frac{1}{1-r^2} \frac{r^4 + 1 - 6r^2 + i(1-r^4) + 2r(1+r^2)(\cos s - \sin s)}{1 + r^2 + i(1-r^2) - 2r(\cos s - \sin s)}.$$

Here $\Gamma_1([0,2\pi])$ and $\Gamma_2([0,2\pi])$ are arcs of circles with the centers $c_1 = i$ and $c_2 = -i$, respectively, and radii $R = \sqrt{2(1+r^4)}/(1-r^2)$. Since $|\Gamma_0(s) - c_j| \leq R$, $j = 1,2$, and since $D(z)$ is a convex Jordan domain, the points $\Gamma_1(s)$, $s \in [q+\pi/2, q_1]$, $\Gamma_2(s)$, $s \in [q, q_2]$ must be situated on the boundary of $D(z)$. Now to see (ii), it is sufficient to notice that $\Gamma_0(q) = \Gamma_2(q), \Gamma_2(q_2) = \Gamma_0(\pi - q), \Gamma_0(\pi + q) = \Gamma_1(q_1)$ and $\Gamma_1(\pi/2 + q) = \Gamma_0(2\pi - q)$.

As an application of Theorem 2 we get sharp estimates for $|zf'(z)/f(z)|$, $|\mathrm{Re}\, zf'(z)/f(z)|$ and $|\mathrm{Im}\, zf'(z)/f(z)|$ as f varies over the class $\mathbf{P}$. The first two are known in a more general form, see [1] and [4].

Theorem 3. *Denote r_0 like in* Proposition 3 *and let* $0 < |z| < 1$. *Then*
(i)

$$\max_{f\in\mathbf{P}} \left|\frac{zf'(z)}{f(z)}\right| = \max_{f\in\mathbf{P}} \mathrm{Re}\frac{zf'(z)}{f(z)} = -\min_{f\in\mathbf{P}} \mathrm{Re}\frac{zf'(z)}{f(z)} = \frac{2|z|}{1-|z|^2}.$$

(ii) *If $r = |z| \leq r_0$, then*

$$\max_{f\in\mathbf{P}} \mathrm{Im}\frac{zf'(z)}{f(z)} = -\min_{f\in\mathbf{P}} \mathrm{Im}\frac{zf'(z)}{f(z)} = \frac{2|z|}{1+|z|^2}.$$

If $1 > |z| > r_0$, then

$$\max_{f\in\mathbf{P}} \mathrm{Im}\frac{zf'(z)}{f(z)} = -\min_{f\in\mathbf{P}} \mathrm{Im}\frac{zf'(z)}{f(z)} = \sqrt{2}\frac{\sqrt{1+r^4}}{1-r^2} - 1.$$

Proof. (i) From Theorem 2 (i) we obtain that

$$\max\{|J_z(f)| : f \in \mathbf{P}\} = \max\{|\Gamma_0(s)| : 0 \leq s \leq 2\pi\} = 2r/(1-r^2) = \Gamma_0(0).$$

(ii) Let $|z| = r \leq r_0$. By Theorem 2 (ii) we get

$$\max(\mathrm{Im}\, J_z(\mathbf{P})) = \max\{\mathrm{Im}\Gamma_0(s) : 0 \leq s \leq 2\pi\} = 2r/(1+r^2) = \Gamma_0(\pi/2)/i.$$

If $|z| = r > r_0$, then by Theorem 2 (ii)

$$\max(\mathrm{Im}\, J_z(\mathbf{P})) = \max\{\mathrm{Im}\,\Gamma_2(s) : q < s < q + \pi/2\} = R - 1,$$

where $R = \sqrt{2(1+r^4)}/(1-r^2)$ is the radius of the circle which contains $\Gamma_2([0,2\pi])$.

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REFERENCES

[1] Bernardi, S.D., *New distortion theorems for functions of positive real part and applications to the partial sums of univalent complex functions*, Proc. Amer. Math. Soc. 45 (1974), 113–118.

[2] B i e r n a c k i, M., *Sur la représentation conforme des domaines linéairement accesssibles* , Prace Mat.-Fiz. 44 (1936), 293-314.

[3] D u r e n, P.L., *Univalent functions*, Springer, New York 1983.

[(] 4) G o l u z i n, G., *Zur Teorie der schlichten konformen Abbildungen* , Mat. Sb. 42 (1935), 169–190.

[5] G o o d m a n, A.W., *Univalent functions* , Mariner, Tampa, Florida 1983.

[6] G r u n s k y, H., *Neue Abschätzungen zur konformen Abbildung ein-und mehrfach zusammen - hängender Bereiche* , Schr. Inst. Angew. Math. Univ. Berlin (1932), 95–140.

[7] K o l o d y n s k i, S., M. S z a p i e l and W. S z a p i e l, *On the functional* $f \mapsto \xi f'(\xi)/f(\xi)$ *within typically real functions* ,Ann. Univ. Mariae Curie-Sklodowska Sect. A 47 (1993), 69–81.

[8] K o l o d y n s k i, S. and W. S z a p i e l, *On a relative growth functional over the class of typically real functions* , Ann. Univ. Mariae Curie - Sklodowska Sect. A 45 (1991), 59–70.

[9] R ø n n i n g, F., *On the range af a certain functional over the class of close-to-convex functions* , Complex Variables 14 (1990), 1–14.

[10] R u s h e w e y h, St., *Nichtlineare Extremalprobleme für holomorphe Stieltjesintegrale* , Math. Z. 142 (1975), 19–23.

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