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On Distributions and Moments of Order Statistics for Random Sample Size

O rozkładach i momentach statystyk porządkowych dla próby o losowej liczebności

O распределениях и моментах порядковых статистик для выборки случайного объема

1. **Introduction.** Let $(X_1, X_2, \dots, X_n)$ be a sample from a population having distribution function (d.f.) $\bar{F}(x)$ and probability density function (p.d.f.) $\bar{f}(x)$. Suppose that $X_1 \leq X_2 \leq \dots \leq X_n$ is an ordered sample of size n . Properties of order statistics (o.s.) for fixed sample size n were widely investigated, while a literature on this subject in the case when n is a value of random variable N is not so rich. Some studies on order statistics for random sample size can be found, e.g. in [1], [2] and [6].

In this note, we will study distributions and moments of order statistics in the case when N has a power series distribution (e.g. [3], [5]) and an inflated power series distribution. In particular cases, some of our result reduces to those of [6]. It is worth mentioning here that such types of distributions are of interest in mathematical statistics ([4], [7], [8], [9], [10]).

2. N has a power series distribution.

(i) Distributions of order statistics.

A random variable N is said to have the power series distribution (PSD), if the probability function of N is of the form

$$p(k; \theta) = P[N = k] = \frac{a(k) \theta^k}{\bar{f}(\theta)} \quad \text{for } k \in T, \quad (1)$$

where $T \in \mathbb{N} \cup \{0\}$, $a(k) > 0$, $\bar{f}(\theta) = \sum_{k \in T} a(k) \theta^k$ for $\theta \in \Omega = \{\theta : 0 < \theta < \rho\}$, the

parameter space, and ρ is the radius of convergence of the power series of $\bar{f}(\theta)$, and $\mathbb{N}$ denotes the set of all integers.

In what follows we write f_i for $f(x_i)$, F_j for $F(x_j)$ etc., and put

$$A_i(\theta) = \sum_{\substack{k \geq i \\ k \in T}} a(k) \theta^k, \quad D_i(\theta, F) = \sum_{\substack{k \in T \\ k \geq i}} \binom{k}{i} a(k) [\theta(1 - F_i)]^k,$$

$$E_i(\theta, F) = \sum_{\substack{k \geq i \\ k \in T}} \binom{k}{i} a(k) [\theta F_{N-i+1}]^k,$$

$$L_j(\theta, F) = \sum_{k \geq j} k(k-1) \binom{k-2}{i-1, j-i-1, k-j} a(k) [\theta(1 - F_j)]^k,$$

where $\binom{n}{k_1, k_2, k_3}$ is the multinomial coefficient.

Lemma 1. If $X_1 \leq X_2 \leq \dots \leq X_N$ is an ordered sample of size N , where N has the power series distribution (1), then:

a) the conditional p.d.f. of X_i , $1 \leq i \leq N$, conditioned on the event $[N \geq i]$ is

$$g(x_i) = \frac{D_i(\theta, F)}{A_i(\theta)} i F_i^{i-1} (1 - F_i)^{-i} f_i, \quad (2)$$

b) if T is finite, then the conditional p.d.f. of X_{N-i+1} , conditioned on the event $[N \geq i]$ is

$$g(x_{N-i+1}) = \frac{E_i(\theta, F)}{A_i(\theta)} i (1 - F_{N-i+1})^{i-1} F_{N-i+1}^{-i} f_{N-i+1}, \quad (3)$$

c) the joint conditional p.d.f. of X_i and X_j , $1 \leq i < j \leq N$, conditioned on the event $[N \geq j]$ is

$$g(x_i, x_j) = \frac{L_j(\theta, F)}{A_j(\theta)} F_i^{i-1} [F_j - F_i]^{j-i-1} [1 - F_j]^{-j} f_i f_j. \quad (4)$$

Proof. Let G denote the conditional cumulative distribution function of X_i , conditioned on the event $[N \geq i]$, i.e. $G(x_i) = P[X_i \leq x_i | N \geq i]$. For a given sample size k let us put $H(x_i | k) = P[X_i \leq x_i | N = k]$ and $h(x_i | k) = H'(x_i | k)$. We have

$$\begin{aligned} G(x_i) &= P[X_i \leq x_i | N \geq i] = \frac{1}{P[N \geq i]} \sum_{\substack{k \geq i \\ k \in T}} P[X_i \leq x_i | N = k] = \\ &= \frac{1}{P[N \geq i]} \sum_{\substack{k \geq i \\ k \in T}} H(x_i | k) P[N = k]. \end{aligned}$$

Hence, we get

$$g(x_i) = \frac{1}{P[N \geq i]} \sum_{\substack{k \geq i \\ k \in T}} h(x_i | k) P[N = k]. \quad (5)$$

By (1), we have

$$\begin{aligned} \sum_{\substack{k \geq i \\ k \in T}} h(x_i | k) P[N = k] &= \sum_{\substack{k \geq i \\ k \in T}} \frac{k!}{(i-1)!(k-i)!} F_i^{i-1} (1-F_i)^{k-i} \frac{a(k) \theta^k}{\bar{f}(\theta)} f_i = \\ &= \frac{i F_i^{i-1} (1-F_i)^{-i} f_i}{\bar{f}(\theta)} D_i(\theta, F), \end{aligned}$$

since $h(x_i | k) = \frac{k!}{(i-1)!(k-i)!} F_i^{i-1} (1-F_i)^{k-i} f_i$, $i = 1, 2, \dots, k$. Moreover,

$$P[N \geq i] = \frac{A_i(\theta)}{\bar{f}(\theta)}.$$

Hence we obtain (2).

The formulae (3) and (4) can be obtained in an analogous way, using

$$h(X_{N-i+1} | k) = \frac{k!}{(i-1)!(k-i)!} (1-F_{N-i+1})^{i-1} F_{N-i+1}^{k-i} f_{N-i+1}$$

and

$$h(x_i, x_j | k) = \frac{k!}{(i-1)!(j-i-1)!(k-j)!} F_i^{i-1} [F_j - F_i]^{j-i-1} [1-F_j]^{k-j} f_i f_j$$

for $i < j$.

The p.d.f. of the smallest and largest (when T is finite) order statistics are directly obtained from (2) and (3) by putting $i = 1$.

Lemma 2. Let R be the range of an ordered sample $X_1 \leq X_2 \leq \dots \leq X_N$, where N is a random variable distributed according to (1). Then

$$g(R) = \frac{2}{A_2(\theta)} \sum_{\substack{k \geq 2 \\ k \in T}} \binom{k}{2} a(k) \theta^k \int_{-\infty}^{\infty} [F(x+R) - F(x)]^{k-2} f(x) f(x+R) dx. \quad (6)$$

Proof. Since the p.d.f. of the range R ([11] p. 248) is, for fixed k ,

$$h(R) = k(k-1) \int_{-\infty}^{\infty} [F(x+R) - F(x)]^{k-2} f(x) f(x+R) dx,$$

therefore the p.d.f. of R for random N with the distribution (1) is given by (6).

(ii) A sample from a population uniformly distributed in $(0; 1)$.

We now consider a sample $(X_1, \dots, X_N)$ of size N from a population having the uniform distribution in $(0; 1)$, i.e. $F(x) = x$ for $x \in (0, 1)$. In this case we can prove

Theorem 1. *If $X_1 \leq X_2 \leq \dots \leq X_N$ is an ordered sample from $F(x) = x$, $x \in (0, 1)$ and N has the distribution given by (1), then:*

$$g(R) = \frac{2(1-R)\theta^2}{A_2(\theta)} \sum_{\substack{k \geq 2 \\ k \in T}} \binom{k}{2} a(k) (\theta R)^{k-2} \quad (7)$$

for $m \geq 1$,

$$E X_i^m = \frac{i}{A_i(\theta)} \sum_{\substack{k \geq i \\ k \in T}} \binom{k}{i} a(k) \theta^k \frac{(m+i-1)! (k-i)!}{(m+k)!} \quad (8)$$

and

$$E X_i X_j = \frac{i(j+1)}{A_j(\theta)} \sum_{\substack{k \geq j \\ k \in T}} \frac{a(k) \theta^k}{(k+1)(k+2)}, \text{ for } i < j. \quad (9)$$

Proof. (7) is a straightforward consequence of (6), while (8) and (9) we obtain from Lemma 1.

(iii) Particular cases. It is known that (1) with $T = \{0, 1, \dots, n\}$, $a(k) = \binom{n}{k}$, $\bar{f}(\theta) = (1+\theta)^n$, $\theta = p/q$, where $0 < p < 1$, $p+q=1$, reduces to the binomial distribution with parameters p and n .

If $T = \mathbb{N} \cup \{0\}$, $a(k) = (-1)^k \binom{-n}{k}$, $\bar{f}(\theta) = (1-\theta)^{-n}$, $\theta = q$, $0 < q < 1$, then (1) gives the negative binomial distribution with parameters q and n .

Putting $T = \mathbb{N} \cup \{0\}$, $a(k) = 1/k!$, $\bar{f}(\theta) = e^\theta$, $\theta = \lambda > 0$, we get the Poisson distribution with parameter λ .

Using the above facts one can get from (2) – (4) and from (6) – (9) the results of [6].

The above given considerations concerning the moments of the ordered statistics lead us to the following combinatorial formulae:

Corollary. *If T , θ , $\bar{f}(\theta)$ and $a(k)$ are quantities determining the binomial, the negative binomial and Poisson distribution, then for $m \geq 1$*

$$\sum_{\substack{k \geq i \\ k \in T}} \frac{k!}{(m+k-1)!} a(k) \theta^k \left[\frac{1}{\theta} - \frac{(k+1)}{m+k} - \frac{a(k+1)}{a(k)} \right] = \frac{i!}{(m+i-1)!} a(i) \theta^{i-1},$$

i.e. explicite

$$\sum_{k=i}^n \frac{k!}{(m+k-1)!} \binom{n}{k} \left(\frac{p}{q}\right)^k \left[\left(\frac{p}{q}\right)^{-1} - \frac{n-k}{m+k}\right] = \frac{i!}{(m+i-1)!} \binom{n}{i} \left(\frac{p}{q}\right)^{i-1},$$

$$\sum_{k=i}^{\infty} \frac{k!}{(m+k-1)!} (-1)^k \binom{n}{k} q^k \left[\frac{1}{q} - \frac{n+k}{m+k}\right] = \frac{i!}{(m+i-1)!} (-1)^i \binom{n}{i} q^{i-1},$$

$$\sum_{k=i}^{\infty} \frac{1}{(m+k-1)!} \lambda^k \left[\frac{1}{\lambda} - \frac{1}{m+k}\right] = \frac{1}{(m+i-1)!} \lambda^{i-1}.$$

Proof. The above formulae we obtain from (8) and from the results of [6].

3. N has an inflated power series distribution.

(i) Distribution of order statistics.

A random variable N is said to have the inflated (at the point $k = 1, 1 \in T$) power series distribution (IPSD), if the probability function of N is of the form

$$p(k; \theta, \alpha) = P[N = k] = \begin{cases} \beta + \alpha \frac{a(k) \theta^k}{\bar{f}(\theta)} & \text{for } k = 1 \\ \alpha \frac{a(k) \theta^k}{\bar{f}(\theta)} & \text{for } k \in T - 1 \end{cases} \quad (10)$$

where $0 < \alpha < 1$, $\alpha + \beta = 1$, and the symbols T , $a(k)$, $\bar{f}(\theta)$, θ are the same as in the definition of PSD.

It is obvious that in the case $\alpha = 1$ IPSD reduces to PSD. Putting $\gamma(\theta) = (\beta/\alpha)\bar{f}(\theta)$, we have

Lemma 3. If $X_1 \leq X_2 \leq \dots \leq X_N$ is an ordered sample of size N , where N has IPSD (10), then:

a) the conditional p.d.f. of X_i , $1 \leq i \leq N$ conditioned on the event $[N \geq i]$ is

$$g(x_i) = \begin{cases} \frac{D_i(\theta, F) + \gamma(\theta) \binom{i}{i} (1 - F_i)^i}{A_i(\theta) + \gamma(\theta)} i F_i^{i-1} (1 - F_i)^{-i} f_i & \text{for } i \leq 1 \\ \frac{D_i(\theta, F)}{A_i(\theta)} i F_i^{i-1} (1 - F_i)^{-i} f_i & \text{for } i > 1; \end{cases} \quad (11)$$

b) if T is finite, then the conditional p.d.f. of X_{N-i+1} , conditioned on the event $[N \geq i]$ is

$$g(x_{N-l+1}) = \begin{cases} \frac{E_l(\theta, F) + \gamma(\theta) \binom{l}{i} F_{N-l+1}^l}{A_l(\theta) + \gamma(\theta)} i(1 - F_{N-l+1})^{i-1} F_{N-l+1}^{-l} f_{N-l+1} & \text{for } i \leq l \\ \frac{E_l(\theta, F)}{A_l(\theta)} i(1 - F_{N-l+1})^{i-1} F_{N-l+1}^{-l} f_{N-l+1} & \text{for } i > l \end{cases} \quad (12)$$

c) the joint conditional p.d.f. of X_i and X_j , $1 \leq i < j \leq N$, conditioned on the event $[N \geq j]$ is

$$g(x_i, x_j) = \begin{cases} \frac{L_j(\theta, F) + \gamma(\theta) \mathcal{L}(l-1)(i-1, j-i-1, l-j)(1-F_j)^l}{A_j(\theta) + \gamma(\theta)} F_i^{l-1} [F_j - F_i]^{j-i-1} [1-F_j]^{-j} f_i f_j & \text{for } j \leq l \\ \frac{L_j(\theta, F)}{A_j(\theta)} F_i^{l-1} [F_j - F_i]^{j-i-1} [1-F_j]^{-j} f_i f_j & \text{for } j > l \end{cases} \quad (13)$$

Proof. Consider the case when $i \leq l$. By (10), we have

$$\begin{aligned} P[N \geq i] &= \sum_{\substack{k \geq i \\ k \in T}} P[N=k] = \sum_{\substack{i \leq k \leq l-1 \\ k \in T}} P[N=k] + P[N=l] + \sum_{\substack{k \geq l \\ k \in T}} P[N=k] = \\ &= \frac{\alpha}{\bar{f}(\theta)} [A_l(\theta) + \gamma(\theta)] \end{aligned}$$

Moreover, in this case, we have

$$\begin{aligned} \sum_{\substack{k \geq i \\ k \in T}} h(x_i | k) P[N=k] &= \frac{\alpha}{\bar{f}(\theta)} \left\{ \sum_{\substack{k \geq i \\ k \in T}} i \binom{k}{i} F_i^{i-1} [1-F_i]^{k-i} f_i a(k) \theta^k + \right. \\ &\quad \left. + \frac{\beta}{\alpha} \bar{f}(\theta) i \binom{l}{i} F_i^{l-1} (1-F_i)^{l-i} f_i = \right. \\ &= \frac{\alpha}{\bar{f}(\theta)} [D_l(\theta, F) + \gamma(\theta) \binom{l}{i} i(1-F_i)^l] F_i^{i-1} i(1-F_i)^{-i} f_i \end{aligned}$$

Hence we obtain the first part of (11).

Similar evaluations allows us to get for $i > l$:

$$P[N > i] = \frac{\alpha}{\bar{f}(\theta)} A_i(\theta)$$

and

$$\sum_{\substack{k \geq i \\ k \in T}} h(x_i | k) P[N = k] = \frac{\alpha}{\bar{f}(\theta)} i F_i^{i-1} (1 - F_i)^{-i} f_i D_i(\theta, F).$$

This leads us to the second part of (11). The formulae (12) and (13) can be obtained in an analogous way.

The p.d.f. of the smallest and largest (when T is finite) order statistics are directly obtained from (11) and (12) by putting $i = 1$.

Lemma 4. Let R be the range of an ordered sample $X_1 \leq X_2 \leq \dots \leq X_N$, where N is a random variable distributed according to (10). Then

$$g(R) = \frac{2}{A_2(\theta) + \gamma(\theta)} \left[\sum_{\substack{k \geq 2 \\ k \in T}} \binom{k}{2} a(k) \theta^k \int_{-\infty}^{\infty} [F(x+R) - F(x)]^{k-2} f(x) f(x+R) dx + \right. \\ \left. + \gamma(\theta) \binom{l}{2} \int_{-\infty}^{\infty} [F(x+R) - F(x)]^{l-2} f(x) f(x+R) dx \right]. \quad (14)$$

Proof. Since the p.d.f. of the range R is, for fixed k ,

$$h(R) = k(k-1) \int_{-\infty}^{\infty} [F(x+R) - F(x)]^{k-2} f(x) f(x+R) dx,$$

then the p.d.f. of R for random N distributed according to (10) is given by (14).

(ii) A sample from a population uniformly distributed in $(0, 1)$.

We now consider a sample $(X_1, X_2, \dots, X_N)$ of size N from a population having a uniform distribution in $(0, 1)$ i.e. $F(x) = x$ for $x \in (0, 1)$.

Theorem 2. If $X_1 \leq X_2 \leq \dots \leq X_N$ is an ordered sample from $F(x) = x$, $x \in (0, 1)$, and N has the distribution given by (10), then

$$g(R) = \frac{2(1-R)}{R^2 [A_2(\theta) + \gamma(\theta)]} \left[\sum_{\substack{k \geq 2 \\ k \in T}} \binom{k}{2} a(k) (\theta R)^k + \gamma(\theta) \binom{l}{2} R^l \right]; \quad (15)$$

for $m \geq 1$

$$E X_m^i = \begin{cases} \frac{i(m+i-1)!}{A_i(\theta) + \gamma(\theta)} \left[\sum_{\substack{k \geq i \\ k \in T}} \binom{k}{i} a(k) \theta^k \frac{(k-i)!}{(m+k)!} + \gamma(\theta) \binom{l}{i} \frac{(l-i)!}{(m+l)!} \right] & \text{for } i \leq l \\ \frac{i(m+i-1)!}{A_i(\theta)} \sum_{\substack{k \geq i \\ k \in T}} \binom{k}{i} a(k) \theta^k \frac{(k-i)!}{(m+k)!} & \text{for } i > l; \end{cases} \quad (16)$$

$$EX_i X_j = \begin{cases} \frac{i(j+1)}{A_j(\theta) + \gamma(\theta)} \left[\sum_{\substack{k \geq j \\ k \in T}} \frac{a(k) \theta^k}{(k+1)(k+2)} + \frac{\gamma(\theta)}{(l+1)(l+2)} \right] & \text{for } j \leq l \\ \frac{i(j+1)}{A_j(\theta)} \sum_{\substack{k \geq j \\ k \in T}} \frac{a(k) \theta^k}{(k+1)(k+2)} & \text{for } j > l. \end{cases} \quad (17)$$

Proof. (15) is a straightforward consequence of (14), while (8) and (9) we obtain from Lemma 3.

(iii) Particular cases.

a) If the random variable N has the inflated binomial distribution with the parameters α, p, n , i.e. the probability function is of the form

$$P[N = k] = \begin{cases} \beta + \alpha \binom{n}{k} p^k q^{n-k} & \text{for } k = l \\ \alpha \binom{n}{k} p^k q^{n-k} & \text{for } k = 0, 1, \dots, l-1, l+1, \dots, n, \end{cases}$$

where $0 < \alpha \leq 1$, $\alpha + \beta = 1$, $0 < p < 1$, $p + q = 1$, $n \in \mathbb{N}$, then the formulae (11) – (17) take the forms:

$$g(x_i) = \begin{cases} \frac{1}{B_i(n, p) + \gamma} [np \binom{n-1}{i-1} (pF_i)^{i-1} (1-pF_i)^{n-i} f_i + \\ + \gamma l \binom{l-1}{i-1} F_i^{l-1} (1-F_i)^{l-i} f_i & \text{for } i \leq l \\ \frac{1}{B_i(n, p)} np \binom{n-1}{i-1} (pF_i)^{i-1} (1-pF_i)^{n-i} f_i & \text{for } i > l, \end{cases} \quad (11a)$$

where $\gamma = \beta/\alpha$ and $B_i(n, p) = \sum_{k=i}^n \binom{n}{k} p^k q^{n-k}$;

$$g(x_{N-i+1}) = \begin{cases} \frac{1}{B_i(n, p) + \gamma} [np^i \binom{n-1}{i-1} (1-F_{N-i+1})^{i-1} (q + pF_{N-i+1})^{n-i} f_{N-i+1} + \\ + \gamma l \binom{l-1}{i-1} (1-F_{N-i+1})^{i-1} F_{N-i+1}^{l-i} f_{N-i+1} & \text{for } i \leq l \\ \frac{1}{B_i(n, p)} np^i \binom{n-1}{i-1} (1-F_{N-i+1})^{i-1} (q + pF_{N-i+1})^{n-i} f_{N-i+1} & \text{for } i > l; \end{cases} \quad (12a)$$

$$g(x_i, x_j) = \begin{cases} \frac{1}{B_j(n, p) + \gamma} [n(n-1)p^j (i-1, j-\frac{n-2}{i-1}, n-j)(1-pF_j)^{n-j} + \\ + \gamma l(l-1)(i-1, j-\frac{l-2}{i-1}, l-j)(1-F_j)^{l-j}] F_i^{l-1} [F_j - F_i]^{j-l-1} f_i f_j & \text{for } j \leq l \\ \frac{1}{B_j(n, p)} n(n-1)p^j F_i^{l-1} [F_j - F_i]^{j-l-1} (i-1, j-\frac{n-2}{i-1}, n-j) \\ (1-pF_j)^{n-j} f_i f_j & \text{for } j > l; \end{cases} \quad (13a)$$

$$g(R) = \frac{1}{B_2(n, p) + \gamma} \left\{ n(n-1)p^2 \int_{-\infty}^{\infty} [p(F(x+R) - F(x)) + q]^{n-2} f(x)f(x+R)dx + \right. \\ \left. + \gamma l(l-1) \int_{-\infty}^{\infty} [F(x+R) - F(x)]^{l-2} f(x)f(x+R)dx \right\}; \quad (14a)$$

$$g(R) = \frac{(1-R)(q+pR)^{-2}}{B_2(n, p) + \gamma} [n(n-1)p^2 (q+pR)^n + \gamma l(l-1)(q+pR)^l]; \quad (15a)$$

$$EX_i^m = \begin{cases} \frac{m+i-1}{(n+m)p} EX_i^{m-1} + \frac{1}{(n+m)[B_i(n, p) + \gamma]} \left[\gamma \frac{(n-l)(m+i-1)l}{p(m+l)} \right. \\ \left. \cdot \left(\frac{l-1}{i-1} \right) \beta(m+i-1, l-i+1) - \left(\frac{n-1}{i-1} \right) np^{l-1} q^{n-l+1} \right] & \text{for } i \leq l \\ \frac{m-i+1}{(n+m)p} EX_i^{m-1} - \frac{n}{(n+m)B_i(n, p)} \left(\frac{n-1}{i-1} \right) p^{i-1} q^{n-l+1} & \text{for } i > l, \end{cases} \quad (16a)$$

$$\text{where } \beta(a, b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx;$$

$$EX_i X_j = \begin{cases} \frac{i(j+1)}{B_j(n, p) + \gamma} \left[\frac{B_{j+2}(n+2, p)}{p^2(n+1)(n+2)} + \frac{\gamma}{(l+1)(l+2)} \right] & \text{for } j \leq l \\ \frac{i(j+2)B_{j+2}(n+2, p)}{(n+1)(n+2)B_j(n, p)} & \text{for } j > l. \end{cases} \quad (17a)$$

b) If the random variable N has the inflated negative binomial distribution with the parameters α, q, n , i.e. the probability function is of the form

$$P[N = k] = \begin{cases} \beta + \alpha \binom{n}{k} p^n (-q)^k & \text{for } k = l \\ \alpha \binom{n}{k} p^n (-q)^k & \text{for } k = 0, 1, \dots, l-1, l+1, \dots \end{cases}$$

where $0 < \alpha \leq 1$, $\alpha + \beta = 1$, $0 < q < 1$, $p + q = 1$, $n \in \mathbb{N}$, then (putting $B_n^*(i, p) = \sum_{k=0}^n \binom{i}{k} p^k q^{i-k}$) the formulae (11) and (13) – (17) take the forms:

$$g(x_i) = \begin{cases} \frac{n \binom{n+i-1}{n-1} p^n q^i F_i^{i-1} f_i + \gamma l \binom{l-1}{i-1} (p + qF_j)^{n+1} [(p + qF_j) F_i]^{i-1} (1 - F_j)^{l-i} f_i}{[B_{n-1}^*(n-i+1, p) + \gamma] (p + qF_j)^{n+i}} & \text{for } i \leq l \\ \frac{n \binom{n+i-1}{n-1} p^n q^i F_i^{i-1} f_i}{B_{n-1}^*(n+i-1, p) (p + qF_j)^{n+i}} & \text{for } i > l; \end{cases} \quad (11b)$$

$$g(x_i, x_j) = \begin{cases} \frac{F_i^{i-1} [F_j - F_i]^{j-i-1} f_i f_j}{[B_{n-1}^*(n+j-1, p) + \gamma] (p + qF_j)^{n+j}} [(i-1, j-i-1, n-1)(n+j-2)(n+j-1) p^n q^j + \gamma(i-1, j-i-1, l-j)(l-1) l (1 - F_j)^{l-j} (p + qF_j)^{n+j}] & \text{for } j \leq l \\ \frac{F_i^{i-1} [F_j - F_i]^{j-i-1} f_i f_j p^n q^j}{B_{n-1}^*(n+j-1, p) (p + qF_j)^{n+j}} (i-1, j-i-1, n-1)(n+j-2)(n+j-1) & \text{for } j > l; \end{cases} \quad (13b)$$

$$g(R) = \frac{p^n q^2}{B_{n-1}^*(n+1, p) + \gamma} \left[\int_{-\infty}^{\infty} \frac{f(x) f(x+R)}{[1 - q(F(x+R) - F(x))]^{n+2}} dx + \right. \\ \left. + \gamma l (l-1) \int_{-\infty}^{\infty} [F(x+R) - F(x)]^{l-2} f(x) f(x+R) dx \right]; \quad (14b)$$

$$g(R) = \frac{p^n q^2 (1-R)}{B_{n-1}^* (n+1, p) + \gamma} \left[\frac{1}{(p+qR)^{n+2}} + \gamma l(l-1) R^{l-2} \right]; \quad (15b)$$

for $m \geq 1$

$$EX_i^m = \begin{cases} \frac{p(m+i-1)}{q(n-m)} EX_i^{m-1} - \frac{1}{(n-m)[B_{n-1}^* (n+i-1, p) + \gamma]} \cdot \left[\gamma \frac{(m-qn+pl)}{q(m+l)} \binom{l-1}{i-1} l(m+i-1) \beta(m+i-1, l-i+1) + \frac{p^n q^{i-1}}{\beta(n, i)} \right] & \text{for } i \leq l \\ \frac{p(m+i-1)}{q(n-m)} EX_i^{m-1} - \frac{p^n q^{i-1}}{(n-m) B_{n-1}^* (n+i-1, p) \beta(n, i)} & \text{for } i > l \end{cases} \quad (16b)$$

$$EX_i X_j = \begin{cases} \frac{i(j+1)}{B_{n-1}^* (n+j-1, p) + \gamma} \left[\frac{p^2 I_9(j+2, n-2)}{q^2 (n-2)(n-1)} + \frac{\gamma}{(l+1)(l+2)} \right] & \text{for } j \leq l \\ \frac{i(j+1) p^2 I_9(j+2, n-2)}{B_{n-1}^* (n+j-1, p) q^2 (n-1)(n-2)} & \text{for } j > l \end{cases} \quad (17b)$$

where

$$\beta(a, b) I_9(a, b) = \int_0^q t^{a-1} (1-t)^{b-1} dt$$

c) If the random variable N has the inflated Poisson distribution with the parameters α, λ , i.e. the probability function is of the form

$$P[N = k] = \begin{cases} \beta + \alpha \frac{\lambda^k}{k!} e^{-\lambda} & \text{for } k = l \\ \alpha \frac{\lambda^k}{k!} e^{-\lambda} & \text{for } k = 0, 1, \dots, l-1, l+1, \dots \end{cases}$$

where $0 < \alpha \leq 1, \alpha + \beta = 1, \lambda > 0$ then the formulae (11) and (13) – (17) take the forms:

$$g(x_i) = \begin{cases} \frac{1}{P_i(\lambda) + \gamma} \left[\frac{\lambda(\lambda F_i)^{i-1} f_i}{(i-1)!} e^{-\lambda F_i} + \gamma l \binom{l-1}{i-1} F_i^{i-1} (1-F_i)^{l-i} f_i \right] & \text{for } i \leq l \\ \frac{\lambda(\lambda F_i)^{i-1} f_i}{P_i(\lambda) (i-1)!} e^{-\lambda F_i} & \text{for } i > l \end{cases} \quad (11c)$$

$$\text{where } P_i(\lambda) = \sum_{k=i}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!},$$

$$g(x_i, x_j) = \begin{cases} \frac{1}{P_j(\lambda) + \gamma} \left[\frac{\lambda^2 (\lambda F_i)^{i-1} e^{-\lambda F_j} [\lambda(F_j - F_i)]^{j-i-1} f_i f_j}{(i-1)! (j-i-1)!} + \right. \\ \left. + \gamma \binom{l-2}{i-1, j-i-1, l-j} (l-1) l F_i^{i-1} [F_j - F_i]^{j-i-1} [1-F_j]^{l-j} f_i f_j \right] & \text{for } j \leq l \\ \frac{\lambda^2 (\lambda F_i)^{i-1} e^{-\lambda F_j} [\lambda(F_j - F_i)]^{j-i-1} f_i f_j}{(i-1)! (j-i-1)! P_j(\lambda)} & \text{for } j > l \end{cases} \quad (13c)$$

$$g(R) = \frac{1}{P_2(\lambda) + \gamma} \left[\lambda^2 e^{-\lambda} \int_{-\infty}^{\infty} e^{\lambda[F(x+R) - F(x)]} f(x) f(x+R) dx + \right. \\ \left. + \gamma l(l-1) \int_{-\infty}^{\infty} [F(x+R) - F(x)]^{l-2} f(x) f(x+R) dx \right] \quad (14c)$$

$$g(R) = \frac{1-R}{P_2(\lambda) + \gamma} [\lambda^2 e^{-\lambda(1-R)} + \gamma l(l-1) R^{l-2}]; \quad (15c)$$

for $m \geq 1$

$$EX_i^m = \begin{cases} \frac{m+i-1}{\lambda} EX_i^{m-1} - \frac{1}{P_i(\lambda) + \gamma} \left[\gamma \frac{(m+i-1)(m+l-\lambda)l}{\lambda(m+l)} \cdot \right. \\ \left. \cdot \binom{l-1}{i-1} \beta(m+i-1, l-i+1) + \frac{\lambda^{i-1}}{(i-1)!} e^{-\lambda} \right] & \text{for } i \leq l \\ \frac{m+i-1}{\lambda} EX_i^{m-1} - \frac{\lambda^{i-1} e^{-\lambda}}{(i-1)! P_i(\lambda)} & \text{for } i > l \end{cases} \quad (16c)$$

$$EX_i X_j = \begin{cases} \frac{i(j+1)}{P_j(\lambda) + \gamma} \left[\frac{P_{j+2}(\lambda)}{\lambda^2} + \frac{\gamma}{(l+1)(l+2)} \right] & \text{for } j \leq l \\ \frac{i(j+1)}{\lambda^2 P_j(\lambda)} P_{j+2}(\lambda) & \text{for } j > l. \end{cases} \quad (17c)$$

Remark. The results of [6] can be obtained from (iii) by setting $\alpha = 1$.

REFERENCES

- [1] Berman, S. M., *Limiting distribution on the maximum term in sequence of dependent random variables*, Ann. Math. Stat. 33 (1962), 894–908.
- [2] Epstein, B., *A modified extreme value problem*, Ann. Math. Stat. 20 (1949), 99–103.
- [3] Noack, A., *Class of random variables with discrete distributions*, Ann. Math. Stat. 21, 1, (1950), 127–132.
- [4] Pandey, K. N., *On generalized Inflated Poisson distribution*, J. Sci. Res. Banares Hindu Univ. 15, 2, (1964–65), 157–162.
- [5] Patil, G. P., *Minimum variance unbiased estimation and certain problems of additive number theory*, Ann. Math. Stat. 34 (1963), 1050–1056.
- [6] Raghundanan K., Patil S. A., *On order statistics for random sample size*, Statist. Neerlandica 26 (1972), nr 4, 121–126.
- [7] Rychlik Z., Szyndal D., *Inflated truncated negative binomial acceptance sampling plan*, Apl. Mat. 22 (1977), 157–165.
- [8] Singh, M. P., *Inflated binomial distribution*, J. Sci. Res. Banares Hindu Univ. 16 (1965–66), 87–90.
- [9] Singh, S. N., *A note of inflated Poisson distribution*, J. Indian Statist. Assoc. 1, 3 (1963), 140–144.
- [10] Singh, S. N., *Probability models for the variation in the number of births per couple*, J. Amer. Statist. Assoc. Vol 58, 33 (1963), 721–727.
- [11] Wilks, S. S., *Mathematical Statistics*, John Wiley and Sons, New York 1962.

STRESZCZENIE

Niech $X_1 < X_2 < \dots < X_N$ będą statystykami porządkowymi dla próby o liczebności N , gdzie N – zmienna losowa o wartościach całkowitych nieujemnych.

W pracy podaje się wzory na rozkłady i momenty statystyk porządkowych, w przypadku, gdy N ma rozkład typu PSD i IPSD.

W szczególności rozpatrzono przypadek, gdy X_1 ma rozkład jednostajny na $(0, 1)$ a N – dwumianowy, ujemny dwumianowy i Poissona oraz powyższe rozkłady uogólnione.

РЕЗЮМЕ

Пусть $X_1 < X_2 < \dots < X_N$ – порядковые статистики выборки объема N , где N – случайная величина, принимающая неотрицательные целые значения.

В работе дается формула для распределений и моментов порядковых статистик в случае, когда N имеет распределение типа степенных рядов и обобщенных степенных рядов.

В частности рассмотрено случай, когда X_1 имеет равномерное распределение на $(0, 1)$ и N – биномиальное, отрицательно-биномиальное и Пуассона распределения и эти же распределения обобщенные.

