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Values Assumed by Gelfer Functions

O zbiorze wartości funkcji Gelfera

О множестве значений функции Гельфера

1. Introductory Remarks

Let Δ be the open unit disk of complex numbers, $\Delta = \{z \in C\colon |z| < 1\}$ and $\mathscr{G}$ the class of "Gelfer functions". $\varphi(z)$ is a Gelfer function if

$$\varphi(z) = 1 + \sum_{k=1}^{\infty} a_k z^k, \quad z \in \Delta, \tag{1.1}$$

and

$$\varphi(z_1) + \varphi(z_2) \neq 0 \quad \text{for all } z_1 \text{ and } z_2 \text{ in } \Delta. \tag{1.2}$$

The latter condition asserts that both a complex number and its negative are never assumed; this property is shared by the class $\mathscr{P}$ of functions of positive real part studied by Carathéodory, therefore $\mathscr{P}$ is a proper subclass of $\mathscr{G}$. Members of $\mathscr{G}$ need not be univalent however, in the present work we will consider only those which are and will denote the univalent subclass of $\mathscr{G}$ by $\mathscr{G}_u$.

The definition and basic properties of $\mathscr{G}$ were given by Gelfer in 1946, [5]. An important tool in his considerations was the relation $\varphi(z) = \dfrac{1+E(z)}{1-E(z)}$ which gives a one-to-one correspondence between Gelfer functions and the Bieberbach-Eilenberg functions. Gelfer showed that $|a_1| \leqslant 2$

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and that $|a_k| \leqslant 2\sqrt{2e}$, for $k \geqslant 2$; subsequently Lebedev and Mamai [12] showed that $|a_2| \leqslant 2.05$ and they conjectured that $|a_2| \leqslant 2$ for all $\varphi(z)$ in $\mathscr{G}$. Hummel [8], using a variational method for the Bieberbach-Eilenberg functions developed jointly with Schiffer [7], arrived at the surprising conclusion that $|a_2| \leqslant 2.00011\cdots$ (which is sharp).

If $\varphi(z)$ is in $\mathscr{G}_u$ and we define $\psi(z)$ by

$$\psi(z)+1 = [\varphi(z)]^2, \tag{1.3}$$

then

$$\psi(0) = 0, \ \psi(z) \neq -1 \tag{1.4}$$

and $\psi(z)$ is regular and univalent for $z \in \varDelta$; we denote the collection of all such $\psi(z)$ by $\mathcal{O}$. An essential property of the class $\mathcal{O}$ which will be exploited below is that its members omit the value -1.

Goluzin [6] has given a variational formula for a univalent function in $\varDelta$ which omits m pre-assigned values. With proper choice of parameters, the case $m = 1$ gives a variational formula for $\mathcal{O}$ which is equivalent to that of Hummel [8]. This variational method is used below to find the region of values (variability region) for the class $\mathcal{O}$ and as a consequence of (1.3) and (1.4) for the class $\mathscr{G}_u$. The techniques used are similar to those of Krzyż [10,11] and Złotkiewicz [15,16].

2. Some Preliminary Results

Here we re-write the formula of Goluzin to conform to our situation, give some auxiliary tools and review the notion of a regular point of the boundary of a region in the plane.

Lemma 1. *If $\psi(z)$ is in $\mathcal{O}$, λ is real, $z_0 \in \varDelta$ and A is any complex number, then*

$$\psi^*(z) = \psi(z)+\lambda A\psi(z)\frac{\psi(z)+1}{\psi(z)-\psi(z_0)} - \lambda Az\psi'(z)\frac{[\psi(z_0)+1]\psi(z_0)}{z_0[\psi'(z_0)]^2[z-z_0]} + \lambda\bar{A}z^2\psi'(z)\frac{\overline{[\psi(z_0)+1]\psi(z_0)}}{\bar{z}_0\overline{[\psi'(z_0)]^2}[1-\bar{z}_0z]} + o(\lambda) \tag{2.1}$$

is likewise in $\mathcal{O}$.

This is obtained from Goluzin [6, p. 109] by choosing $m = 1$ and $a_1 = -1$.

Replacing $\psi(z_0)$ by a value σ exterior to the set of all $\psi(z)$ and retaining the first two terms of (2.1) gives an "exterior" variation of the form

$$\tilde{\psi}(z) = \psi(z)+B\psi(z)\frac{\psi(z)+1}{\psi(z)-\sigma}, \tag{2.2}$$

where B is a complex number close to the origin [6, p. 108].

We will require two other variations which are summarized in the following statement.

Lemma 2. *If $\psi(z)$ is in $\mathcal{O}$, then so are the functions*

$$\tilde{\psi}(z) = \psi(e^{i\theta}z) = \psi(z) + i\theta z\psi'(z) + o(\theta), \tag{2.3}$$

for θ real, and

$$\psi^0(z) = \psi(z) - z\psi'(z)\,\frac{1-z_0 z}{1+z_0 z}\,t + \mathcal{O}(t), \tag{2.4}$$

or $0 \leqslant t < 1$.

Let $k(z)$ be the Koebe function $k(z) = z(1-z)^{-2}$ and $g(z, t) = k^{-1}[(1-t)k(z)]$ be the corresponding quasi-starlike function (see [13], for example) which maps Δ onto itself out along the negative axis. The variation (2.4) is given by $\psi(g(z, t))$.

Finally we review the notion of a non-singular (or regular) boundary point of a region given by Biernacki [4] and Schaeffer and Spencer [14]. Let D be a domain in the complex plane and ∂D its boundary; a point b in ∂D is a non-singular boundary point of D if there exists a point a in $C \setminus \overline{D}$ and a non-degenerate disk $\Delta(a; r)$ centered at a and with radius r which has the property that ∂D meets $\overline{\Delta(a; r)}$ in only the point b. ($\overline{D}$ denotes the closure of set D.)

It is known [14] that the set of non-singular boundary points of the region of values assumed by a compact family of univalent functions is a dense subset of that boundary. Now $\mathcal{G}_u$ is not compact, however adjoining the function identically 1 yields a compact family which we will again denote by $\mathcal{G}_u$ whenever we need compactness; $\mathcal{O}$ is treated analagously by adjoining the function which is identically zero. It follows that the boundary of the variability region of either class has the property asserted above. In the remainder of this paper we aim characterizing the boundaries of these variability regions by determining their non-singular points.

3. The Differential Equation.

In this section we derive the fundamental equation satisfied by functions corresponding to boundary points of the variability region of $\mathcal{O}$ for z' in Δ. Let R, or more precisely $R(z')$, be this region, i.e.,

$$R = \{w \in C \mid w = \psi(z') \text{ for a } \psi(z) \text{ in } \mathcal{O}\}. \tag{3.1}$$

Suppose b is a non-singular boundary point of R and suppose $\psi(z)$ is an extremal which gives b, i.e., $b = \psi(z)$ for the given z in Δ. If a (dependent on $\psi(z)$) is a point in $C \setminus R$ corresponding to $\psi(z)$ in the definition above, then $\psi(z)$ provides the sharp lower bound for

$$\min\{|g(z)-a|:\ g(z) \text{ in } \mathcal{O}\}. \tag{3.2}$$

This means that $|\psi(z)-a| \geqslant |g(z)-a|$ if $g(z)$ is any one of the functions in $\mathcal{O}$ given by (2.1), (2.2), (2.3) or (2.4). In particular, choosing $\psi^*(z)$ from (2.1), we may write

$$|\psi^*(z)-a|^2 \geqslant |\psi(z)-a|^2. \tag{3.3}$$

Letting $\arg\{\psi(z)-a\} = \alpha$, using the representation (2.1), letting $\lambda \to 0$ and performing appropriate computations we obtain

$$e^{-i\alpha}\left\{\psi(z)\frac{\psi(z)+1}{\psi(z)-\psi(z_0)} - z\psi'(z_0)\frac{\big(\psi(z_0)+1\big)\psi(z_0)}{z_0[\psi'(z_0)]^2(z-z_0)}\right\} + e^{i\alpha}\frac{\bar{z}^2\overline{\psi'(z)}\psi(z_0)\big(\psi(z_0)+1\big)}{z_0[\psi'(z_0)]^2(1-z_0\bar{z})} = 0, \tag{3.4}$$

or in a more symmetric form

$$e^{-i\alpha}\frac{\psi(z)\big(\psi(z)+1\big)}{\psi(z_0)\big(\psi(z_0)+1\big)}\cdot\frac{z_0[\psi'(z_0)]^2}{\psi(z)-\psi(z_0)} = e^{-i\alpha}\frac{z\psi'(z)}{z-z_0} - e^{i\alpha}\frac{\overline{\bar{z}^2\psi'(z)}}{1-z_0\bar{z}}. \tag{3.5}$$

Now we simplify the right side of (3.5). By applying the variation given by (2.3) as in (3.3) and in the successive computations specifically replacing $\psi^*(z)$ in (3.3) by $\hat{\psi}(z)$, we find that the number $\big(e^{-i\alpha}z\psi'(z)\big)$ is real. Applying the variation given by (2.4) in the same fashion we conclude that

$$\operatorname{Re}\left\{e^{-i\alpha}z\psi'(z)\frac{1-zz_0}{1+zz_0}\right\} \leqslant 0,$$

and that

$$\big(e^{-i\alpha}z\psi'(z)\big) < 0. \tag{3.6}$$

By taking (3.6) into account and simplifying (3.5) we arrive at the differential equation

$$e^{-i\alpha}\frac{w_0(1+w_0)\,dw^2}{w(1+w)(w_0-w)} = \frac{c\,d\zeta^2}{\zeta(z-\zeta)(1-\zeta\bar{z})}, \tag{3.7}$$

where $c = e^{-i\alpha}z\psi'(z)(1-|z|^2)$, w_0 corresponds to $\psi(z)$ and w to $\psi(z_0)$. z is fixed and we have replaced z_0 by ζ. [Note that by rotating Δ we may choose z to be real and replace $\bar{z}$ by z in the last form if we wish.]

We state the main results of this section.

Theorem 1. *If $\psi(\zeta)$ corresponds to a non-singular boundary point of R_A, different from ∞ where R is the variability region of $\mathcal{O}$, then $\psi(\zeta)$ satisfies equation (3.7). Furthermore $\psi(\zeta)$ maps Δ onto the slit from -1 to ∞ along an analytic arc.*

To justify the last statement we apply variation (2.2) under the assumption that the complement of $\psi[\Delta]$ contains a neighborhood. This gives a contradiction which shows that this complement is a union of arcs containing ∞ and that $\psi(\eta) = -1$ for some $\eta \in \partial\Delta$. Rewriting (3.7) in the form

$$(3.7)' \qquad \frac{[\psi'(\zeta)]^2}{\psi(\zeta)(1+\psi(\zeta))(\psi(z)-\psi(\zeta))} = \frac{A}{\zeta(r-\zeta)(1-\zeta r)},$$

where A is a suitable constant dependent on z and $r = |z|$, and comparing both sides of (3.7)′ for regularity we conclude that $\psi'(\zeta)$ has only a simple zero at η_0 and consequently that -1 is the end of a simple arc. We see in the same way that $\psi(\zeta)$ has a simple pole on $\partial\Delta$. Then the conclusion follows by further direct analysis of (3.7)′ or by an application of Theorem 3.2, [9].

4. The Region R.

Here solutions to (3.7) are given in closed form in terms of the pe function of Weierstrass, then the boundary of R is obtained as the solution of an equation given in terms of a related elliptic modular function.

We begin by finding solutions of (3.7) which lie in $\mathcal{O}$. To ease the representation we let

$$(4.1) \qquad P(\zeta) = \zeta(r-\zeta)(1-\zeta r),$$

having made the assumption $z = r$. Now choosing and fixing either branch of $\sqrt{P(\zeta)}$ in the disk Δ cut along the segment $[0, r]$ we define the modules

$$\Omega_1 = 2\int_0^r \frac{\sqrt{c}}{\sqrt{P(\zeta)}}\,d\zeta,$$

$$(4.2) \qquad \Omega_2 = 2\int_r^\eta \frac{\sqrt{c}}{\sqrt{P(\zeta)}}\,d\zeta,$$

$$\Omega_3 = \Omega_1 + \Omega_2,$$

where $\eta = e^{i\theta}$ and $w(\eta) = -1$.

A solution of (3.7) can now be written as

$$(4.3)\qquad w(\zeta) = A\mathscr{P}\left[\int_{\tau}^{\zeta} \frac{\sqrt{c}}{\sqrt{P(\zeta)}}\,d\zeta;\ \Omega_1,\ \Omega_2\right] + B,$$

with $w(\tau) = \infty$, $\tau = e^{i\gamma}$ and A and B being constants to be determined to guarantee that $w(\zeta)$ lies in $\mathcal{O}$. [Hereafter, if no confusion arises, we will write the pe function with but a single argument.]

We now show that $w(\zeta)$ is single-valued. Because $\dfrac{c}{P(\zeta)}\,d\zeta^2$ is real and positive when $\zeta \in \partial\Delta$ it follows from the extremal character of $w(\zeta)$, see ([14], [16]), that

$$(4.4)\qquad \int_{\gamma}^{\beta} \sqrt{\left|\frac{c}{P(\zeta)}\right|}\,d\theta = \int_{\gamma}^{\beta+2\pi} \sqrt{\left|\frac{c}{P(\zeta)}\right|}\,d\theta$$

where $\zeta = e^{i\theta}$. Then if we consider closed paths inside Δ enclosing segment $[0, r]$ and homotopic to $\partial\Delta$ we have

$$(4.5)\qquad 2\int_{\gamma}^{\beta} \sqrt{\left|\frac{c}{P(\zeta)}\right|}\,d\theta = \int_{\gamma}^{\gamma+2\pi} \sqrt{\left|\frac{c}{P(\zeta)}\right|}\,d\theta = \int_{0}^{2\pi} \sqrt{\left|\frac{c}{P(\zeta)}\right|}\,d\theta = \Omega_1.$$

Let $[\eta, \zeta]$ denote the segment joining η to ζ when either 0 or r does not lie on the segment and otherwise let it denote the segment with suitable semi-circular paths about the singular points 0 or r. Then

$$(4.6)\qquad I(\zeta) = \int_{[\eta,\zeta]} \sqrt{\frac{c}{P(\zeta)}}\,d\zeta$$

is single-valued in Δ. Now, for any path in Δ joining η to ζ, we have

$$(4.7)\qquad \int_{\eta}^{\zeta} \sqrt{\left|\frac{c}{P(\zeta)}\right|}\,d\zeta = \begin{cases} I(\zeta) + m\Omega_1 \\ \Omega_2 - I(\zeta) + n\Omega_1, \end{cases}$$

for suitable integers m and n. Then the integral appearing in (4.3) can be written as

$$(4.8)\qquad \int_{\tau}^{\zeta} \sqrt{\frac{c}{P(\zeta)}}\,d\zeta = \int_{\tau}^{\eta} \sqrt{\frac{c}{P(\zeta)}}\,d\zeta + \int_{\eta}^{\zeta} \sqrt{\frac{c}{P(\zeta)}}\,d\zeta = \begin{cases} \left(I(\zeta) + \dfrac{\Omega_1}{2}\right) + m\Omega_1 \\[2ex] -\left(I(\zeta) - \dfrac{\Omega_1}{2}\right) + \Omega_2 + n\Omega_1, \end{cases}$$

having used the fact that the first integral in (4.5) corresponds to an integral from τ to η on $\partial\Delta$. The last form in (4.8) can be written as $-\left(I(\zeta)+\dfrac{\Omega_1}{2}\right)+\Omega_2+(n-1)\Omega_1$. Finally, because the pe function is both periodic and even, we have

$$\mathscr{P}\left(\int_\tau^\zeta \sqrt{\frac{c}{P(\zeta)}}\,d\zeta\right)=\mathscr{P}\left(I(\zeta)+\frac{\Omega_1}{2}\right). \tag{4.9}$$

This shows that $w(\zeta)$ is single-valued in Δ; the univalence of $w(\zeta)$ is a deeper consequence of the form of the quadratic differential (3.7) due to Teichmuller (see [2], for example).

Using the mapping properties of (4.3) we arrive at

$$w(\zeta)=\frac{\mathscr{P}\left(I(\zeta)+\frac{\Omega_1}{2}\right)-\mathscr{P}\left(I(0)+\frac{\Omega_1}{2}\right)}{\mathscr{P}\left(I(0)+\frac{\Omega_1}{2}\right)-\mathscr{P}\left(I(\eta)+\frac{\Omega_1}{2}\right)}. \tag{4.10}$$

It should be noted that because the pe function is homogeneous all constants other than w_0 in (3.7), play no role in the form (4.10) and the one remaining parameter is η. By (4.2) we may write $I(0)=\frac{1}{2}(\Omega_1+\Omega_2)$,

$$w(\zeta)=\frac{\mathscr{P}\left(I(\zeta)+\frac{\Omega_1}{2}\right)-\mathscr{P}\left(\frac{\Omega_2}{2}\right)}{\mathscr{P}\left(\frac{\Omega_2}{2}\right)-\mathscr{P}\left(\frac{\Omega_1}{2}\right)}, \tag{4.11}$$

and finally that a boundary point of R is given by

$$w(r;\eta)=\frac{\mathscr{P}\left(\frac{\Omega_3}{2}\right)-\mathscr{P}\left(\frac{\Omega_2}{2}\right)}{\mathscr{P}\left(\frac{\Omega_2}{2}\right)-\mathscr{P}\left(\frac{\Omega_1}{2}\right)} \tag{4.12}$$

where the notation was altered to emphasize the dependence on η. [Also one may write $\Omega_1=\Omega_1(r)$ and $\Omega_2=\Omega_2(\eta, r)$ to clarify the roles played by η and r.]

By appealing to standard notation used for elliptic functions (see [1, 3], for example) we observe that

$$w(r;\eta)=\frac{e_3-e_2}{e_2-e_1}. \tag{4.13}$$

Now, by writing

$$t(\eta) = \frac{\Omega_2}{\Omega_1}, \tag{4.14}$$

we see that

$$w(r;\eta) = \frac{\lambda(t)}{1-\lambda(t)} = -\lambda(1+t), \tag{4.15}$$

where $\lambda(t)$ is the elliptic modular function defined in terms of $\mathscr{P}$, [1, 3]. If $t(\eta)$ is real (this corresponds to the case when $\eta = 1$), then the solution of (3.7) corresponds to the Koebe function.

Summarizing of the results of this section gives rise to the following result.

Theorem 2. *For fixed z the region of values R of functions in $\mathcal{O}$ is the set bounded by the points satisfying the equation*

$$w = \lambda(1+t). \tag{4.16}$$

Relation (1.3) now makes it possible to state our principal conclusion.

Theorem 3. *If $\varphi(\zeta)$ is a univalent Gelfer function and z is fixed in Δ, then $\varphi(z)$ lies in the closure of the domain bounded by the curve given by*

$$w = \sqrt{1-\lambda(1+t)}, \tag{4.17}$$

where λ and t have meanings given in (4.2), (4.14) *and* (4.16) *and the branch in* (4.17) *is taken so that* -1 *is not covered.*

Finally, if we make use of the connection between Gelfer functions and the Bieberbach-Eilenberg functions mentioned above, we may draw the following conclusion.

Corollary. *If $E(\zeta)$ is a univalent Bieberbach-Eilenberg function and z is fixed in Δ, then $E(z)$ lies in the closure of the region bounded by the curve whose points satisfy the equation*

$$w = \frac{\sqrt{1-\lambda(1+t)}-1}{\sqrt{1-\lambda(1+t)}+1}, \tag{4.18}$$

where λ and t have the meaning given above.

It is known [5] that every Gelfer function and every Bieberbach-Eilenberg function is subordinate to a univalent one of the same variety, therefore Theorem 3 and the corollary hold true without the assumption of univalence.

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STRESZCZENIE

Niech G oznacza rodzinę funkcji analitycznych w kole jednostkowym Δ i takich, że $f(z)+f(u)\neq 0$ dla $z,u\in\Delta$ i niech G_u będzie podklasą funkcji jednolistnych. Wykorzystując związki między funkcjami klasy G_u i funkcjami nie przyjmującymi w kole Δ wartości -1 znaleziono wzory wariacyjne w klasie G_u a następnie wyznaczono dokładny zbiór wartości funkcjonału $f(z)$, z ustalone, $f\in G$.

РЕЗЮМЕ

Пусть G обозначает семейство аналитических функций в единичном круге Δ и таких, что $f(z)+f(u) \neq 0$ для $z, u \in \Delta$ и пусть G_u будет подклассом однолистных функций. Используя связи между функциями класса G_u и функциями не принимающими в круге Δ значения -1 найдено вариационные формулы в классе G_u а потом определено точное множество значений функционала $f(z)$, z фиксировано, $f \in G$.