ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN-POLONIA
VOL. XXXI, 8 SECTIO A 1977

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On a Probabilistic Generalization of Banach's Fixed Point Theorem

O probabilistycznym uogólnieniu twierdzenia Banacha o punkcie stałym.

О вероятностном обобщении теоремы Банаха о неподвижной точке.

1. The content of the well-known theorem of Banach is that a contraction mapping on a complete metric space has a fixed point and that every sequence of iterates converges to this point. Some probabilistic analogues of that theorem and extension to a complete Menger space can be found e.g. in [1], [2], [4], and [5].

The aim of this note is to give a generalization of the main result of [2].

The axiomatic characterization of probabilistic metric spaces PM is quite similar to that of a metric space. Definitions given here can be found in [3], [4], and [5].

Let I denote the closed unit interval $[0, 1]$ and Δ the set of all distribution functions F with $F(0) = 0$. $H \in \Delta$ is defined by

$$H(x) = \begin{cases} 0, & \text{if } x \leqslant 0, \\ 1, & \text{if } x > 0. \end{cases}$$

By t-norm we mean a function T mapping $I \times I$ into I such that:

1°. $\bigwedge_{a \in I} T(a, 1) = a$

2°. $\bigwedge_{a,b \in I} T(a, b) = T(b, a)$

3°. $\bigwedge_{a,b,c \in I} T(a, T(b, c)) = T(T(a, b), c)$

4°. $\bigwedge_{a,b,c,d \in I} [(a \leqslant c \wedge b \leqslant d) \Rightarrow T(a, b) \leqslant T(c, d)]$.

A probabilistic metric space is an ordered pair $(S, \mathscr{F})$ where S is an abstract set, and

$$\mathscr{F}: S \times S \to \Delta \text{ with } \mathscr{F}(p, q) = F_{pq},$$

where the functions F_{pq} are assumed to satisfy the following conditions:

(I) $\bigwedge_{p,q\in S} F_{pq} = H \Leftrightarrow p = q$, i.e. $\bigwedge_{p,q\in S}\bigwedge_{x>0} F_{pq}(x) = 1 \Leftrightarrow p = q$,

(II) $\bigwedge_{p,q\in S} F_{pq} = F_{qp}$,

(III) $\bigwedge_{p,q,r\in S}\bigwedge_{x,y\in R} [(F_{pq}(x) = 1 \wedge F_{qr}(x) = 1) \Rightarrow F_{pr}(x+y) = 1]$.

$F_{pq}(x)$ is interpreted as the probability that the distance between p and q is less than x.

A probabilistic Menger space or shortly Menger space, M-space, is an ordered triple $(S, \mathscr{F}, T)$, where $(S, \mathscr{F})$ is an PM-space, and T is a t-norm such that

(IVm) $\bigwedge_{p,q,r\in S}\bigwedge_{x,y\in R} F_{pr}(x+y) \geqslant T(F_{pq}(x), F_{qr}(y))$.

Let $(S, \mathscr{F})$ be a PM-space. A mapping $M: S \to S$ is a contraction map on $(S, \mathscr{F})$ if and only if there exists an $k \in (0, 1)$ such that

$$\bigwedge_{p,q\in S}\bigwedge_{x>0} F_{M(p)M(q)}(x) \geqslant F_{pq}(x/k).$$

Let $(S, \mathscr{F}, T)$ be any M-space with T such that $\sup_{x<1} T(x, x) = 1$. Sequential convergence on any M-space is defined in a natural way.

A sequence $\{p_n, n \geqslant 1\}$ is sequentially convergent, we say convergent to $p \in S$, if

$$\bigwedge_{x>0} \lim_{n\to\infty} F_{p_n p}(x) = 1.$$

A sequence $\{p_n, n \geqslant 1\}$ is said to be a Cauchy sequence if

$$\bigwedge_{x>0} \lim_{\substack{n\to\infty \\ m\to\infty}} F_{p_n p_m}(x) = 1.$$

A M-space is said to be complete if every Cauchy sequence is convergent.

Let $\mathscr{U} \subset 2^{S\times S}$ be a class of sets defined as follows

$$\mathscr{U} = \{U(\varepsilon, \lambda): \varepsilon > 0,\ 0 < \lambda < 1\} =$$
$$= \{[(p, q): F_{pq}(\varepsilon) > 1 - \lambda];\ \varepsilon > 0,\ 0 < \lambda < 1\}.$$

It has been shown in [3] that $\mathscr{U}$ is a base of neighbourhoods of a Hausdorff uniform structure. This uniform structure induces a metrizable topology $\tau_{\varepsilon,\lambda}$ on S [3].

Then

$$p_n \xrightarrow[n\to\infty]{\tau_{\varepsilon,\lambda}} p \quad \bigwedge_{0<\varepsilon,\lambda<1}\ \bigvee_{n_{\varepsilon,\lambda}}\ \bigwedge_{n\geqslant n_{\varepsilon,\lambda}} F_{p_n p}(\varepsilon) > 1 - \lambda.$$

It is known that:

a) A sequence $\{p_n, n \geqslant 1\}$ on an M-space is a Cauchy sequence if and only if for any $0 < \varepsilon, \lambda < 1$, there exists $n_{\varepsilon,\lambda}$ such that for all $m, n \geqslant n_{\varepsilon,\lambda}$ $F_{p_m p_n}(\varepsilon) > 1 - \lambda$;

b) An M-space $(S, \mathscr{F}, T)$ is complete if and only if every Cauchy sequence converges in $\tau_{\varepsilon,\lambda}$ topology in S.

Let $(\Omega, \mathscr{A}, P)$ be a probability space. A sequence $\{U_n, n \geqslant 1\}$ of mappigs U_n: $S \times \Omega \to S$ is called to be global almost surely convergent to a funcnion $U: S \times \Omega \to S$ if

$$\bigvee_{A \in \mathscr{A}, P(A)=1} \bigwedge_{p \in S} \bigwedge_{\omega \in A} \lim_{n\to\infty} U_n(p, \omega) = U(p, \omega).$$

2. We now prove the following

Theorem. Let $(S, \mathscr{F}, \mathrm{Min})$ *be a complete M-space. Suppose that* M_{ni}, $n = 1, 2, \ldots; i = 1, 2, \ldots, s$ $n \geqslant 1$ *is a rectangular array of mappings* M_{ni}: $S \to S$, $n \geqslant 1$; $i = 1, 2, \ldots, s$ *satisfying the conditions:*

1°. $$\bigwedge_{n\geqslant 1} \bigwedge_{1\leqslant i\leqslant s} \bigvee_{k_{ni}} \bigwedge_{x>0} F_{M_{ni}(p) M_{ni}(q)}(x) \geqslant F_{pq}(x/k_{ni})$$

(the Lipschitz condition),

2°. $$\bigwedge_{n\geqslant 1} \bigwedge_{m\geqslant 1} \bigvee_{k_{nm}} \bigwedge_{1\leqslant i,j\leqslant s} \bigwedge_{p\in S} \bigwedge_{x>0} F_{M_{ni}(p)p}(x) \geqslant F_{M_{mj}(p)p}(x/k_{nm}).$$

Furthermore, let $\{X_n, n \geqslant 1\}$ *be a sequence of random variables such that*

$$X_n: \Omega \to \{k_{n1}, k_{n2}, \ldots, k_{ns}\}, \ n \geqslant 1.$$

Define a sequence $\{U_n, n \geqslant 1\}$ *of random iterates by* U_n: $S \times \Omega \to S$, *where*

$$U_n(p, \omega) = \begin{cases} p, & n = 0 \\ M_{ni_n}(U_{n-1}(p, \omega)), & \text{if } X_n(\omega) = k_{ni_n}, \ i_n = 1, 2, \ldots, s, \ n \geqslant 1. \end{cases}$$

If $\lim\limits_{n\to\infty} (\prod\limits_{k=1}^{n} X_k)^{1/n} = \tau$ *with probability* 1, *where* τ *is a constant belonging to* $[0, 1)$, *then the sequence* $\{U_n, n \geqslant 1\}$ *converges uniformly almost surely to a constant-value function* p_0, *and* $M_{ni}(p_0) = p_0$, $n \geqslant 1$, $i = 1, 2, \ldots, s$.

Proof. Because $\lim\limits_{n\to\infty} (\prod\limits_{k=1}^{n} X_k)^{1/n} < 1$, there must exist a $k_{n_0 i_{n_0}} < 1$, $i_{n_0} \in \{1, 2, \ldots, s\}$, $n_0 \geqslant 1$. Therefore, $M_{n_0 i_{n_0}}$ is a contraction mapping on the complete M-space $(S, \mathscr{F}, \mathrm{Min})$. Hence, by [4], the mapping $M_{n_0 i_{n_0}}$ has a unique fixed point, say $p_0 \in S$. Now from 2°., we see that

$$\bigwedge_{n\geqslant 1} \bigvee_{k_{nn_0}} \bigwedge_{1\leqslant i\leqslant s} \bigwedge_{x>0} F_{M_{ni}(p_0)p_0}(x) \geqslant F_{M_{n_0 i_{n_0}}(p_0)p_0}(x/k_{nn_0}) = F_{p_0 p_0}(x/k_n) = 1$$

and by the axioms of M-space, $M_{ni}(p_0) = p_0$ $n \geq 1$, $i \in \{1, 2, \ldots, s\}$ Take now $A \in \mathscr{A}$ with $P(A) = 1$ such that

$$\bigwedge_{\omega \in A} \lim_{n\to\infty} \Big(\prod_{k=1}^{n} X_k(\omega)\Big)^{1/n} = \tau.$$

We shall show now that

$$\bigwedge_{p \in S} \bigwedge_{\omega \in A} \bigwedge_{x>0} \lim_{n\to\infty} F_{U_n(p,\omega)p_0}(x) = 1 .$$

Assume that $X_n(\omega) = k_{ni_n^0}$, $n \geq N$, $i_n^0 \in \{1, 2, \ldots, s\}$. Then, using $M_{ni_n^0}(p_0) = p_0$, $n \geq 1$, $i_n^0 \in \{1, 2, \ldots, s\}$, and 1°., we have for $n \geq N$

$$F_{U_n(p,\omega)p_0}(x) = F_{U_n(p,\omega)M_{ni_n^0}(p_0)}(x) = F_{M_{ni_n^0}(U_{n-1}(p,\omega))M_{ni_n^0}(p_0)}(x)$$

$$\geq F_{U_{n-1}(p,\omega)p_0}(x/k_{ni_n^0}) = F_{U_{n-1}(p,\omega)p_0}\left(\frac{x}{X_n(\omega)}\right) \geq \ldots \geq F_{pp_0}\left(\frac{x}{\prod_{k=1}^{n} X_k}\right).$$

But $\lim_{n\to\infty} \prod_{k=1}^{n} X_k(\omega) = 0$, $\omega \in A$. Since F_{pp_0} is a distribution function, then

$$\lim_{n\to\infty} F_{U_n(p,\omega)p_0}(x) = \lim_{n\to\infty} F_{pp_0}\left(\frac{x}{\prod_{k=1}^{n} X_k}\right) = 1,\ \omega \in A,$$

which proves that $U_n(p, \omega)$ converges global almost surely to the constant $p_0 \in S$, and moreover, $M_{ni}(p_0) = p_0$ $n \geq 1$, $i = 1, 2, \ldots, s$.

The above Theorem with $M_{ni} \equiv M_i$, $n \geq 1$, $i = 1, 2, \ldots, s$, $X_n\colon \Omega \to \{k_1, k_2, \ldots, k_s\}$, and

$$U_n(p, \omega) = \begin{cases} p, & n = 0 \\ M_i\big(U_{n-1}(p, \omega)\big), & \text{if } X_n(\omega) = k_i, \end{cases}$$

is an extension of the main result of [2] on a complete M-space.

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STRESZCZENIE

Twierdzenie Banacha o punkcie stałym, w którym iteracje zależą od wartości zmiennych losowych zostało dowiedzione w [2]. Celem tej noty jest rozszerzenie tego twierdzenia z przestrzeni metrycznej na probabilistyczną przestrzeń metryczną.

РЕЗЮМЕ

Теорема Банаха о неподвижной точке, в которой интерации зависимы от значения случайных величин доказано в [2]. Целью настоящей заметки является расширение этой теоремы из метрического пространства на вероятностное метрическое пространство.