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Instytut Matematyki, Uniwersytet Marii Curie Skłodowskiej, Lublin

JAN SZYNAL, JÓZEF WANIURSKI

# Some Problems for Linearly Invariant Families*

Pewne problemy dla rodzin liniowo-niezmienniczych

Некоторые проблемы для линейно-инвариантных семейств

## 1. Introduction. Notations

One of the most interesting family of locally univalent functions

(1) $$\varphi(z) = z+b_2z^2+\ldots$$

which are analytic in the unit disk $K = \{z: |z|<1\}$ is the class $V_k (k \geqslant 2)$ of functions with bounded boundary rotation at most $k\pi$ in $K$. Namely, we say that $\varphi \in V_k$ if for every $r \in [0, 1)$

$$\int_0^{2\pi} \left| \mathrm{Re}\left(1+re^{i\theta}\frac{\varphi''(re^{i\theta})}{\varphi'(re^{i\theta})}\right)\right| d\theta \leqslant k\pi.$$

It is well known that $\varphi \in V_k$ if and only if

$$\varphi'(z) = \exp\left\{-\int_0^{2\pi} \log(1-ze^{-it})\,d\mu(t)\right\},$$

where $\mu$ is real valued function with bounded variation on $[0; 2\pi]$ with

$$\int_0^{2\pi} d\mu(t) = 2, \int_0^{2\pi} |d\mu(t)| \leqslant k.$$

Another very useful necessary and sufficient condition for $\varphi$ to be an element of the class $V_k$ has been done by Brannan [1].

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**Lemma 1.** [1]. *A function $\varphi$ belongs to $V_k$ if and only if there exist two normalized starlike functions $s_1, s_2$ such that*

$$\varphi'(z) = \left[\frac{s_1(z)}{z}\right]^{\frac{k+2}{4}} \Big/ \left[\frac{s_2(z)}{z}\right]^{\frac{k-2}{4}}, \quad z \in K. \tag{2}$$

The condition (2) can be written in the following form

$$\varphi'(z) = [g_1'(z)]^{\frac{2+k}{4}} [g_2'(z)]^{\frac{2-k}{4}}, \quad z \in K, \tag{3}$$

where $g_1, g_2$ are two normalized convex functions. In this way we may say that the class $S^c$ of convex normalized functions generates the class $V_k$ by the formula (3). In the same manner we can define another class of functions using instead of $S^c$ for example the class $S$ of normalized univalent functions in $K$. Looking at this problem somewhat more generally, we consider the following class $\mathscr{F}$.

**Definition 1.** We say that $F \in \mathscr{F}$ if $F$ is analytic in $K$ and its derivative $F'$ has the form

$$F'(z) = \prod_{j=1}^{n} [f_j'(z)]^{a_j}, \quad a_j \text{ is real}, \quad \sum_{j=1}^{n} a_j = 1, \quad z \in K, \tag{4}$$

where $f_j \in \mathfrak{M}_j$, $j = 1, 2, \ldots, n$, and $\mathfrak{M}_j$ is linearly invariant family in the sense of Pommerenke [10].

For some known families $\mathfrak{M}_j$ we will determine here the region

$$D(z, a) = \left\{ w \in C\colon\ w = \log\frac{F'(z)}{F'(a)},\ F \in \mathscr{F} \right\}, \quad D(z, 0) = D(z), \tag{5}$$

for fixed $z, a \in K$. As a corollary we obtain the sharp estimates for $|F'(z)|$ and $|\arg F'(z)|$.

Moreover, we will determine the region $D(z)$ for the class of $\beta$-close-to-$V_k$ functions defined below.

The fact that $\mathscr{F}$ is linearly invariant family for which we know $\max|\arg F'(z)|$, $z \in K$, allow us to find the radii of univalence and close-to-convexity of $\mathscr{F}$. In particular we show that if $f$ is univalent ($f \in S$) and $a \in [0; 1]$ then the integral $F(z) = \int_0^z (f'(\xi))^a d\xi$ is univalent at least in the disk $|z| < 0{,}81$.

**Definition 2.** Let $\beta \geqslant 0$. An analytic function $f$ of the form

$$f(z) = z + a_2 z^2 + \ldots, \quad z \in K \tag{6}$$

is $\beta$-close-to-$V_k$ function if there exist a real number $a$, and a function $\varphi \in V_k$ such that

$$\left|\arg e^{ia}\frac{f'(z)}{\varphi'(z)}\right| \leqslant \beta\frac{\pi}{2}, \quad z \in K. \tag{7}$$

The class of $\beta$-close-to-$V_k$ functions which we will denote by $LV(\beta, k)$ has been considered in [3].

In particular, we have
$LV(\beta, 2) = L\beta$ = the class of $\beta$-close-to-convex functions;
$LV(1, 2) = L$ = the class of close-to-convex functions;
$LV(0, k) = V_k$; $LV(0, 2) = S^c$, e.g. [5], [7], [8], [13] respectively.

We will need the class $\mathscr{P}$ of all analytic functions in $K$ of the form

$$(8)\qquad p(z) = e^{i\delta}+p_1z+p_2z^2+\ldots \ (\delta \text{ is real})$$

which satisfy the condition

$$\operatorname{Re} p(z) > 0, \ z \in K.$$

Finally, let us define the function which will play further an important role:

$$(9)\quad F_{\beta,k}(z) = \int_0^z \frac{(1-\xi e^{i\theta_2})^{\beta+\frac{k}{2}-1}}{(1-\xi e^{i\theta_1})^{\beta+\frac{k}{2}+1}} = \frac{2}{(2\beta+k)(e^{i\theta_1}-e^{i\theta_2})}\left[\left(\frac{1-ze^{i\theta_2}}{1-ze^{i\theta_1}}\right)^{\beta+\frac{k}{2}}-1\right],$$

$\theta_1, \theta_2 \in [0, 2\pi]$, $e^{i\theta_1} \neq e^{i\theta_2}$.

## 2. Statement of results

Let us denote

$$(10)\qquad D_j(z) = \{w\colon w = \log f_j'(z),\ f_j \in \mathfrak{M}_j\},\ j = 1, 2, \ldots, n,$$

and

$$(11)\qquad \eta = \left(\frac{1-|a|^2}{1-\bar{a}z}\right)^2,\ \xi = \frac{z-a}{1-\bar{a}z},\ z,\ a \in K.$$

We have

**Theorem 1.** *If $F \in \mathscr{F}$, then*

$$(12)\qquad D(z, a) = \bigoplus_{j=1}^{n} D_j(\xi) \cup \{\log\eta\},$$

*where $\oplus$ denotes the geometric sum of sets.*

From Theorem 1 it follows that in order to find the set $D(z, a)$ it is sufficient to determine the sets $D_j(\xi)$. The set $D(z)$ is known in the case of the classes $S$ [12], $L$ [7] and $S^c$ [13]. Here we will find this set for the class $LV(\beta, k)$.

**Theorem 2.** *The set $D(z) = \{w\colon w = \log f'(z),\ f \in LV(\beta, k)\}$ is a closed and convex set whose boundary has the equation*

$$(13)\qquad w(t) = \log\frac{(1-re^{i\theta_2})^{\gamma-1}}{(1-re^{i\theta_1})^{\gamma+1}},\ t \in [0; 2\pi],\ r = |z| < 1,$$

*where* $\gamma = \beta + \frac{k}{2}$ *and*

(14) $\theta_1 = \theta_1(t) = t - \arcsin(r\sin t),\quad \theta_2 = \theta_2(t) = \pi + t + \arcsin(r\sin t).$

*The functions corresponding to the boundary points of* $D(z)$ *have the form* (9) *with* $\theta_1$, $\theta_2$ *qiven by* (14).

Putting $\beta = 0$ in Theorem 2 we have

**Theorem 2′.** *The set* $D(z) = \{w\colon w = \log\varphi'(z),\ \varphi \in V_k\}$ *is a closed and convex set whose boundary has the equation*

$$w(t) = \log\frac{(1-re^{i\theta_2})^{k/2-1}}{(1-re^{i\theta_1})^{k/2+1}},\quad t \in [0; 2\pi],\ r = |z| < 1,$$

*where* $\theta_1$ *and* $\theta_2$ *are given by* (14). *The functions corresponding to the boundary points of* $D(z)$ *have the form* (9) *with* $\beta = 0$.

By putting $\beta = 0$ and $k = 2$ in Theorem 2 we obtain the result for close-to-convex functions [7].

From Theorem 2 we can get exact estimates for $|f'(z)|$ and $|\arg f'(z)|$ if $f \in LV(\beta, k)$.

**Theorem 3.** *If* $f \in LV(\beta, k)$ *then for* $|z| = r < 1$

(15) $$|\arg f'(z)| \leqslant (2\beta + k)\arcsin r \quad [3],$$

(16) $$\frac{(1-r)^{\gamma-1}}{(1+r)^{\gamma+1}} \leqslant |f'(z)| \leqslant \frac{(1+r)^{\gamma-1}}{(1-r)^{\gamma+1}}.$$

*The extremal function has the form* (9) *with* $\theta_1$, $\theta_2$ *given by* (14) *with appropriate* $t$.

So far as the set $D(\xi)$ has been determined for $f \in LV(\beta, k)$ we can find the set $D(z, a)$ for the following class $\mathscr{F}$.

**Theorem 4.** *Let*

$$\mathscr{F}_{LV} = \Big\{F\colon F'(z) = \prod_{j=1}^{n} [f_j'(z)]^{a_j},\ f_j \in LV(\beta_j, k_j)\Big\}.$$

*Then* $D(z, a)$ *is a closed and convex set whose boundary has the equation*

(17) $$w(t) = \log\eta\prod_{j=1}^{n}\frac{(1-|\xi|e^{i\theta_2})^{(\gamma_j-1)a_j}}{(1-|\xi|e^{i\theta_1})^{(\gamma_j+1)a_j}},\quad t \in [0, 2\pi],$$

*where* $\gamma_j = \beta_j + k_j/2$ *and* $\eta$, $\xi$ *are given by* (11) *and*

(18) $$\theta_1 = \theta_1(t) = t - \arcsin(|\xi|\sin t),$$
$$\theta_2 = \theta_2(t) = \pi + t + \arcsin(|\xi|\sin t).$$

*The functions $F$ corresponding to the boundary of $D(z, a)$ are given by*

(19) $$\frac{F'(z)}{F'(a)} = \eta \prod_{j=1}^{n} \frac{(1-\xi e^{i\theta_2})^{(\gamma_j-1)\alpha_j}}{(1-\xi e^{i\theta_1})^{(\gamma_j-1)\alpha_j}}.$$

Putting in Theorem 4 $n = 1$ and $k = 2$ or $\beta = 0$ we obtain

**Corollary 1.** *The boundary of*

$D(z, a) = \left\{w\colon w = \log \frac{f'(z)}{f'(a)},\ f \in L_\beta\right\}$ *has the equation*

$$w(t) = \log \eta \frac{(1-|\xi| e^{i\theta_2})^{\beta}}{(1-|\xi| e^{i\theta_1})^{\beta+2}},\quad t \in [0; 2\pi].$$

**Corollary 2.** *The boundary of*

$D(z, a) = \left\{w\colon w = \log \frac{\varphi'(z)}{\varphi'(a)},\ \varphi \in V_k\right\}$ *has the equation*

$$w(t) = \log \eta \frac{(1-|\xi| e^{i\theta_2})^{k/2-1}}{(1-|\xi| e^{i\theta_1})^{k/2+1}},\quad t \in [0; 2\pi].$$

Moreover, from Theorem 4 it follows

**Corollary 3.** *If $F \in \mathscr{F}_{LV}$ then*

(20) $$\left|\arg \frac{F'(z)}{F'(a)} \eta^{-1}\right| \leqslant \left(2 \sum_{j=1}^{n} \gamma_j\right) \arcsin |\xi|,$$

(21) $$|\eta| \frac{(1-|\xi|)^{\sum_1^n (\gamma_j-1)\alpha_j}}{(1+|\xi|)^{\sum_1^n (\gamma_j+1)\alpha_j}} \leqslant \left|\frac{F'(z)}{F'(a)}\right| \leqslant |\eta| \frac{(1+|\xi|)^{\sum_1^n (\gamma_j-1)\alpha_j}}{(1-|\xi|)^{\sum_1^n (\gamma_j-1)\alpha_j}}.$$

**Theorem 5.** *The radius of univalence of $LV(\beta, k)$ is equal to* $\tan \dfrac{\pi}{2\beta+k}$. *The extremal function is given by* (9) *with appropriate $\theta_1$ and $\theta_2$.*

**Corollary 4.** *The radius of univalence of $\mathscr{F}_{LV}$ is equal to* $\tan \dfrac{\pi}{\Sigma(2\beta_j+k_j)}$.

**Theorem 6.** *The radius of $\varkappa$-close-to-convexity of $LV(\beta, k)$ is the unique root of the equation*

(22) $$2 \operatorname{arccot} w - (2\beta+k) \operatorname{arccot}\left[\left(\beta+\frac{k}{2}\right) w\right] = -\varkappa\pi$$

*where $w = (1-r^2)[(2\beta+k)^2 r^2 - (1-r^2)^2]^{-1/2}$.*
*The result is sharp.*

**Corollary 5.** *$LV(\beta,k)\subset L_{\beta+k/2-1}$, in particular $V_k\subset L_{k/2-1}$* [2].

**Remark 1.** If $2\beta+k\leqslant 4$ then $LV(\beta,k)$ consists only of univalent close-to-convex functions.

An interesting example of the class $\mathscr{F}$ is the following one:

$$\mathscr{F}_s=\Big\{F\colon F(z)=\int_0^z[f'(\xi)]^a[g'(\xi)]^{1-a}d\xi,\ f,g\in S\Big\}. \tag{23}$$

The considerations of properties of the class $\mathscr{F}_S$ are close to the problem concerning univalence of the integral $\int_0^z(f'(\zeta))^a d\zeta$. We will prove.

**Theorem 7.** *If $a\in[0,1]$ then every function $F\in\mathscr{F}_S$ is univalent at least in the disk $|z|<r_u$ where*

$$r_u=\frac{e^{\pi/2}}{1+\sqrt{1+e^{\pi}}},\ (r_u>0{,}81). \tag{24}$$

**Remark 2.** The integral $\int_0^z(f'(\zeta))^a d\zeta$, $f\in S$, $a\in[0,1]$ is univalent at least for $|z|<r_u$.

### 3. Lemmas

**Lemma 1′.** *A function $f\in LV(\beta,k)$ if and only if there exist a function $p\in\mathscr{P}$ and functions $g_1,g_2\in S^c$ such that*

$$f'(z)=e^{-i\alpha}p^{\beta}(z)[g_1'(z)]^{\frac{2+k}{4}}[g_2'(z)]^{\frac{2-k}{4}},\ z\in K. \tag{25}$$

The proof of Lemma 1′ follows from Lemma 1 and formula (7).

**Lemma 2.** *For every $\beta\geqslant 0$ and $k\geqslant 2$ the class $LV(\beta,k)$ is linearly invariant family of order $\gamma=\beta+\dfrac{k}{2}$.*

**Proof.** Let $\beta,k$ be admissible and $f\in LV(\beta,k)$. From the definition 2 $f'(z)\neq 0$, $z\in K$, so in order to prove the linear invariance of $LV(\beta,k)$ we should show that for arbitrary $a\in K$

$$F(z)=\frac{f\left(\dfrac{z+a}{1+\bar a z}\right)-f(a)}{(1-|a|^2)f'(a)}\in LV(\beta,k). \tag{26}$$

Robertson has proved [11] that $V_k$ is linearly invariant family which implies

$$\Phi(z)=\frac{\varphi\left(\dfrac{z+a}{1+\bar a z}\right)-\varphi(a)}{(1-|a|^2)\varphi'(a)}\in V_k \text{ if } \varphi\in V_k. \tag{27}$$

From (26) and (27) we obtain

$$\frac{F'(z)}{\Phi'(z)} = \frac{\varphi'(a)}{f'(a)} \cdot \frac{f'\left(\dfrac{z+a}{1+\bar{a}z}\right)}{\varphi'\left(\dfrac{z+a}{1+\bar{a}z}\right)}.$$

Because $\left|\arg e^{i\alpha}\dfrac{f'(a)}{\varphi'(a)}\right| \leqslant \beta\dfrac{\pi}{2}$, $a \in K$, we can choose $\tilde{a}$ such that $\left|\arg e^{i\tilde{a}}\dfrac{F'(z)}{\Phi'(z)}\right| \leqslant \beta\dfrac{\pi}{2}$, $z \in K$, and hence $F \in LV(\beta, k)$.

From (7) we can write

$$e^{i\alpha}\frac{f'(z)}{\varphi'(z)} = p^{\beta}(z) = (e^{i\delta} + p_1 z + \ldots)^{\beta} \tag{28}$$

where $p \in \mathscr{P}$. Comparing the coefficients in (28) we find that

$$a_2 = b_2 + \frac{1}{2}\beta p_1 e^{-i\delta} \tag{29}$$

Taking into account that $|b_2| \leqslant \dfrac{k}{2}$ and $|p_1| \leqslant 2$ we find that the order of famiy $LV(\beta, k)$ is equal to $\gamma = \sup|a_2| = \beta + \dfrac{k}{2}$.

**Lemma 3.** *The class $\mathscr{F}$ is linearly invariant family.*

**Proof.** Let $F \in \mathscr{F}$ be given by (4). Then $f_j \in \mathfrak{M}_j$ as well as

$$F_j(z) = \frac{f_j\left(\dfrac{z+a}{1+\bar{a}z}\right) - f_j(a)}{(1-|a|^2)f_j'(a)} \in \mathfrak{M}_j,\ j = 1, 2, \ldots, n. \tag{30}$$

If we put $\psi(z) = \dfrac{F\left(\dfrac{z+a}{1+\bar{a}z}\right) - F(a)}{(1-|a|^2)F'(a)}$, then

$$\psi'(z) = \frac{F'\left(\dfrac{z+a}{1+\bar{a}z}\right)}{(1+\bar{a}z)^2 F'(a)} = \frac{1}{(1+\bar{a}z)^2}\prod_{j=1}^{n}\left[\frac{f_j'\left(\dfrac{z+a}{1+\bar{a}z}\right)}{f_j'(a)}\right]^{a_j} =$$

$$\frac{1}{(1+\bar{a}z)^2}\prod_{j=1}^{n}[(1+\bar{a}z)^2 F_j'(z)]^{a_j} = \prod_{j=1}^{n}[F_j'(z)]^{a_j}$$

which completes the proof.

**Remark 3.** It is easy to observe that if $f_j(z) \in \mathfrak{M}_j$ then $e^{-i\theta} f_j(e^{i\theta} z) \in \mathfrak{M}_j$ for arbitrary real $\theta$. This implies that if $\gamma_j$ is the order of $\mathfrak{M}_j$ then the order of $\mathscr{F}$ is equal to $\gamma = \sum_{j=1}^{n} |a_j| \gamma_j$.

**Remark 4.** The result of Pommerenke [10] implies that radius of convexity of the family $\mathscr{F}$ is equal to

$$r_c = \Big(\sum_{j=1}^{n} |a_j| \gamma_j + \sqrt{\Big(\sum_{j=1}^{n} |a_j| \gamma_j\Big)^2 - 1}\Big)^{-1}.$$

## 4. Proofs of theorems

**Proof of Theorem 1.** Let $F \in \mathscr{F}$. Then from (4) and (30) we have

$$\frac{F'(z)}{F'(a)} = \prod_{j=1}^{n} \left[\frac{f_j(z)}{f_j'(a)}\right]^{a_j} = \prod_{j=1}^{n} \left[\left(\frac{1-|a|^2}{1-\bar{a}z}\right)^2 F_j'\left(\frac{z-a}{1-\bar{a}z}\right)\right]^{a_j} =$$

$$= \eta \prod_{j=1}^{n} [F_j'(\xi)]^{a_j}, \quad \xi \in K.$$

Now, we see that

$$\log \frac{F'(z)}{F'(a)} = \log \eta + \sum_{j=1}^{n} a_j \log F_j'(\xi),$$

which implies (12) because $F_j$ is ranging independently over $\mathfrak{M}_j$, $j = 1, 2, \ldots n$.

It is worthwhile to mention that convexity of $D_j$, $j = 1, 2, \ldots, n$ (which occurs for example if $\mathfrak{M}_j$ is one of the classes $S^c$, $L$, $S$, $LV(\beta, k)$) implies convexity of $D(z, a)$.

**Proof of Theorem 2.** First of all it is convenient to observe that $D(ze^{i\theta}) = D(|z|) = D$, $|z| = r$. The set $D$ is closed because the class $LV(\beta, k)$ is compact. The convexity of $D$ follows from the fact that for $\lambda \in [0, 1]$ and $f_1, f_2$ $LV(\beta, k)$ the function

$$f_\lambda(z) = \int_0^z [f_1'(t)]^\lambda \cdot [f_2'(t)]^{1-\lambda} dt \in LV(\beta, k).$$

Let $f \in LV(\beta, k)$. From Lemma 1′ we have

$$f'(r) = e^{-i\alpha} p^\beta(r) [\varphi_1'(r)]^{\frac{2+k}{k}} [\varphi_2'(r)]^{\frac{2-k}{4}}, \tag{31}$$

$0 < r < 1$, is real, where $p \in \mathscr{P}$, $\varphi_1, \varphi_2 \in S^c$.

It is well-known that functions corresponding to the boundary points of $\{w\colon w = p(r),\ p \in \mathscr{P}\}$ have the form

$$p(r) = \frac{e^{i\delta} - re^{i\tau}}{1 - re^{i\tau}}, \quad \tau \in [0; 2\pi),$$

as well as the functions corresponding to the boundary points of $\{w\colon w = \varphi_j'(r),\ \varphi_j \in S^c,\ j = 1, 2\}$ have the form

$$\varphi_j(r) = \frac{r}{1-re^{it_j}},\ j = 1, 2,\ [13].$$

From the definition of the class $LV(\beta, k)$ it follows that $e^{ia} = e^{i\delta\beta}$. This fact and the facts cited above imply together with (31) that the function $f$ corresponding to the boundary points of $D$ has the following form

$$(32) \qquad f'(r) = \left(\frac{1-\varepsilon_1 r}{1-\varepsilon_2 r}\right)^{\beta}(1-\varepsilon_3 r)^{-\frac{2+k}{2}}(1-\varepsilon_4 r)^{-\frac{2-k}{2}},$$

$\varepsilon_j = e^{i\theta_j}$, $\theta_j \in [0, 2\pi]$, $j = 1, 2, 3, 4$.
The convexity of $D$ implies that finding the boundary of $D$ is equivalent to determining the maximum of the function

$$(33) \quad \operatorname{Re}[e^{-it}\log f'(r)] = \operatorname{Re}\Big\{e^{-it}\Big[\beta\log(1-re^{i\theta_1}) - \beta\log(1-re^{i\theta_2}) - \frac{2+k}{2}\log(1-re^{i\theta_3}) - \frac{2-k}{2}\log(1-re^{i\theta_4})\Big]\Big\}$$

with respect to $\theta_j \in [0, 2\pi]$, for fixed $t \in [0, 2\pi]$. The number $t$ denotes the angle between the imaginary axis and supporting line to $D$. Moreover, we may observe from (24) that $D$ is symmetric with respect to the real axis because the image of the circle $\xi = 1-ze^{i\theta}$, $\theta \in [0, 2\pi]$, $z \in K$, under the mapping $w = \log\zeta$ is a convex curve symmetric with respect to the real axis.
So, we may assume that $t \in [0; \pi]$.
It is easy to check that the function

$$\chi(\theta) = \operatorname{Re}\{e^{-it}\log(1-re^{i\theta})\}$$

attains its minimum for $\theta = \theta_1(t)$ and maximum for $\theta = \theta_2(t)$, where $\theta_1$ and $\theta_2$ are given by (14). Substituting (14) into (33) we obtain (13).

**Proof of Theorem 3.** Taking into account the convexity and symmetry with respect to the real axis of $D(z)$ we see that $\max|f'(z)|$ is attained for $t = 0$, $\min|f'(z)|$ for $t = \pi$ and $\max|\arg f'(z)|$ for $t = \pi/2$. Putting these special values of $t$ into (14) and (32) we get (15) and (16).

**Proof of Theorem 4.** This result follows immediately from Theorems 1 and 2.

**Proof of Theorem 5.** In order to find the radius of univalence $r_u$ of linearly invariant family of it is sufficient to find the largest disk $|z| < r_0$ in which every function from $LV(\beta, k)$ is different from zero because then we have [10]

$$r_u = \frac{r_0}{1+\sqrt{1-r_0^2}} \tag{34}$$

The reasoning as in [10, Satz 2.6] implies that if $f \in LV(\beta, k)$ and $f(r_0) = 0$ then $\arg f'(r_0) = \pm 2\pi$. Since from (15) we have $|\arg f'(z)| \leqslant (2\beta+k)\arcsin|z|$ we conclude that $r_0 \geqslant \sin\frac{2\pi}{2\beta+k} = \varrho$. By considering the function $F\beta, k$ given by (9) with $\theta_1 = \arccos\varrho$, $\theta_2 = -\arccos\varrho$ we find that $F_{\beta,k}\left(\sin\frac{2\pi}{2\beta+k}\right) = 0$. Thus $r_0 \leqslant \varrho = \sin\frac{2\pi}{2\beta+k}$ which complete the proof after using formula (34).

**Proof of Theorem 6.** The estimate of $|\arg f'(z)|$, $f \in LV(\beta, k)$ help us to find the radius of $\varkappa$-close-to-convexity, $\varkappa \geqslant 0$, of $LV(\beta, k)$. Goodman [5] has proved that $f \in L_\varkappa$ if for each $r \in (0, 1)$ and for each pair $\theta_1, \theta_2$, $0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi$

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(1 + re^{i\theta}\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right) d\theta \geqslant -\varkappa\pi$$

Using this condition and the method originated in [6] (see also [3]) one can prove Theorem 6. Corollary 5 follows from the fact that the left hand side of the equation (22) is strictly decreasing function of $r \in (0, 1)$ and its value for $r = 1$ is equal to $\frac{\pi}{2}(2-2\beta-k)$.

**Proof of Theorem 7.** Let $F \in \mathscr{F}_S$ be given by (23) and $a \in [0, 1]$. Then

$$|\arg F'(z)| \leqslant \begin{cases} 4\arcsin|z|, & |z| \leqslant \frac{1}{\sqrt{2}}. \\ \pi + \log\frac{|z|^2}{1-|z|^2}, & |z| \geqslant \frac{1}{\sqrt{2}} \end{cases} \tag{35}$$

The estimate (35) is exact and the sign of equality holds if $f = g = F_0 \in S$, where $F_0$ is extremal function for $\max|\arg F'(z)|$ [4, p. 116] within class $S$. Let $r_0 = \inf_{F\in\mathscr{F}_S} \sup_{r>0}\{r: F(z) \neq 0 \text{ for } 0 < |z| < r\}$ The fact that $\mathscr{F}_S$ is linearly invariant family allow us to apply the method of Pommerenke [10, Satz 2.6] like in the Proof of Theorem 5. In this manner we find that $r_0$ is the root of the equation $\log\frac{r^2}{1-r^2} = \pi$ which implies (24) after using (34). It may be checked that $r_u(S) > 0.81$.

From (35) and from linearly invariance of $\mathscr{F}_S$ follows that the exact radius of close-to-convexity $r_L(\mathscr{F}_S) = r_L(S) \in (0{,}80; 0{,}81)$ [6].

If $g(z) \equiv z$ then the class $\mathscr{F}_S$ reduces to

$$\left\{F\colon F(z) = \int_0^z (f'(\xi))^a d\xi,\ f \in S\right\}, \tag{36}$$

which is no more linearly invariant.

The problem of finding $\max|a|$ for which (36) consists of univalent functions has been considered by many authors. Recently Pfaltzgraff [9] proved that (36) is univalent for $|a| \leqslant \frac{1}{4}$ ($a$ may be a complex number).

Since the class (36) for $a \in (0, 1)$ is a subclass of $\mathscr{F}_S$ given by (23) we obtain from Theorem 7 that the radius of univalence for (36) is at least 0,81.

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## STRESZCZENIE

Niech $\mathcal{F}$ oznacza rodzinę funkcji holomorficznych $F$ w kole $K = \{z: |z| < 1\}$ danych wzorem

$$F'(z) = \prod_{j=1}^{n} [f_j'(z)]^{a_j}, \quad z \in K,$$

gdzie $a_j$ jest liczbą rzeczywistą, $\sum_{j=1}^{n} a_j = 1$ a funkcje $f_j$ należą do ustalonej rodziny $\mathfrak{M}_j$, która jest liniowo-niezmiennicza w sensie Pommerenke [10].

W pracy wyznaczono obszar zmienności

$$D(z, a) = \left\{w: \ w = \log \frac{F'(z)}{F'(a)}, \ F \in \mathcal{F}\right\}, \quad z, a \in K$$

dla kilku znanych rodzin $\mathfrak{M}_j$.

W szczególności wyznaczono obszar wartości

$$D(z) = \{w: \ w = \log f'(z)\}$$

dla ustalonego $z \in K$ i funkcji $f$ zmieniającej się w klasie funkcji $\beta$-prawie- $V_k$ (definicja 2).

Podano również promienie jednolistności i prawie-wypukłości pewnych rodzin $\mathcal{F}$, a jako wniosek otrzymano, że jeśli $f$ jest funkcją jednolistną, to całka

$$F(z) = \int_0^z (f'(t))^a dt, \quad a \in [0, 1]$$

jest jednolistna przynajmniej w kole $|z| < 0,81$.

## РЕЗЮМЕ

Пусть $F$ обозначает класс голоморфных функций в кругу $K = \{z: |z| < 1\}$ данных формулой

$$F'(z) = \prod_{j=1}^{n} [f'(z)]^{a_j}, \quad z \in K,$$

где $a_j$ вещественное число, $\sum a_j = 1$ и функция $f_j$ принадлежит к фиксированному семейству $\mathfrak{M}_j$, которое есть линейно-инвариантное в смысле Поммеренке [10].

В этой работе определено область изменения

$$D(z,a) = \left\{w: w = \log \frac{F'(z)}{F'(a)}, F \in \mathcal{F}\right\}, \quad z, a \in k$$

для некоторых известных семейств $\mathfrak{M}_j$.

В частности определено область изменения

$$D(z) = \{w: w = \log f'(z)\}$$

для фиксированного $z \in K$, когда функция $f$ изменяется в класс $\beta$ – почти – $V_k$ (определение 2).

Кроме того поданы радиусы однолистности и почти-выпуклости некоторых семейств $\mathcal{F}$.

В следствии получен результат, что для однолистной функции $f$ интеграл

$$F(z) = \int_0^z (f(t))^a dt, \quad a \in [0,1]$$

однолистный по крайней мере в круге $|z| < 0,81$.

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