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Two Remarks on Typically-Real Functions

Dwie uwagi o funkcjach typowo-rzeczywistych

Две заметки о типично-вещественных функциях

Introduction. Denote by TR the class of functions $f(z) = z+\sum_{k=2}^{\infty} a_k z^k$ analytic in the unit disc D that assume real values on the segment $(-1;1)$ and satisfy

$$(\operatorname{Im} z)\cdot(\operatorname{Im} f(z)) > 0$$

otherwise in D.

Functions of this class are said to be typically-real. It is well-known, cf. e.g. [3], that $f \in TR$ if and only if it has the representation

$$f(z) = \int_0^{\pi} \frac{z}{(z-e^{i\theta})(z-e^{-i\theta})}\, d\mu(\theta), \tag{1}$$

where μ is a probability measure on the interval $\langle 0;\pi\rangle$.

Suppose $f(z) = z+\sum_{k=2}^{\infty} a_k z^k$, $g(z) = z+\sum_{k=2}^{\infty} b_k z^k$ are analytic in D. Then the function

$$h(z) = f*g(z) = z+\sum_{k=2}^{\infty} a_k b_k z^k$$

is said to be the convolution, or Hadamard's product of f and g.

The aim of this note is to give new and elementary proofs of two results concerning typically-real functions:

Theorem A. [4]. *If f, g belong to TR, so does*

$$H(z) = \int_0^z u^{-1} f * g(u) \cdot du = z + \sum_{k=2}^{\infty} k^{-1} a_k b_k z^k.$$

Theorem B. [2]. *If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in TR$, then*

$$n - a_n \leqslant \tfrac{1}{6} n(n^2 - 1)(2 - a_2), \quad n = 2, 3, \ldots$$

and the result is best possible.

The original proof of Theorem A was based on a lemma of Fejér [1]. The second result was obtained by making use of some extremal properties of Tchebycheff polynomials. Our proofs of both theorems make use of (1) and some simple observations.

In order to prove the Convolution Theorem A, we first establish the following

Lemma. *Let w_1, w_2 be points in the half-plane $\{w\colon \operatorname{Re} w < 1\}$. Then*

$$\operatorname{Re} \frac{\log(1 - w_1) - \log(1 - w_2)}{w_1 - w_2} < 0.$$

Proof. It is easy to see that the function

$$W = \varphi(w) = (1 + w)(1 - w)^{-1}$$

maps the half-plane $\{w\colon \operatorname{Re} w < 1\}$ onto the half-plane $\{W\colon \operatorname{Re} W > -1\}$. If the segment $[w_1; w_2]$ lies in the halfplane $\{w\colon \operatorname{Re} w < 1\}$, then $\operatorname{Re}\varphi(w) > -1$ for any $w \in [w_1; w_2]$ and hence

$$-1 < \int_0^1 \operatorname{Re}\varphi[w_1 + t(w_2 - w_1)]dt$$

$$= \operatorname{Re} \int_{[w_1; w_2]} \varphi(w) \frac{dw}{w_2 - w_1} = \operatorname{Re}\left\{\frac{1}{w_2 - w_1} \log \frac{1 - w_1}{1 - w_2} - 1\right\}.$$

and this proves the Lemma.

Proof of Theorem A. Suppose that the functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ belong to TR. It follows from (1) that there exist probability measures μ, ν on $\langle 0; \pi \rangle$ such that

$$f(z) = \int_0^\pi \frac{z d\mu(\theta)}{(z - e^{i\theta})(z - e^{-i\theta})}, \quad g(z) = \int_0^\pi \frac{z d\nu(\varphi)}{(z - e^{i\varphi})(z - e^{-i\varphi})}$$

We thus have

$$H(z) = \int_0^z u^{-1} f * g(u)\,du = \left(\int_0^z u^{-1} f(u)\,du\right) * g(z) = F(z) * g(z).$$

Suppose that $|z| < |u| = r < 1$. Then

$$H(z) = \frac{1}{2\pi i} \int_{|u|=r} F(u) g(u^{-1}z) u^{-1}\,du$$

$$= \int_0^\pi d\mu(\theta) \int_0^\pi d\nu(\varphi) \left[\frac{1}{2\pi i} \int_{|u|=r} \frac{z}{2i\sin\theta(u - ze^{i\varphi})(u - ze^{-i\varphi})} \log\frac{1-ue^{-i\theta}}{1-ue^{i\theta}}\,du\right]$$

The integral over the circumference may be computed readily by means of residua and we obtain

$$H(z) = \int_0^\pi \int_0^\pi K(z, \theta, \varphi)\,d\mu(\theta)\,d\nu(\varphi)$$

where

$$K(z, \theta, \varphi) = \frac{1}{4\sin\theta\sin\varphi} \log\frac{1-2z\cos(\theta+\varphi)+z^2}{1-2z\cos(\theta-\varphi)+z^2}$$

In order to prove that $F * g \in TR$ it is sufficient to show that

$$\mathrm{Re}\,\frac{1-z^2}{z} K(z, \theta, \varphi) > 0$$

for any $z \in D$ and any real θ, φ, cf. [3].
To this end, set

$$G(z, t) = 2z(1-z^2)^{-1}(t-z),$$

$(z, t) \in D \times \langle -1; 1\rangle$.
Since, for any fixed $z \in D$, $\mathrm{Re}\,G(z, t)$ is a linear function of $t \in \langle -1; 1\rangle$, it attains extreme values for $t = \mp 1$. Hence we conclude that

$$\mathrm{Re}\,G(z, t) < 1$$

in the set $D \times \langle -1; 1\rangle$.
Putting $t_1 = \cos(\theta+\varphi)$, $t_2 = \cos(\theta-\varphi)$ and $w_k = G(z, t_k)$, $k = 1, 2$ we obtain

$$\mathrm{Re}\,w_k < 1, \quad k = 1, 2$$

and

$$w_2 - w_1 = 4z(1-z^2)^{-1}\sin\theta\cdot\sin\varphi.$$

Hence

$$\frac{1-z^2}{z}K(z,\theta,\varphi)=\frac{\log(1-w_1)-\log(1-w_2)}{w_2-w_1}$$

and the result follows.

Proof of Theorem B. In view of (1) we find

$$a_n=\int_0^\pi \frac{\sin n\theta}{\sin\theta}\,d\mu(\theta),\ n=1,2,\ldots \tag{2}$$

Setting $a_0=0$ and

$$\alpha_n=a_n-a_{n-1},\ n=1,2,\ldots$$

we have

$$a_n=\sum_{k=1}^{n}\alpha_k,\ A_n=a_{n+1}-2a_n+a_{n-1}=\alpha_{n+1}-\alpha_n.$$

By (2) we obtain

$$A_n=\int_0^\pi\frac{\sin(n+1)\theta-2\sin n\theta+\sin(n-1)\theta}{\sin\theta}\,d\mu(\theta)$$

$$=-2\int_0^\pi\frac{\sin n\theta}{\sin\theta}(1-\cos\theta)\,d\mu(\theta)$$

i.e.

$$A_n=n\beta_n(2-a_2),\ n=1,2,3,\ldots \tag{3}$$

where $\{\beta_n\}$ is a sequence of real numbers $\beta_n\in\langle -1;1\rangle$, $n=1,2,3,\ldots$. Thus

$$\sum_{k=1}^{n-1}A_k=\alpha_n-1=(2-a_2)\sum_{k=1}^{n-1}k\beta_k$$

and

$$a_n-n=\sum_{k=1}^{n}(\alpha_k-1)=(2-a_2)\sum_{k=1}^{n-1}k(n-k)\beta_k.$$

Since $|\beta_k|\leqslant 1$ we obtain

$$n-a_n\leqslant(2-a_2)\sum_{k=1}^{n}k(n-k)=6^{-1}(2-a_2)n(n^2-1)$$

Considering the function $f(z)=z(z-e^{i\theta})^{-1}(z-e^{-i\theta})^{-1}$ for small values of $|\theta|$ we can verify that the sequence $6^{-1}n(n^2-1)$ is best possible.

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STRESZCZENIE

W artykule podano są nowe i proste dowody dwóch twierdzeń o funkcjach typowo-rzeczywistych.

Twierdzenie A. *Jeżeli* $f, g \in TR$, *to* $f * g(z) = z + \sum_{k=2}^{\infty} \frac{a_k b_k}{k}$ *należy do tej samej klasy.*

Twierdzenie B. *Jeżeli* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in TR$, *to* $n - a_n < 6^{-1} n (n^2 - 1)(2 - a_2)$.

РЕЗЮМЕ

В этой работе представлены новые и простые доказательства двух теорем об типично-вещественных функциях.

Теорема А. *когда* $f, g \in TR$, то $f * g(z) = z + \sum \frac{a_n b_n}{n} \in TR$

Теорема В. *когда* $f(z) = z + \sum_{2}^{\infty} a_n z^n \in TR$, *то*

$$n - a_n < 6^{-1} n (n^2 - 1)(2 - a_2)$$