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On a Fixed Point Theorem for Multivalued Nonexpansive Mappings

O pewnym twierdzeniu o punkcie stałym wielowartościowych operacji nieoddalających

О некоторой теореме о неподвижной точке для многозначных слабосжимающих отображений

Let C be a nonempty weakly compact convex subset of a Banach space X and denote by $\mathfrak{N}$ the family of all nonempty compact subsets of C. Let H be the Hausdorff metric on $\mathfrak{N}$ and let $T\colon C\to\mathfrak{N}$ be a nonexpansive mapping, i.e. such that for $x, y\in C$

$$H(Tx, Ty) \leqslant \|x-y\|. \tag{1}$$

A point $x\in C$ is said to be a fixed point of T if $x\in Tx$.

If T satisfies the condition

$$H(Tx, Ty) \leqslant k\|x-y\| \tag{2}$$

then the existence of a fixed point of T is a consequence of the result of Nadler [7]. The fixed point theorems for mappings satisfying (1) have been established by Markin [6] in case of Hilbert space, Browder [1] in case of spaces having weakly continuous duality mappings and by Lami Dozo [3] for spaces satisfying Opial's condition and recently by Lim [5] for uniformly convex spaces.

The proof given by Lim is based on the use of the notion of asymptotic center of a sequence in Banach space and on rather delicate transfinite induction. The aim of this paper is to give another, in our opinion simpler, proof of Lim's theorem. We shall also use the asymptotic center technique.

Let $\{x_i\}$ be a bounded sequence of elements of X. Denote

$$r\{x_i\} = \inf[\limsup\|x_i-y\|\colon y\in C]$$

$$A\{x_i\} = [z\in C\colon \limsup\|x_i-z\| = r\{x_i\}]$$

and call them: the asymptotic radius and the asymptotic center of $\{x_i\}$

in C respectively. The asymptotic center is a closed convex subset of C which in case of uniformly convex space consists of exactly one point [2], [4]. Obviously if two sequences differ only by a finite number of elements then their asymptotic radii and centers are the same. It is also obvious that if $\{y_i\}$ is a subsequence of $\{x_i\}$ then $r\{y_i\} \leqslant r\{x_i\}$ but it is hard to find any simple relations between $A\{y_i\}$ and $A\{x_i\}$

For our purpose, let us call the sequence $\{x_i\}$ regular if all its subsequences have the same asymptotic radius and almost convergent if all its subsequences have the same asymptotic center. Let us prove two simple lemmas:

Lemma 1: *If X is uniformly convex then each regular sequence is almost couvergent.*

Proof: Let $\{x_i\}$ be a regular sequence. Put $z = A\{x_i\}$ and denote the common value of radii of subsequences of $\{x_i\}$ by r. Suppose $\{y_i\}$ is a subsequence of $\{x_i\}$ such that $A\{y_i\} = y \neq z$. Then

$$\limsup \|y_i - z\| \leqslant \limsup \|x_i - z\| = r$$

$$\limsup \|y_i - y\| = r$$

and then because of the uniform convexity of X

$$\limsup \left\| y_i - \frac{y+z}{z} \right\| < r = r\{y_i\}$$

which is a contradiction.

Lemma 2: *Any bounded sequence $\{x_i\}$ contains a regular subsequence.*

Proof: For arbitrary bounded sequence $\{z_i\}$ denote $r_0\{z_i\} = \inf[r\{v_i\}:$ $:\{v_i\}$ is a subsequence of $\{z_i\}]$ Obviously $r_0\{z_i\} \leqslant r\{z_i\}$ and if $\{w_i\}$ is a subsequence of $\{z_i\}$ then $r_0\{z_i\} \leqslant r_0\{w_i\}$.

Now let $\{x_i\}$ be a given bounded sequence. Let us construct the family of sequences $\{x_i^n\}$ $n = 1, 2, \ldots$ to satisfy the following conditions

$$\{x_i^1\} = \{x_i\}$$

$$\{x_i^{n+1}\} \text{ is a subsequence of } \{x_i^n\}$$

$$r\{x_i^{n+1}\} \leqslant r_0\{x_i^n\} + \frac{1}{n}$$

It is easy to see that the diagonal sequence $\{x_i^i\}$ is regular.

Now we can prove the theorem.

Theorem 1: /Lim [5]/ *Let X be an uniformly convex Banach space and let C be a closed bounded and convex subset of X. Suppose $T\colon C \to \mathfrak{N}$ be a nonexpansive mappings. Then T has a fixed point.*

Proof. Notice first that

$$\inf[\operatorname{dist}(x, Tx) \colon x \in C] = 0.$$

It is so because for arbitrary $0 < \varepsilon < 1$ and any $u \in C$ the mapping $T_\varepsilon x = \varepsilon u + (1-\varepsilon) Tx$ satisfies (2) with $k = 1-\varepsilon$ and then has a fixed point and $H(T_\varepsilon x, Tx) \to 0$ when $\varepsilon \to 0$.

Take then a sequence $\{x_n\}$ such that disc (x_n, Tx_n) converges to zero. In view of our lemmas we can assume that $\{x_n\}$ is regular so almost convergent.

Let $z = A\{x_i\}$ and $r = r\{x_i\}$. For any x_i find $y_i \in Tx_i$ such that $\|x_i - y_i\|$ tends to zero. Now, for each y_i find $z_i \in Tz$ such that

$$\|y_i - z_i\| \leqslant H(Tx_i, Tz) \leqslant \|x_i - z\|$$

It is possible because of compactness of Tx_i, Tz. Once again in view of compactness of Tz we can find a subsequence $\{z_{i_k}\}$ of $\{z_i\}$ convergent to an element $v \in Tz$. Since the regularity of $\{x_i\}$ we have $r\{x_{i_k}\} = r\{x_i\} = r$ and $A\{x_{i_k}\} = A\{x_i\} = z$.

On the other hand we have.

$$\|x_{i_k} - v\| \leqslant \|v - z_{i_k}\| + \|z_{i_k} - y_{i_k}\| + \|y_{i_k} - x_{i_k}\|,$$

and

$$\|z_{i_k} - y_{i_k}\| \leqslant \|z - x_{i_k}\|$$

implying

$$\limsup \|x_{i_k} - v\| \leqslant r$$

meaning $v = A\{x_{i_k}\} = z$, what ends the proof.

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STRESZCZENIE

W pracy podany jest prosty dowód wyniku T. C. Lima [5] o punkcie stałym.

РЕЗЮМЕ

В работе представлено простое доказательство результата Т. Ц. Лима [5] о неподвижной точке.