ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN — POLONIA
VOL. XXIX, 4 SECTIO A 1975

Instytut Ekonomii Politycznej i Planowania
Uniwersytet Marii Curie — Skłodowskiej, Lublin

CZESŁAWA BUCKA, KRYSTYNA CIOZDA

Some Estimations and Problems of the Majorization in the Classes of Functions $S^k_{(\alpha,\beta)}$

Pewne oszacowania i problemy majoryzacji w klasach funkcji $S^k_{(\alpha,\beta)}$

Некоторые оценки и вопросы можорации в классах функций $S^k_{(\alpha,\beta)}$

1. Let S denote the class of functions $f(z)$ of the form

$$f(z) = z + a_2 z^2 + \ldots \tag{1.1}$$

regular and univalent in the unit disc K_1, where $K_r = \{z\colon |z| < r\}$. And let $S_{(\alpha,\beta)}$, $\alpha \in (0, 2)$, $\beta \in (-2, 0)$, $\alpha - \beta \leqslant 2$, denote the class of functions of the form (1.1) and satisfying the condition

$$\beta\frac{\pi}{2} < \arg\frac{zf'(z)}{f(z)} < \alpha\frac{\pi}{2}$$

for every $z \in K_1$.

This condition means that $w = zf'(z)/f(z)$ is in the angle of the vertex at the origin of the coordinate system, which includes the point $w = 1$ and equals to $(\alpha-\beta)\pi/2$. In some cases the class $S_{(\alpha,\beta)}$ coincides with the well-known subclasses of functions of the class S:

$$S_{(1,-1)} = \left\{f\colon \operatorname{Re}\frac{zf'(z)}{f(z)} > 0 \quad \bigwedge z \in K_1\right\} = S^*$$

$$S_{(\alpha,-\alpha)} = \left\{f\colon \left|\arg\frac{zf'(z)}{f(z)}\right| < \alpha\frac{\pi}{2} \quad \bigwedge z \in K_1, \alpha \in (0, 1\rangle\right\} = S_\alpha$$

$$S_{(\alpha,\alpha-2)} = \left\{f\colon \operatorname{Re}\left\{e^{-i\delta}\frac{zf'(z)}{f(z)}\right\} > 0 \quad \bigwedge z \in K_1, \delta = \frac{\pi}{2}(\alpha-1)\right\} = \tilde{S}_\delta$$

The class $S_{(\alpha,\beta)}$ has been investigated in the paper [3]. In this paper we deal with the subclass $S^k_{(\alpha,\beta)}$, ($k \geqslant 1$ is an arbitrary positive integer) of the class $S_{(\alpha,\beta)}$. $S^k_{(\alpha,\beta)}$ denotes the class of functions of the form

$$f(z) = z + a_{k+1}z^{k+1} + a_{2k+1}z^{2k+1} + \ldots, \ z \in K_1 \tag{1.2}$$

k-symmetric and univalent in K_1.

Let P denote the class of functions p of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + \ldots, \ z \in K_1$$

regular in K_1 and satisfying the condition: $\operatorname{Re} p(z) > 0$, and let P_k denote the class of functions h of the form

$$h(z) = 1 + h_k z^k + h_{2k} z^{2k} + \ldots, \ z \in K_1 \tag{1.3}$$

regular in K_1 and satisfying the condition: $\operatorname{Re} h(z) > 0$. There is the following relation between P_k and P: if $h \in P_k$ then there exists a function of the class P such that $h(z) = p(z^k)$.

2. Theorem 2.1. *A function f belongs to $S_{(\alpha,\beta)}$ if and only if there exists a function $p \in P$ such that*

$$\frac{zf'(z)}{f(z)} = \{1 + e^{i\gamma}\cos\gamma[p(z) - 1]\}^{\frac{\alpha-\beta}{2}} \quad \bigwedge z \in K_1, \ \gamma = \frac{\pi}{2} \cdot \frac{\alpha+\beta}{\alpha-\beta}$$

This theorem was given in the paper [3] as the theorem I.

Theorem 2.2. *A function f belongs to $S^k_{(\alpha,\beta)}$ if there exists a function $p(z^k) \in P_k$ such that*

$$f(z) = z \exp\left\{\int_0^z \frac{1}{\zeta}\left[\{1 + e^{i\gamma}\cos\gamma[p(\zeta^k) - 1]\}^{\frac{\alpha-\beta}{2}} - 1\right] d\zeta\right\} \tag{2.1}$$

Proof. If $f \in S^k_{(\alpha,\beta)}$ then there exists a function $F \in S_{(\alpha,\beta)}$ such that $f(z) = \sqrt[k]{F(z^k)}$.

Hence

$$\frac{zf'(z)}{f(z)} = \frac{z^k F'(z^k)}{F(z^k)} = \frac{\zeta F'(\zeta)}{F(\zeta)},$$

where $\zeta = z^k$. As we know, a function $F \in S_{(\alpha,\beta)}$ satisfies the following condition

$$\frac{\zeta F'(\zeta)}{F(\zeta)} = \{1 + e^{i\gamma}\cos\gamma[p(\zeta) - 1]\}^{\frac{\alpha-\beta}{2}}, \ |\zeta| < 1, \ \zeta \in K_1, \ \gamma = \frac{\pi}{2}\frac{\alpha+\beta}{\alpha-\beta}$$

and so

$$\frac{zf'(z)}{f(z)} = \{1+e^{i\gamma}\cos\gamma[p(z^k)-1]\}^{\frac{\alpha-\beta}{2}}$$

hence we get (2.1).

Theorem 2.3. *In the class $S^k_{(\alpha,\beta)}$, the set of variability of the functional $w = [zf'(z)/f(z)]^{\frac{2}{\alpha-\beta}}$, $z \in K_1$ is the disc*

$$\left| w - \frac{1+r^{2k}e^{2i\gamma}}{1-r^{2k}} \right| \leqslant \frac{2r^k\cos\gamma}{1-r^{2k}}, \quad |z| = r,\ \gamma = \frac{\pi}{2}\frac{\alpha+\beta}{\alpha-\beta}$$

The extremal functions are of the of form

$$f(z) = z\exp\left\{\int_0^z \frac{1}{\zeta}\left[\left(1+\frac{2e^{i\gamma}\zeta^k\cos\gamma}{1-\zeta^k}\right)^{\frac{\alpha-\beta}{2}} - 1\right]d\zeta\right\}$$

Proof. From the definition of the class $S^k_{(\alpha,\beta)}$ we have

$$\frac{zf'(z)}{f(z)} = \{1+e^{i\gamma}\cos\gamma[h(z)-1]\}^{\frac{\alpha-\beta}{2}}, \ h \in P_k$$

whence

$$w = \left[\frac{zf'(z)}{f(z)}\right]^{\frac{2}{\alpha-\beta}} = 1+e^{i\gamma}\cos\gamma[h(z)-1] = \int_{-\pi}^{\pi} \frac{1+z^k e^{-i(t-2\gamma)}}{1-z^k e^{-it}}\, d\mu(t)$$

because functions of the class P_k can be introduced by means of integral formula of Herglotz-Stieltjes in the following way

$$h(z) = \int_{-\pi}^{\pi} \frac{1+z^k e^{-it}}{1-z^k e^{-it}}\, d\mu(t)$$

where $\mu(t)$ is a real non-decreasing function in $\langle -\pi, \pi\rangle$ satisfying conditions: $\int_{-\pi}^{\pi} d\mu(t) = 1$, $\mu(-\pi+0) = \mu(-\pi)$, $\mu(\pi) = 1$. The set of variability of the functional w is the convex hull of the domain bounded by the curve

$$\zeta = \frac{1+z^k e^{-i(t-2\gamma)}}{1-z^k e^{-it}}, \quad t \in \langle -\pi, \pi\rangle$$

The equation of this curve can be written in the form $|(\zeta-1)/(\zeta+e^{2i\gamma})| = r^k$, $r = |z|$.

It is the circle about $\zeta_0 = (1+r^{2k}e^{2i\gamma})/(1-r^{2k})$ and the radius $\varrho = 2r^k\cos\gamma\,/\,(1-r^{2k})$.

Corollary. *If* $f \in S^k_{(a,\beta)}$, *then*

$$\left[\frac{\sqrt{1+2r^{2k}\cos 2\gamma+r^{4k}}-2r^k\cos\gamma}{1-r^{2k}}\right]^{\frac{a-\beta}{2}} \leqslant \left|\frac{zf'(z)}{f(z)}\right|$$

$$\leqslant \left[\frac{\sqrt{1+2r^{2k}\cos 2\gamma+r^{4k}}+2r^k\cos\gamma}{1-r^{2k}}\right]^{\frac{a-\beta}{2}}$$

$$\frac{a-\beta}{2}\left[\operatorname{arctg}\frac{r^{2k}\sin 2\gamma}{1+r^{2k}\cos 2\gamma}-\arcsin\frac{2r^k\cos\gamma}{\sqrt{1+2r^{2k}\cos 2\gamma+r^{4k}}}\right] \leqslant \arg\frac{zf'(z)}{f(z)}$$

$$\leqslant \frac{a-\beta}{2}\left[\operatorname{arctg}\frac{r^{2k}\sin 2\gamma}{1+r^{2k}\cos 2\gamma}+\arcsin\frac{2r^k\cos\gamma}{\sqrt{1+2r^{2k}\cos 2\gamma+r^{4k}}}\right]$$

$$\frac{1+r^{2k}\cos 2\gamma-2r^k\cos\gamma}{1-r^{2k}} \leqslant \operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right]^{\frac{2}{a-\beta}} \leqslant \frac{1+r^{2k}\cos 2\gamma+2r^k\cos\gamma}{1-r^{2k}}$$

where $\gamma = \frac{\pi}{2}\frac{a+\beta}{a-\beta}$.

For suitable a and β we obtain above estimations in the classes S^*_k, S^k_a, $\check{S}^k_\delta$ and for $k = 1$ in the class $S_{(a,\beta)}$, and for $k = 1$ and suitable a and β in the classes S^*, S_a and $\check{S}_\delta$.

Theorem 2.4. *If* $f(z) = z+\sum_{n=1}^{\infty} a_{nk+1}z^{nk+1}$ *belongs to the class* $S^k_{(a,\beta)}$ *and* λ *is an arbitrary complex number, then*

(2.2) $$|a_{2k+1}-\lambda a^2_{k+1}| \leqslant$$
$$\leqslant \frac{(a-\beta)\cos\gamma}{2k}\max\left(1,\ \left|e^{i\gamma}\cos\gamma\left[1-\frac{(a-\beta)(k+2-4\lambda)}{2k}\right]-1\right|\right)$$

For each λ *there exist functions:*

$$f_1(z) = z\exp\int_0^z \frac{1}{\zeta}\left[\left\{1+\frac{2\zeta^k e^{i\gamma}\cos\gamma}{1-\zeta^k}\right\}^{\frac{a-\beta}{2}}-1\right]d\zeta$$

$$f_2(z) = z\exp\int_0^z \frac{1}{\zeta}\left[\left\{1+\frac{2\zeta^{2k} e^{i\gamma}\cos\gamma}{1-\zeta^{2k}}\right\}^{\frac{a-\beta}{2}}-1\right]d\zeta$$

belonging to the class $S^k_{(a,\beta)}$ *such that the inequality* (2.2) *becames an equality.*

To prove this theorem, we need the following

Lemma. *If* $h(z) = 1+h_kz^k+h_{2k}z^{2k}+\ldots$, $z \in K_1$, $h \in P_k$ *and* τ *is an arbitrary complex number, then*

(2.3) $$|h_{2k}-\tau h^2_k| \leqslant 2\max(1,\ |2\tau-1|)$$

Proof. First, we'll prove that if $p(z) = 1+p_1z+p_2z^2+\ldots \in P$, then

(2.4) $$|p_2-\eta p_1^2| \leqslant 2\max(1, |2\eta-1|)$$

where η is an arbitrary complex number.

It's known that

(2.5) $$p(z) = \frac{1+\omega(z)}{1-\omega(z)}$$

where $\omega(z) = a_1z+a_2z^2+\ldots$, $z\in K_1$ is regular function in K_1 and satisfying the condition: $|\omega(z)|<1$ for $z\in K_1$.

For the function ω, the following inequality is well known [4]

(2.6) $|a_2-\lambda a_1^2| \leqslant \max(1, |\lambda|)$, λ is any real number.

From (2.5) it follows that $p_1 = 2a_1$, $p_2 = 2(a_2+a_1^2)$, and so

$$|p_2-\eta p_1^2| = 2|a_2-(2\eta-1)a_1^2| \leqslant 2\max(1, |2\eta-1|)$$

If $h(z) = 1+h_kz^k+h_{2k}z^{2k}+\ldots \in P_k$ then there exists a function $p\in P$ such, that $h(z) = p(z^k)$, hence $h_k = p_1$, $h_{2k} = p_2$. Now, making use of (2.4) we get (2.3). Equalities in (2.3) occur when the functions h_1 and h_2 take the form:

$$h_1(z) = \frac{1+z^k}{1-z^k}$$

$$h_2(z) = \frac{1+z^{2k}}{1-z^{2k}}$$

Proof of the theorem 2.4.

If $f(z) = z+a_{k+1}z^{k+1}+a_{2k+1}z^{2k+1}+\ldots \in S^k_{(\alpha,\beta)}$, *then*

$$\frac{zf'(z)}{f(z)} = \{1+e^{i\gamma}\cos\gamma[h(z)-1]\}^{\frac{\alpha-\beta}{2}}, \ h\in P_k, \gamma = \frac{\pi}{2}\frac{\alpha+\beta}{\alpha-\beta}.$$

From this it follows that

$$a_{k+1} = \frac{\alpha-\beta}{2k}e^{i\gamma}\cos\gamma\, h_k$$

$$a_{2k+1} = \frac{\alpha-\beta}{4k}e^{i\gamma}\cos\gamma\left[h_{2k}+\frac{e^{i\gamma}\cos\gamma[(k+2)(\alpha-\beta)-2k]}{4k}\cdot h_k^2\right]$$

or

$$|a_{2k+1}-\lambda a_{k+1}^2| = \frac{\alpha-\beta}{4k}\cos\gamma\left|h_{2k}-\frac{e^{i\gamma}\cos\gamma}{2}\left[1-\frac{(\alpha-\beta)(k+2-4\lambda)}{2k}\right]h_k^2\right|$$

Making use of (2.3) we have

$$|a_{2k+1}-\lambda a_{k+1}^2| \leqslant \frac{a-\beta}{2k}\cos\gamma \max\left(1, \left|e^{i\gamma}\cos\gamma\left[1-\frac{(a-\beta)(k+2-4\lambda)}{2k}\right]-1\right|\right)$$

Corollaries.

1. *If* $f\in S^k_{(1,-1)} = S^*_k$ *then* $|a_{2k+1}-\lambda a_{k+1}^2| \leqslant \frac{1}{k}\max\left(1, \left|\frac{k+2-4\lambda}{k}\right|\right)$

2. *If* $f\in S^k_{(a,-a)} = S^k_a$ *then* $|a_{2k+1}-\lambda a_{k+1}^2| \leqslant \frac{a}{k}\max\left(1, \left|\frac{a(k+2-4\lambda)}{k}\right|\right)$

3. *If* $f\in S^k_{(a,a-2)} = \check{S}^k_\delta$ *then* $|a_{2k+1}-\lambda a_{k+1}^2| \leqslant \frac{1}{k}\cos\delta\max\left(1, \left|e^{i\delta}\cos\delta \times\right.\right.$ $\left.\left.\times\left[1-\frac{k+2-4\lambda}{k}\right]-1\right|\right)$, *where* $\delta = \frac{\pi}{2}(a-1)$. For $k=1$ we get the estimation of $|a_3-\lambda a_2^2|$, λ is an arbitrary complex number, in the class $S_{(a,\beta)}$, and for the suitable values of a and β also in S^*, S_a, $\check{S}_\delta$.

Theorem 2.5. *If* $f\in S^k_{(a,\beta)}$, *then*

a) $|a_{k+1}| \leqslant \frac{a-\beta}{k}\cos\gamma$

b) $|a_{2k+1}| \leqslant \frac{(a-\beta)\cos\gamma}{2k}\max\left(1, \left|e^{i\gamma}\cos\gamma\left[1-\frac{(k+2)(a-\beta)}{2k}\right]-1\right|\right)$

Proof. The inequality a) follows from the facts that $a_{k+1} = \frac{a-\beta}{2k}e^{i\gamma}\cos\gamma\, h_k$ and $|h_k|\leqslant 2$, and if in (2.2) we put $\lambda=0$, we'll get the inequality b).

The inequality b) can be written in the form:

$$|a_{2k+1}| \leqslant \begin{cases} \frac{a-\beta}{2k}\cos\gamma & \text{for } 0 < a-\beta \leqslant \frac{2k}{k+2} \\ \frac{a-\beta}{2k}\cos\gamma\left|e^{i\gamma}\cos\gamma\left[1-\frac{(k+2)(a-\beta)}{2k}\right]-1\right| & \text{for } \frac{2k}{k+2} \leqslant a-\beta\leqslant 2 \end{cases}$$

The conditions: $0 < a-\beta \leqslant \frac{2k}{k+2}$ and $\frac{2k}{k+2} < a-\beta\leqslant 2$ follow from inequalities: $\left|e^{i\gamma}\cos\gamma\left[1-\frac{(k+2)(a-\beta)}{2k}\right]-1\right| < 1$ and

$$\left|e^{i\gamma}\cos\gamma\left[1-\frac{(k+2)(a-\beta)}{2k}\right]-1\right| > 1$$

respectively.

Corollary. *For suitable values of α and β we get estimations of $|a_{k+1}|$ and $|a_{2k+1}|$ in the classes S_k^*, S_α^k, $\check{S}_\delta^k$ and namely:*

1. *In the class $S_k^* = S_{(1,-1)}^k$* $\quad |a_{k+1}| \leqslant \frac{2}{k}$

$$|a_{2k+1}| \leqslant \frac{1}{k}\max\left(1, \frac{k+2}{k}\right)$$

2. *In the class $S_\alpha^k = S_{(\alpha,-\alpha)}^k$* $\quad |a_{k+1}| \leqslant \frac{2\alpha}{k}$

$$|a_{2k+1}| \leqslant \frac{\alpha}{k}\max\left(1, \alpha\frac{k+2}{k}\right)$$

3. *In the class $\check{S}_\delta^k = \check{S}_{(\alpha,\alpha-2)}^k$* $\quad |a_{k+1}| \leqslant \frac{2}{k}\cos\delta$

$$|a_{2k+1}| \leqslant \frac{\cos\delta}{k}\max\left(1, \left|\frac{2}{k}e^{i\delta}\cos\delta+1\right|\right)$$

where $\delta = \frac{\pi}{2}(\alpha-1)$

For $k=1$ and suitable α and β we get the estimations of $|a_2|$ and $|a_3|$ in the classes $S_{(\alpha,\beta)}$, S^*, S_α, $\check{S}_\delta$.

3. Let $S^k \subset S$ denote the class of k-symmetric, univalent in K_1 functions of the form

$$f(z) = z + a_{k+1}z^{k+1} + a_{2k+1}z^{2k+1} + \ldots, \; z \in K_1$$

Let $f \in S_{(\alpha,\beta)}^k$, $\alpha \in (0, 2)$, $\beta \in (-2, 0)$, $\alpha - \beta \leqslant 2$. Denote by

$$r_k(\alpha, \beta) = \sup_r \left\{r\colon \bigwedge_{f \in S^k} \bigwedge_{|z| \leqslant r} \beta\frac{\pi}{2} < \arg\frac{zf'(z)}{f(z)} < \alpha\frac{\pi}{2}\right\}$$

and

$$r_k^*(\alpha, \beta) = \sup_r \left\{r\colon \bigwedge_{f \in S_{(\alpha,\beta)}^k} \bigwedge_{|z| \leqslant r} \left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}\right\}$$

Theorem 3.1. $r_k(\alpha, \beta) = \sqrt[k]{\operatorname{th}\eta\frac{\pi}{4}}$, $\eta = \min\{\alpha, -\beta\}$.

Proof. If $f \in S^k$ then

$$\left|\arg\frac{zf'(z)}{f(z)}\right| \leqslant \log\frac{1+r^k}{1-r^k}, \qquad |z| \leqslant r$$

and the estimation is sharp.

The condition

$$\beta\frac{\pi}{2} \leqslant \log\frac{1-r^k}{1+r^k} \leqslant \arg\frac{zf'(z)}{f(z)} \leqslant \log\frac{1+r^k}{1-r^k} \leqslant a\frac{\pi}{2}$$

will be satisfied for every function $f \in S^k$, if $\log\frac{1+r^k}{1-r^k} \leqslant \eta\frac{\pi}{2}$, where $\eta = \min\{a, -\beta\}$.

From this it follows that $r_k(a, \beta)$ is the solution of the equation

$$\log\frac{1+r^k}{1-r^k} = \eta\frac{\pi}{2}$$

Hence $r_k(a, \beta) = \sqrt[k]{\operatorname{th}\eta\frac{\pi}{4}}$.

Theorem 3.2. $r_k^*(a, \beta) = 1$, *when* $a \in (0, 1\rangle$, $\beta \in \langle -1, 0)$ *and* $r_k^*(a, \beta)$ *is the root of the equation*

$$(3.1)\quad \frac{a-\beta}{2}\left[\left|\operatorname{arctg}\frac{r^{2k}\sin 2\gamma}{1+r^{2k}\cos 2\gamma}\right| + \arcsin\frac{2\cos\gamma r^k}{\sqrt{1+2r^{2k}\cos 2\gamma + r^{4k}}}\right] = \frac{\pi}{2}$$

otherwise.

Proof. If $a \in (0, 1\rangle$, $\beta \in \langle -1, 0)$ and $f \in S^k(a, \beta)$ then

$$-\frac{\pi}{2} \leqslant \beta\frac{\pi}{2} < \arg\frac{zf'(z)}{f(z)} < a\frac{\pi}{2} \leqslant \frac{\pi}{2}$$

and so $\operatorname{Re}\frac{zf'(z)}{f(z)} > 0$ for $z \in K_1$.

This means that in this case $r_k^*(a, \beta) = 1$.

If $f \in S^k_{(a,\beta)}$ then

$$\left|\arg\frac{zf'(z)}{f(z)}\right| \leqslant \frac{a-\beta}{2}\left[\left|\operatorname{arctg}\frac{r^{2k}\sin 2\gamma}{1+r^{2k}\cos 2\gamma}\right| + \arcsin\frac{2\cos\gamma r^k}{\sqrt{1+2r^{2k}\cos 2\gamma + r^{4k}}}\right].$$

where

$$\gamma = \frac{\pi}{2}\,\frac{a+\beta}{a-\beta}.$$

The right hand member of this inequality is strictly increasing function of r which is bounded by $\frac{\pi}{2}$ if $r \in (0, r_k^*(a, \beta))$, $a \in (1, 2)$, $\beta \in (-2, -1)$, where $r_k^*(a, \beta)$ is the unique, positive root of the equation (3.1).

4. In this part of the paper we deal with a relation between module and domain majorization of the functions of the class $S^k_{(a,\beta)}$. The function $f(z) = a_1 z + a_2 z^2 + \ldots, z \in K_1$, is said to be module subordinated to the function $F(z) = A_1 z + A_2 z^2 + \ldots$ if $|f(z)| \leqslant |F(z)|$ for every $z \in K_r$. This fact will be written in the following way: $|f, F, r|$. If $f(z) = F(\omega(z))$ for every $z \in K_r$, where the function $\omega(z)$ is holomorphic in K_r and such that $\omega(0) = 0$, $|\omega(z)| < r$ for $z \in K_r$, then f is said to be domain subordinated to the function F in K_r and we write it (f, F, r). In the case, when F is univalent function, the above condition means that

$$f(K_r) \subset F(K_r)$$

Now suppose that $F \in S^k_{(a,\beta)}$ and $f(z)/f'(0) \in S^k_{(a,\beta)}$. We deal with the following problems:

1. Find possibly greatest number $\tilde{r}_0 \in (0, 1)$ such that independently of the choice of functions f and F, the following implication is satisfied:

$$(f, F, 1) \Rightarrow |f, F, \tilde{r}_0|$$

2. Find possibly greatest number $r_0 \in (0, 1)$ such, that independently of the choice of functions f and F, the following implication is satisfied:

$$|f, F, 1| \Rightarrow (f, F, r_0)$$

Let S_v denote the class of functions $F(z) = z + A_2 z^2 + \ldots$ holomorphic and univalent in K_1 and satisfying the following condition for every $r \in \langle 0, 1)$:

$$\left| \arg \frac{zF'(z)}{F(z)} \right| \leqslant v(r) \quad \text{for } |z| \leqslant r < 1$$

where

$$v(r) = \sup_{F \in Sv} \left\{ \sup_{|z| \leqslant r} \left| \arg \frac{zF'(z)}{F(z)} \right| \right\}$$

is the continuous function in $\langle 0, 1)$.
From this it follows that the function $v(r)$ is strictly increasing in $\langle 0, 1)$ and $v(0) = 0$, provided that this class doesn't contain only an identity.

Let

$$r(v) = \sup_{r \in \langle 0,1)} \left\{ r\colon\ v(r) + 2\operatorname{arc\,tg} r < \frac{\pi}{2} \right\}$$

The number $r(v)$ is the unique positive root of the equation

$$v(r) + 2\operatorname{arc\,tg} r = \frac{\pi}{2}$$

In the papers [1], [2] the following theorems have been proved:

Theorem 4.1. *If $F \in S_v$ and $f(z) = a_1 z + a_2 z^2 + \ldots, a_1 > 0$ is holomorphic function in K_1 and $f(z) \neq 0$ for $z \neq 0, z \in K_1$ and if $(f, F, 1)$ then $|f, F, r(v)|$, where $r(v)$ is the unique root of the equation*

$$v(r) + 2\operatorname{arctg} r = \frac{\pi}{2}$$

The number $r(v)$ can't be replaced by any greater one.

Theorem 4.2. *If $F \in S_v$ and $f(z)/f'(0) \in S_v, f'(0) > 0$, and $|f, F, 1|$ then $(f, F, r(v))$, where $r(v)$ is the unique root of the equation*

$$v(r) + 2\operatorname{arctg} r = \frac{\pi}{2}$$

The number $r(v)$ can't be replaced by any greater one.

Let $S_v^k \subset S_v$ denote the class of functions of the form

$$F(z) = z + A_{k+1} z^{k+1} + A_{2k+1} z^{2k+1} + \ldots, z \in K_1$$

holomorphic, univalent and k-symmetric in K_1.
If $F \in S_v^k$, then

$$\left| \arg \frac{zF'(z)}{F(z)} \right| \leqslant v(r^k)$$

It easily follows from the fact that $F(z) = \sqrt[k]{G(z^k)}, z \in K_1$, where $G \in S_v$, whence

$$\frac{zF'(z)}{F(z)} = \frac{z^k G'(z^k)}{G(z_k)}$$

and

$$\left| \arg \frac{zF'(z)}{F(z)} \right| = \left| \arg \frac{z^k G'(z^k)}{G(z^k)} \right| \leqslant v(r^k), \text{ where } |z| = r.$$

And in the class S_v^k we can state the following theorems:

Theorem 4.1'. *If $F \in S_v^k$ and $f(z) = a_1 z + a_2 z^2 + \ldots, a_1 > 0$ is holomorphic function in the circle K_1 and $f(z) \neq 0$ for $z \neq 0, z \in K_1$, and if $(f, F, 1)$ then $|f, F, r(v)|$, where $r(v)$ is the unique root of the equation*

$$v(r^k) + 2\operatorname{arctg} r = \frac{\pi}{2}$$

The number $r(v)$ can't be replaced by any greater one.

Theorem 4.2'. *If $F \in S_v^k$ and $f(z)/f'(0) \in S_v^k, f'(0) > 0$ and $|f, F, 1|$ then $(f, F, r(v))$, where $r(v)$ is the unique root of the equation*

$$v(r^k) + 2\operatorname{arctg} r^k = \frac{\pi}{2}$$

The number $r(v)$ can't be replaced by any greater one. Now we give applications of these theorems to the class $S^k_{(\alpha,\beta)}$.

If $F \in S^k_{(\alpha,\beta)}$, then as it follows from theorem 2.3

$$\left|\arg\frac{zF'(z)}{F(z)}\right| \leqslant v(r^k) \tag{4.1}$$

where

$$v(r^k) = \sup_{F \in S^k_{(\alpha,\beta)}} \left\{ \sup_{|z| \leqslant r^k} \left| \arg \frac{zF'(z)}{F(z)} \right| \right\}$$

$$= \frac{\alpha-\beta}{2}\left[\left|\operatorname{arctg}\frac{r^{2k}\sin 2\gamma}{1+r^{2k}\cos 2\gamma}\right| + \arcsin\frac{2r^k\cos\gamma}{\sqrt{1+2r^{2k}\cos 2\gamma + r^{4k}}},\right.$$

$$\gamma = \frac{\pi}{2}\frac{\alpha+\beta}{\alpha-\beta}.$$

Making use of the theorems 4.1′ and 4.2′ and the estimation (4.1) we have

Theorem 4.3. *Let $F \in S^k_{(\alpha,\beta)}$ and $f(z) = a_1 z + a_2 z^2 + \ldots, a_1 > 0$ is holomorphic function in K_1 and $f(z)/z \neq 0$, $z \in K_1$ and $(f, F, 1)$. Then $|f, F, \tilde{r}_0|$, where $\tilde{r}_0$ is the unique root of the equation*

$$\frac{\alpha-\beta}{2}\left[\left|\operatorname{arctg}\frac{r^{2k}\sin 2\gamma}{1+r^{2k}\cos 2\gamma}\right| + \arcsin\frac{2r^k\cos\gamma}{\sqrt{1+2r^{2k}\cos 2\gamma + r^{4k}}} + 2\operatorname{arctg} r = \frac{\pi}{2},\right.$$

$$\gamma = \frac{\pi}{2}\frac{\alpha+\beta}{\alpha-\beta},$$

and can't be replaced by any greater number.

Theorem 4.4. *If $F \in S^k_{(\alpha,\beta)}$ and $f(z)/f'(0) \in S^k_{(\alpha,\beta)}$, $f'(0) > 0$ and $|f, F, 1|$ then (f, F, r_0), where r_0 is the unique root of the equation*

$$\frac{\alpha-\beta}{2}\left[\left|\operatorname{arctg}\frac{r^{2k}\sin 2\gamma}{1+r^{2k}\cos 2\gamma}\right| + \arcsin\frac{2r^k\cos\gamma}{\sqrt{1+2r^{2k}\cos 2\gamma + r^{4k}}} + 2\operatorname{arctg} r^k = \frac{\pi}{2}\right.$$

$$\gamma = \frac{\pi}{2}\frac{\alpha+\beta}{\alpha-\beta},$$

and can't be replaced by any greater number.

For suitable α and β we get the radii of subordination in the classes $S^k_{(1,-1)} = S^*_k$, $S^k_{(\alpha,-\alpha)} = S^k_\alpha$, $S^k_{(\alpha,\alpha-2)} = \mathfrak{S}^k_\delta$.

Theorem 4.5. *If $F \in S^*_k$ and $f(z) = a_1 z + a_2 z^2 + \ldots, a_1 > 0$ is holomorphic function in K_1 and $f(z)/z \neq 0$, $z \in K_1$, and $(f, F, 1)$ then $|f, F, \tilde{r}_0|$,* *where $\tilde{r}_0$ is the unique root of the equation*

$$\arcsin\frac{2r^k}{1+r^{2k}}+2\,\mathrm{arctg}\,r=\frac{\pi}{2}$$

Theorem 4.6. *If $F\in S_k^*$ and $f(z)/f'(0)\in S_k^*$, $f'(0)>0$ and $|f,F,1|$ then (f,F,r_0), where r_0 is the unique root of the equation*

$$\arcsin\frac{2r^k}{1+r^{2k}}+2\,\mathrm{arctg}\,r^k=\frac{\pi}{2}$$

Theorem 4.7. *If $F\in S_a^k$ and $f(z)=a_1z+a_2z^2+\ldots$, $a_1>0$ is holomorphic function in K_1 and $f(z)/z\neq 0$, $z\in K_1$, and $(f,F,1)$ then $|f,F,\tilde{r}_0|$, where $\tilde{r}_0$ is the unique root of the equation*

$$a\arcsin\frac{2r^k}{1+r^{2k}}+2\,\mathrm{arctg}\,r=\frac{\pi}{2}$$

Theorem 4.8. *If $F\in S_a^k$ and $f(z)/f'(0)\in S_a^k$, $f'(0)>0$ and $|f,F,1|$ then (f,F,r_0), where r_0 is the unique root of the equation*

$$a\arcsin\frac{2r^k}{1+r^{2k}}+2\,\mathrm{arctg}\,r^k=\frac{\pi}{2}$$

Theorem 4.9. *If $F\in \breve{S}_\delta^k$ and $f(z)=a_1z+a_2z^2+\ldots$, $a_1>0$ is holomorphic function in K_1 and $f(z)/z\neq 0$, $z\in K_1$, and $(f,F,1)$ then $\tilde{r}_0$ is the unique root of the equation*

$$\left|\mathrm{arctg}\frac{r^{2k}\sin 2\delta}{1+r^{2k}\cos 2\delta}\right|+\arcsin\frac{2r^k\cos\delta}{\sqrt{1+2r^{2k}\cos 2\delta+r^{4k}}}+2\,\mathrm{arctg}\,r=\frac{\pi}{2}$$

$$\delta=\frac{\pi}{2}(a-1).$$

Theorem 4.10. *If $F\in \breve{S}_\delta^k$ and $f(z)/f'(0)\in \breve{S}_\delta^k$, $f'(0)>0$ and $|f,F,1|$ then $(f,F,\mathring{r}_0)$, where r_0 is the unique root of the equation*

$$\left|\mathrm{arctg}\frac{r^{2k}\sin 2\delta}{1+r^{2k}\cos 2\delta}\right|+\arcsin\frac{2r^k\cos\delta}{\sqrt{1+2r^{2k}\cos 2\delta+r^{4k}}}+2\,\mathrm{arctg}\,r^k=\frac{\pi}{2}$$

$$\delta=\frac{\pi}{2}(a-1).$$

For $k=1$ we get the radii of subordinations in the classes $S_{(a,\beta)}$, S^*, S_a and $\breve{S}_\delta$.

REFERENCES

[1] Bielecki A., Lewandowski Z., *Sur une généralisation de quelques théorèmes de M. Biernacki sur les fonctions analytiques*, Ann. Polon. Math. 12 (1962), 65-70.
[2] —, *Sur certaines majorantes des fonctions holomorphes dans cercle unité*, Coll. Math. 9 (1962), 299-300.
[3] Bucka Cz., Ciozda K., *On a new subclass of the class S*, Ann. Polon. Math. 28 (1973), 153-161.
[4] Keogh F.R., Merkes E.P., *A coefficient inequality for certain classes of analytic functions*, Proc. Amer. Math. Soc. Vol 20, (1), (1969), 8-12.

STRESZCZENIE

Niech $S_{(\alpha,\beta)}$, $\alpha\in(0,2)$, $\beta\in(-2,0)$, $\alpha-\beta\leqslant 2$ oznacza klasę funkcji postaci $f(z) = z+a_2z^2+\ldots$, spełniających warunek: $\beta\pi/2<\arg[zf'(z)/ /f(z)]<\alpha\pi/2$, dla każdego $z\in K_1$. W niniejszej pracy rozpatrywana jest podklasa $S^k_{(\alpha,\beta)}$, $(k\geqslant 1)$ klasy $S_{(\alpha,\beta)}$. $S^k_{(\alpha,\beta)}$ jest klasą funkcji postaci (1.2) k-symetrycznych i jednolistnych w K_1. W pracy podany jest wzór strukturalny, obszar zmienności funkcjonału $w = [zf'(z)/f(z)]^{1/\alpha-\beta}$ oraz oszacowania $|zf'(z)/f(z)|$, $\arg\left(zf'(z)/f(z)\right)$, $\mathrm{Re}[zf'(z)/fz)]^{2/\alpha-\beta}$, $|a_{2k+1}-\lambda a^2_{k+1}|$, $|a_{k+1}|$, $|a_{2k+1}|$. Następnie zostały wyliczone promienie $r_k(\alpha,\beta)$ i $r^*_k(\alpha,\beta)$ w $S^k_{(\alpha,\beta)}$ W dalszej części pracy zbadano relację między podporządkowaniem modułowym a obszarowym funkcji klasy $S^k_{(\alpha\,\beta)}$.

РЕЗЮМЕ

Пусть $S_{(\alpha,\beta)}$, $\alpha\in(0,2)$, $\beta\in(-2,0)$, $\alpha-\beta\leqslant 2$ обозначает класс функций (1.1) удовлетворяющих условию $\beta\pi/2<\arg zf'(z)/f(z)<\alpha\pi 2\wedge z\in K_1$.

В настоящей работе рассмотрен подкласс $S^k_{(\alpha,\beta)}(k\geqslant 1)$ класса $S(\alpha,\beta)$. $S^k(\alpha,\beta)$ это класс функций вида (1.2) к-симметрических и однолистных в K_1.

В работе дается структуральную формулу, область изменения функционала $w = [zf'(z)|f(z)]2/\alpha-\beta$ и оценки: $|zf'(z)/f(z)$, $\arg zf'(z)/f(z)$, $\mathrm{Re}\,[zf'(z)/f(z)]2/\alpha-\beta$, $|a_{2k+1}-\lambda^2_{ak+1}|$, $|a_{k+1}|$, $|a_{2k+1}|$. Далее вычислены радиусы: $r^*_k(\alpha,\beta)$ и $r^*_k(\alpha,\beta)$ в $S^k_{(\alpha,\beta)}$ Кроме того исследовано зависимость между подчинением по модулю и по области функций класса $S^k_{(\alpha,\beta)}$