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On Starlike Functions

O funkcjach gwiaździstych

О звездообразных функциях

1. Introduction

The following theorem was a main tool in the proof of the Pólya-Schoenberg conjecture [3] and reflects the trivial fact that a starlike univalent function is starlike in every direction (in the sense of Robertson [2]):

Let $f(z)$ be a starlike univalent regular function in the unit disc $U = \{z \mid |z| < 1\}$, $f(0) = 0$. Then for every $t \in \mathbf{R}$ there exist t^*, $\varphi \in \mathbf{R}$ such that

$$\mathrm{Re}\left[e^{i\varphi}(1-ze^{-it})(1-ze^{-it^*})\frac{f(z)}{z}\right] > 0, \; z \in U.$$

Ranges for t^*, φ, depending on $f(z)$ and t, have also been given. The aim of the present paper is to generalize this theorem to functions starlike of order α. In the cases $\alpha = 1-n/2$, $n \in \mathbf{N}$, our conditions are also sufficient for these functions, thus creating a *characterization* of them. Furthermore our theorem, in one version, may be considered as an approximation result for starlike functions (of order 0).

2. Definitions and statement of results

By S_α, $\alpha < 1$, we denote the class of functions $f(z)$ starlike of order α. This means, by definition, that $f(z)$ is regular in U, $f(0) = 0$, and that

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > \alpha, \; z \in U. \tag{2.1}$$

These classes are connected by the following obvious relation:

$$f \in S_\alpha \Leftrightarrow g = z\left(\frac{f(z)}{z}\right)^{1/(1-\alpha)} \in S_0. \tag{2.2}$$

For every $f \in S_a$ the function

$$V(t) = \lim_{r\to 1} \arg f(re^{it}) \tag{2.3}$$

exists for all $t \in \mathbf{R}$ and has period 2π. Using (2.2) it is easily seen that theorems A and B below are equivalent.

Theorem A. *Let $f(z)$ be regular in U, $f(0) = 0$. Let $n \in \mathbf{N}$. Then $f \in S_o$ if, and only if, for every $t_o \in \mathbf{R}$ there exist numbers $t_k \in \mathbf{R}$, $k = 1, \ldots, n-1$, and $\varphi \in \mathbf{R}$ such that with*

$$P(z) = z \prod_{k=0}^{n-1} (1 - ze^{-it_k})^{-2/n} \tag{2.4}$$

we have

$$\left| \arg\left[e^{i\varphi} \frac{f(z)}{P(z)} \right] \right| < \frac{\pi}{n}, \; z \in U. \tag{2.5}$$

Remarks: i) It should be stressed that even the 'sufficiency' part of theorem A works for every fixed $n \in \mathbf{N}$.
ii) For $f \in S_o$ let $V(t)$ be as in (2.3). Then in (2.4), (2.5) we can choose

$$\sup\left\{t \,|\, V(t) < V(t_o) + \frac{2k\pi}{n}\right\} \leqslant t_k \leqslant \sup\left\{t \,|\, V(t) \leqslant V(t_o) + \frac{2k\pi}{n}\right\}, \tag{2.6}$$

$k = 1, \ldots, n-1$, and

$$\varphi = \frac{1-n}{n}\pi - V(t_o) + \frac{1}{n}\sum_{k=o}^{n-1} t_k. \tag{2.7}$$

Theorem B. *Let $f(z)$ be regular in U, $f(0) = 0$. Then $f \in S_{(2-n)/2}$, $n \in \mathbf{N}$, if, and only if, for every $t_o \in \mathbf{R}$ there exist numbers $t_k \in \mathbf{R}$, $k = 1, \ldots, n-1$, and $\varphi \in \mathbf{R}$ such that*

$$\mathrm{Re}\left[e^{i\varphi} \prod_{k=o}^{n-1} (1 - ze^{-it_k}) \frac{f(z)}{z} \right] > 0, \; z \in U. \tag{2.8}$$

Remark: Let $f \in S_{(2-n)/2}$. Possible ranges for t_k and φ in (2.8) may be found from (2.5), (2.6) and (2.7) via (2.2).
Theorem B has the following corollary.

Corollary. *Let $f \in S_a$, $a < 1$. Then for every $t_o \in \mathbf{R}$ there exist numbers $t_k \in \mathbf{R}$, $k = 1, \ldots, [1-2a]+1$, and $\varphi \in \mathbf{R}$ such that*

$$\operatorname{Re}\left[e^{i\varphi}(1-ze^{-it_o})^{\gamma}\prod_{k=1}^{[1-2a]+1}(1-ze^{-it_k})\frac{f(z)}{z}\right] > 0,\quad z \in U, \tag{2.9}$$

where $\gamma = 1-2a-[1-2a]$.[1])

Remark: The 'sufficiency' part of theorem B fails to be true for the corollary.

3. One preliminary lemma

Let A be the class of functions $a(t)$, $t \in \mathbf{R}$, with the following properties:

$$\left\{\begin{array}{l} \text{i) } a(t) \text{ is nondecreasing,} \\ \text{ii) } a(t) - t \text{ has period } 2\pi, \\ \text{iii) } a(t) = \tfrac{1}{2}\big(a(t+0)+a(t-0)\big),\ t \in \mathbf{R}. \end{array}\right. \tag{3.1}$$

For $s \in \mathbf{R}$ let

$$p_s(t) = \left|\begin{array}{l} 2k\pi \text{ for } 2(k-1)\pi < t-s < 2k\pi,\ k \in \mathbf{Z}, \\ (2k+1)\pi \text{ for } t = s+2k\pi,\ k \in \mathbf{Z}. \end{array}\right.$$

Obviously $p_s(t) \in A$ for all $s \in \mathbf{R}$.

Lemma. *Let $a(t) \in A$, $t_0 \in \mathbf{R}$, $n \in \mathbf{N}$. Let*

$$\sup\left\{t \mid a(t) < a(t_o) + \frac{2k\pi}{n}\right\} \leqslant t_k \leqslant \sup\left\{t \mid a(t) \leqslant a(t_o) + \frac{2k\pi}{n}\right\}, \tag{3.2}$$

$k = 1, \ldots, n-1$, *and*

$$\mu = a(t_0) - \frac{\pi}{n}. \tag{3.3}$$

Then, with

$$h(t) = \frac{1}{n}\sum_{k=0}^{n-1} p_{t_k}(t)$$

we have for $t \in \mathbf{R}$

$$|a(t) - h(t) - \mu| \leqslant \frac{\pi}{n}. \tag{3.5}$$

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[1]) We use the following conventions: i) $[a]$ denotes the greatest integer less (!) than a. ii) The empty product equals one.

Proof. Obviously $t_{k-1} \leqslant t_k$ and $t_k \in [t_o, t_o + 2\pi]$, $k = 1, \ldots, n-1$. We put $t_n = t_o + 2\pi$, $I_k = (t_{k-1}, t_k)$. Then, if I_k is nonempty, we find

$$h(t) = \frac{2k\pi}{n}, \quad t \in I_k. \tag{3.6}$$

Using the relation

$$\sup\{t | a(t) < \eta\} = \inf\{t | a(t) \geqslant \eta\},$$

valid for every nondecreasing function and every $\eta \in \mathbf{R}$, one easily deduces

$$a(t_o) + \frac{2(k-1)\pi}{n} \leqslant a(t) \leqslant a(t_o) + \frac{2k\pi}{n}, \quad t \in I_k, \tag{3.7}$$

for every nonempty I_k. Combining (3.6), (3.7) we obtain (3.5) for $t \in \bigcup_{k=1}^{n} I_k$. But $a(t) - h(t) - \mu$ has period 2π and possesses the property iii) of (3.1). Thus (3.5) holds for all $t \in \mathbf{R}$.

4. Proof of Theorems A and B

Theorems A and B are equivalent. Thus we can prove the 'necessity' part of theorem A and the 'sufficiency' part of theorem B to obtain a complete proof of both theorems.

Let $f \in S_o$, $t_o \in \mathbf{R}$, $n \in \mathbf{N}$. It is well-known that

$$V(t) = \lim_{r \to 1} \arg f(re^{it}) \in A$$

(cf. Pommerenke [1]). To $V(t)$ we apply the lemma which yields a function $h(t)$ (given by (3.4)) with

$$|V(t) - h(t) - \mu| \leqslant \frac{\pi}{n}, \quad t \in \mathbf{R}. \tag{4.1}$$

It is easily seen that $P(z)$ (given by (2.4)) may be represented by

$$P(z) = z\exp\left[-\frac{1}{\pi}\int_0^{2\pi} \log(1 - ze^{-it})\,dh(t)\right].$$

Obviously $P \in S_o$ and another result of Pommerenke [1] implies

$$\lim_{r \to 1} \arg P(re^{it}) = h(t) + C, \tag{4.2}$$

where C is a constant. With $\varphi = C - \mu$ we obtain from (4.1), (4.2)

$$\lim_{r \to 1} \left| \arg\left[e^{i\varphi} \frac{f(re^{it})}{P(re^{it})}\right] \right| \leqslant \frac{\pi}{n}, \quad t \in \mathbf{R}. \tag{4.3}$$

Since

$$\arg\left[e^{i\varphi}\frac{f(z)}{P(z)}\right]=\frac{1}{2\pi}\int_{o}^{2\pi}\left(\arg\left[e^{i\varphi}\frac{f(re^{it})}{P(re^{it})}\right]\right)\operatorname{Re}\frac{re^{it}+z}{re^{it}-z}\,dt$$

for $|z|<r$, (2.5) follows from (4.3) in combination with Lebesgue's dominated convergence theorem.

An evaluation of (4.2) at $t=o$ gives

$$C=-\pi+\frac{1}{n}\sum_{k=0}^{n-1}t_k,$$

so that (2.6) and (2.7) follow.

Now let $f(z)$ fulfil (2.8). Writing

$$Q(z,t_o)=z\prod_{k=0}^{n-1}(1-ze^{-it_k})^{-1},\ t\in\mathbf{R},$$

$h(z)=f(z)/Q(z,t_0)$, we have $\operatorname{Re}e^{i\varphi}h(z)>0,\ z\in U$, which implies, by a well-known result,

$$\left|\frac{zh'(z)}{h(z)}\right|\leqslant\frac{2|z|}{1-|z|^2},\ z\in U.$$

Thus

$$\left|\frac{zf'(z)}{f(z)}-\frac{zQ'(z,t_0)}{Q(z,t_0)}\right|\leqslant\frac{2|z|}{1-|z|^2},\ z\in U. \tag{4.4}$$

For a fixed $z\in U$ we have from (4.4)

$$\begin{aligned}\operatorname{Re}\frac{zf'(z)}{f(z)}&\geqslant\max_{t_0\in\mathbf{R}}\operatorname{Re}\frac{zQ'(z,t_o)}{Q(z,t_o)}-\frac{2|z|}{1-|z|^2}\\&\geqslant\max_{t_0\in\mathbf{R}}\operatorname{Re}(1-ze^{-it_0})^{-1}-\frac{n-1}{2}-\frac{2|z|}{1-|z|^2}\\&=\frac{1}{1+|z|}-\frac{n-1}{2}>\frac{2-n}{2},\end{aligned}$$

which implies $f\in S_{(2-n)/2}$.

5. Proof of corollary

Let $f\in S_a$, $a<1$. A straightforward calculation gives

$$g(z)=(1-ze^{-it_o})^{\delta}f(z)\in S_{-[1-2a]/2},$$

where $\delta=-2a-[1-2a]$. Applying theorem B to $g(z)$ gives the result.

For every $t_0 \in \mathbf{R}$ there exist $\varphi \in \mathbf{R}$ such that

$$\operatorname{Re}\left[e^{2i\varphi}\frac{1-ze^{-it_0}}{1-z}\right] > 0, \quad z \in U.$$

Thus

$$\left|\arg\left[e^{2i\varphi}\frac{1-ze^{-it_0}}{(1-z)^2}\right]\right| \leqslant \pi$$

or

$$\operatorname{Re}\left[e^{i\varphi}\frac{(1-ze^{-it_0})^{1/2}}{1-z}\right] > 0, \quad z \in U.$$

Since $z/(1-z) \notin S_{3/4}$ our example shows that (2.9) is not sufficient for a regular function $f(z)$ with $f(0) = 0$ to be in S_α.

REFERENCES

[1] Pommerenke Ch., *On Starlike and Convex Functions*, J. London Math. Soc., 37 (1962), 209-224.

[2] Robertson M. S., *Analytic Functions Starlike in one Direction*, Amer. J. Math. 58 (1936), 465-472.

[3] Ruscheweyh St. and Sheil-Small T., *Hadamard Products for Schlicht Functions and the Pólya-Schoenberg Conjecture*, Comment. Math. Helv., 48 (1973), 119-135.

STRESZCZENIE

Niech S_α będzie klasą funkcji α-gwiaździstych ze zwykłym unormowaniem. Nawiązując do jednej z prac poprzednich, autor podał dwa warunki konieczne i dostateczne na to, by $f \in S_\alpha$. Warunki te dotyczą przypadków $\alpha = 0$, $\alpha = 1-n/2$, przy czym pierwszy przypadek prowadzi do pewnego wyniku aproksymacyjnego dla funkcji gwiaździstych.

РЕЗЮМЕ

Пусть S_α будет классом α-звездообразных функций с обычным нормированием. Обращаясь к одной из предыдущих работ, автор дает два условия, необходимые и достаточные для того, чтобы $f \in S_\alpha$.

Эти условия относятся к случаям $\alpha = 0$, $\alpha = 1-n/2$, при этом первый случай приводит к некоторому результату, апроксимативному для звездообразных функций.