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# Functions with Bounded Mocanu Variation II

Funkcje z ograniczoną wariacją Mocanu II

Функции с ограниченной вариацией по Мокану II

## I. Introduction

Let $M$ denote the class of functions $f(z)$ which are analytic in the unit disk $\Delta$, normalized by the conditions $f(0) = 0$ and $f'(0) = 1$, and which satisfy the condition $f(z)f'(z)/z \neq 0$, $z \in \Delta$. If $f(z)$ is in $M$ and $a$ is a real non-negative number then the Mocanu angle $\Psi$ is defined in [2] as

$$\Psi = (1-a)\arg\{f(z)\} + a\arg\{izf'(z)\};$$

$f(z)$ is said to have bounded Mocanu variation if the total variation of this angle on every circle $|z| = r$, $0 < r < 1$, remains bounded as $r \to 1$. The collection of all functions $f(z)$ for which this variation is bounded by $k\pi$ ($k \geqslant 2$) is denoted by $MV[a, k]$. Equivalently, this condition can be expressed as $f(z) \in MV[a, k]$ if $f(z)$ is in $M$ and

$$\int_0^{2\pi} \left| \mathrm{Re}\left\{(1-a)\frac{zf'(z)}{f(z)} + a\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\}\right| d\theta \leqslant k\pi.$$

For convenience we adopt the notation

$$J(a, f(z)) = (1-a)zf'(z)/f(z) + a(1 + zf''(z)/f'(z)).$$

The motivation for the study of these classes in [2] was, in part, due to the article of P. T. Mocanu [7]. In both [2] and [7] $a$ is assumed to be non-negative. Eenigenburg [4], Miller, Mocanu and Reade [6] have extended the work of [7] to the case where $a$ is real. It is the purpose of this note to extend the definition of $MV[a, k]$ to the case where $a$

is real and also to observe some other interesting properties of these classes of functions. Thus in what follows $a$ is assumed to be real unless specifically restricted.

## II. Basic results

The following two theorems and their corollaries are straightforward generalizations of results in [2]. The proofs have been omitted since they require only minor modifications of the earlier results.

**Theorem 1.** *If $f(z)$ is in $M$ and $z = re^{i\theta}$, $0 \leqslant r < 1$ then*

$$\int_0^{2\pi} |\mathrm{Re}\{zf'(z)/f(z)\}|\,d\theta \leqslant \int_0^{2\pi} \big|\mathrm{Re}\{J(a, f(z))\}\big|\,d\theta$$

*for all real $a$.*

**Theorem 2.** *If $f(z)$ is in $M$, $a \neq 0$, $\beta$ is real, and $z = re^{i\theta}$, $0 \leqslant r < 1$, then*

$$\int_0^{2\pi} \big|\mathrm{Re}\{J(\beta, f(z))\}\big|\,d\theta \leqslant \frac{|\beta| + |a-\beta|}{|a|} \int_0^{2\pi} \big|\mathrm{Re}\{J(a, f(z))\}\big|\,d\theta$$

**Corollary 1.** *If $a\beta > 0$ and $|a| \geqslant |\beta|$ then $MV[a, k] \subset MV[\beta, k]$; if $a\beta > 0$ and $|a| \leqslant |\beta|$ then $MV[a, k] \subset MV[\beta, (2\beta - a)k/a]$; and if $a\beta < 0$ then $MV[a, k] \subset MV[\beta, (a - 2\beta)k/a]$.*

**Corollary 2.** *If $f(z)$ is in $MV[a, k]$ for $a \neq 0$ then $f(z)$ has bounded boundary rotation.*

**Corollary 3.** *If $f(z)$ is in $M$ and is a convex univalent function then*

$$\int_0^{2\pi} \big|\mathrm{Re}\{J(\beta, f(z))\}\big|\,d\theta \leqslant \begin{cases} 2\pi & \left|\beta - 1/2\right| \leqslant 1/2 \\ 2\pi|2\beta - 1|; & \left|\beta - 1/2\right| > 1/2. \end{cases}$$

*These inequalities are sharp for* $f(z) = \frac{1}{2}\log\dfrac{1+z}{1-z}$.

## III. $MV[a, k]$ and univalent functions

Let $S$ denote the subclass of $M$ consisting of univalent functions and let $m(a) = \max[2, |2 + 2a|]$.

**Theorem 3.** *If* $k \leqslant m(a)$ *then* $MV[a, k] \subset S$.

**Proof.** If $a \geqslant 0$ the result follows from Theorem 4 in [2]. For $a < 0$ we will make use of the following result of Ogawa [8]. If $f(z)$ is in $M$, $a > -3/2$, $z = re^{i\theta}$, and

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\{1 + zf''(z)/f'(z) + azf'(z)/f(z)\}\, d\theta \geqslant -\pi \tag{3.1}$$

for each $r$, $0 \leqslant r \leqslant R$, and all $\theta_1, \theta_2$, $0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi$ then $f(z)$ is univalent in $|z| < R$. Now let $f(z) \in MV[a, k]$, $a < 0$. Since

$$\int_0^{2\pi} \left|\operatorname{Re}\{J(a, f(z))\}\right| d\theta \leqslant k\pi \text{ and} \tag{3.2}$$

$$\int_0^{2\pi} \operatorname{Re}\{J(a, f(z))\}\, d\theta = 2\pi$$

for every $r$, $0 \leqslant r < 1$, we have

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\{(1-a)zf'(z)/f(z) + a(1 + zf''(z)/f'(z))\}\, d\theta \leqslant (k+2)\pi/2$$

or

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\{\alpha zf'(z)/f(z) + 1 + zf''(z)/f'(z)\}\, d\theta \geqslant (k+2)\pi/2a$$

for each $r$, $0 \leqslant r < 1$ and all $\theta_1$ and $\theta_2$ satisfying $0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi$ where we have used $(1-a)/a = \alpha$. Thus if $(1-a)/a > -3/2$ or equivalently $a < -2$ then Ogawa's theorem shows that $f(z)$ is univalent when $(k+2)/2a \geqslant -1$ or $k \leqslant -2a-2 = m(a)$. Finally if $-2 \leqslant a \leqslant 0$ then $m(a) = 2$ and the only admissible value of $k$ satisfying $k \leqslant m(a)$ is $k = 2$. Using $k = 2$ in (3.2) shows that $\operatorname{Re}\{J(a, f(z))\} \geqslant 0$ and functions satisfying this condition are known to be univalent [6].

**Comment.** For all $a$, a routine calculation for the Koebe function $F(z) = \dfrac{z}{(1-z)^2}$ shows that $F(z) \in MV[a, m(a)]$, but $F(z) \notin MV[a, k]$ if $k < m(a)$.

For $k > m(a)$ and $a < -2$ we may make further use of Ogawa's theorem to estimate the radius of univalence for $MV[a, k]$.

Suppose now that $f \in MV[a, k]$ with $a < -2$ and $k > -2a-2$. By Ogawa's theorem $f$ will be univalent in $|z| < r$ if

$$\int_{\theta_1}^{\theta_2} \left\{\operatorname{Re}(1/a - 1)\frac{zf'(z)}{f(z)} + 1 + \frac{zf''(z)}{f'(z)}\right\} d\theta \geqslant -\pi$$

$0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi$. Note that we must have $1/a - 1 > -3/2$, i.e., $a < -2$.

Now given $f \in MV[\alpha, k]$ there is a $G \in V_k$ such that

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + \frac{zG''(z)}{G'(z)}.$$

Defining, for $\theta = \theta_2 - \theta_1$,

$$\Delta(r,\theta) = \inf_{f \in MV[\alpha,k]} \int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{(1/\alpha - 1)\frac{zf'(z)}{f(z)} + 1 + \frac{zf''(z)}{f'(z)}\right\} d\theta$$

$$\geqslant \inf_{G \in V_k} \frac{1}{\alpha} \int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{1 + \frac{zG''(z)}{G'(z)}\right\} d\theta = \frac{1}{\alpha} \sup_{G \in V_k} \int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{1 + \frac{zG''(z)}{G'(z)}\right\} d\theta,$$

it suffices to solve the inequality

$$\sup_{V_k} \int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{1 + \frac{zG''(z)}{G'(z)}\right\} d\theta \leqslant -\alpha\pi$$

(Note that $-\alpha\pi > 2\pi$).

Referring to the proof of Theorem 1 in [3] we find

$$\gamma(r,\theta) = \sup_{V_k} \int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{1 + \frac{zG''(z)}{G'(z)}\right\} d\theta$$

$$= 2\cot^{-1}\left[\frac{1-r^2}{1+r^2}\cot\frac{\theta}{2}\right] + k\cot^{-1}\left[\frac{1-r^2}{r\{2(1-\cos\theta)\}^{1/2}}\right],$$

(3.3) $$\frac{\partial\gamma}{\partial\theta} = \frac{1-r^2}{1-2r^2\cos\theta + r^4}\left[1 + r^2 + \frac{kr\sin\theta}{[2(1-\cos\theta)]^{1/2}}\right].$$

Noting that all zeros for (3.3) must occur for $\pi < \theta < 2\pi$ we have $\frac{\partial\gamma}{\partial\theta} = 0$ when $\cos\frac{\theta}{2} = \frac{1+r^2}{-kr}$. Let $\theta_0 \in (\pi, 2\pi)$ be chosen so that $\cos\frac{\theta_0}{2} = \frac{1+r^2}{-kr}$. Then $\cot\frac{\theta_0}{2} = \frac{1+r^2}{\sqrt{k^2r^2-(1+r^2)^2}}$ and $[2(1-\cos\theta)]^{1/2} = \frac{2}{kr} \times$ $\times [k^2r^2 - (1+r^2)^2]^{1/2}$. Thus

$$\max_{\theta} \gamma(r,\theta) = \gamma(r,\theta_0) = k\cot^{-1}\left(\frac{k\omega}{2}\right) - 2\cot^{-1}(\omega) + 2\pi$$

where $\omega = (1-r^2)[k^2r^2 - (1+r^2)^2]^{-1/2}$ and $f$ is univalent whenever

(3.4) $$k\cot^{-1}\left(\frac{k\omega}{2}\right) - 2(\cot^{-1}\omega) \leqslant -\alpha\pi - 2\pi.$$

The left hand side of (3.4) is an increasing function of $r$ so if $k \leqslant -2a-2 = m(a)$, $r = 1$ as expected from Theorem 3. If, however, $k > m(a)$ then there is a unique solution $r(a, k)$ to the equation $k \cot^{-1}\left(\frac{k\omega}{2}\right) - 2\cot^{-1}(\omega) = -\pi(a+2)$ and $f$ is univalent at least in $|z| < r(a, k)$.

## IV. $MV[a, k]$ and close-to-convex functions

In this section we restrict $a$ to $a \geqslant 0$. It is well-known that if $a = 0$ or $a = 1$ and $f \in MV[a, m(a)]$ then $f \in K$, the class of close-to-convex functions. We now show that these are the only values of $a$ for which $MV[a, m(a)] \subset K$.

**Lemma.** *$g \in MV[a, k]$ if and only if $f \in MV[ap, k]$, where* $g(z) = [f(z^p)]^{1/p}$.

The proof is essentially the same as that of Theorem 1 [1].

We first observe that there is a function $f \in MV[2, 6]$ which is not close-to-convex. To this end, let $g$ be a function in $M$ which maps $\Delta$ onto the complement of two slits symmetric with respect to the origin in the w-plane, but not pointing at the origin. Then $g \in MV[1, 6]$ (e.g., see [5]). Computing $f$ from the lemma, we see that $f$ maps $\Delta$ onto the complement of part of a parabola. Clearly $f$ is not close-to-convex but, by the lemma, $f \in MV[2, 6]$.

More generally, there exists for each positive $a (a \neq 1)$ a function in $MV[a, m(a)] - K$. To obtain such a function $f$, we require that $f$ map $\Delta$ onto the complement of a single, smooth, twice differentiable slit which has the property that $(1-a)\arg P + a \arg T$ is a constant function of $\Phi = \arg P$; $P$ and $T$ are the position and tangent vectors, respectively. If the slit is defined locally by $r = r(\varphi)$, then the differential equation

$$ar\ddot{r} - (a+1)(\dot{r})^2 - r^2 = 0 \tag{4.1}$$

must be satisfied. On reducing (4.1) to a first order equation we obtain as solutions

$$r = A \sec^a\left(\frac{\varphi + B}{a}\right).$$

A range of values of $\varphi$ can be specified to obtain an infinite slit. The constants $A$ and $B$ allow sufficient freedom to bring $f$ to normalization, and geometric considerations show $f \in MV[a, m(a)]$.

Note that for $a = 1$ we obtain a ray, in agreement with $MV[1, 4] \subset K$; and for $a = 2$ we obtain a parabolic slit as cited in the example above. Clearly, if $a \neq 1$, the curve is not a ray and so $f \notin K$.

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## STRESZCZENIE

Autorzy wprowadzają klasę funkcji $MV[a, k]$ $(a \geq 0, k \geq 2)$, która jest zdefiniowana w ten sposób, iż wariacja wzdłuż okręgu $|z| = r$ tzw. kąta Mocanu $\psi$ jest ograniczona przez $k\pi$. Kat $\psi$ jest określony równaniem $\psi = (1-a)\arg f(z) + a \arg izf'(z)$. W szczególności, autorzy otrzymali relacje zawierania się pomiędzy klasami $MV[a, k]$ odpowiadającymi różnym wartościom parametrów.

## РЕЗЮМЕ

Вводится класс функций $MV[a, k](a \geq 0,\ k \geq 2)$ определенный таким способом, что вариация вдоль окружности $|z| = r$ так называемого угла Мокану $\psi$ ограничена $k\pi$. Угол $\psi$ определяется уравнением $\psi = (1-a)\arg f(z) + a \arg izf'(z)$. В частности, авторы получили реляцию содержания между классами $MV[a, k]$ соответствующим разным значениям параметров.