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Construction of an Object of Center-Projective Connection

Konstrukcja obiektu koneksji środkowo-rzutowej

Конструкция объекта центро-проективной связности

In this paper we present a method of a semiholonomic prolongation of linear connections in principal bundle of the first order $H(M)$ of linear frames to connection in bundle $\bar{H}^2(M)$ of semiholonomic frames of the second order.

Next, from this prolongation we obtain an object of center-projective connection which has been investigated in [4].

A connection in a bundle $P(M)$ of center-projective frames having an n-dimensional manifold as a basis, and the group of center-projective transformations as its structure group is called center-projective connection (cf. [4])

We derive infinitesimal equations for the objects of prolonged connection and of center-projective connection.

1. Let M be an n-dimensional manifold of the class C^∞. Let $H(M)$ be the bundle of the first order linear frames over M. $H(M)$ is the principal bundle with structure group L_1^n.

We start with some linear connection Γ on M. The value of the connection object Γ at the point $x \in M$ on the local cross-section:

$$M \supset U \xrightarrow[\mu]{} H(M)$$

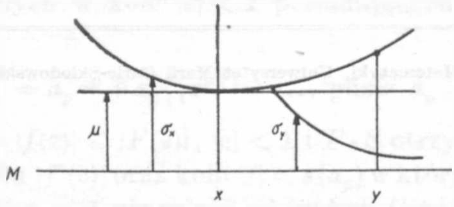
is considered as the first order jet of the local cross-section:

$$\sigma_x\colon \text{neighbourhood of point } x \to H(M),$$

such that: $\sigma_x(x) = \mu(x)$,

$$\Gamma(x) = j^1_{t|x}\sigma_x(t). \tag{1}$$

The right hand-member of the formula (1) denotes the value of the first order jet of the mapping: $t \to \sigma_x(t)$, taken at the point x.



Let y be an arbitrary point of a neighbourhood of the point x. Let u denote a mapping of the neighbourhood of x into L_1^n such that:

$$\sigma_y(y)\cdot u_y = \sigma_x(y).$$

Let us consider the following mapping:

$$(t, y) \mapsto \sigma_y(t)\cdot u_y.$$

We define a semiholonomic prolongation $\bar{\Gamma}$ of the object Γ by means of formula:

(2) $$\bar{\Gamma}(x) = j^1_{y|x} j^1_{t|y} \sigma_y(t)\cdot u_y.$$

If we express this formula in a coordinate system then we'll get the following formula for components of the object $\bar{\Gamma}$:

(2′) $$\bar{\Gamma}^i_{jkl} = \partial_k \Gamma^i_{jl} + \Gamma^i_{jh}\Gamma^h_{kl}.$$

$[\Gamma^i_{jk}, \bar{\Gamma}^i_{jkl}]$ are components of the connection object in the bundle of the second order semiholonomic frames over M with structure group $\bar{L}^n_2$.

2. We set:

(3) $$\Gamma^0_{jl} = \bar{\Gamma}^s_{jsl}.$$

We have to prove that the object Γ^0 with components $[\Gamma^i_{jl}, \Gamma^0_{jl}]$ is an object of a center-projective connection [4]. As the use of the rule transformation leads to some ambiguous formulas we shall use the method of infinitesimal equations for geometric objects.

Later, we shall use the infinitesimal equations for the object of the connection, Γ^i_{jk}, [3]:

(4) $$d\Gamma^i_{jk} - \Gamma^i_{sk}\omega^s_j - \Gamma^i_{js}\omega^s_k + \Gamma^s_{jk}\omega^i_s - \omega^i_{jk} = \Gamma^i_{jk,s}\omega^s$$

We write Cartan's structure equations:

(5)
$$\begin{aligned}
d\omega^i &= -\omega^i_s \wedge \omega^s,\\
d\omega^i_j &= -\omega^i_s \wedge \omega^s_j - \omega^i_{js}\wedge\omega^s,\\
d\omega^i_{jk} &= -\omega^i_s\wedge\omega^s_{jk} - \omega^i_{js}\wedge\omega^s_k - \omega^i_{sk}\wedge\omega^s_j - \omega^i_{sjk}\wedge\omega^s,\\
d\omega^i_{jkl} &= -\omega^i_{sl}\wedge\omega^s_{jk} - \omega^i_s\wedge\omega^s_{jkl} - \omega^i_{sjl}\wedge\omega^s_k - \omega^i_{sj}\wedge\omega^s_{kl} -\\
&\quad - \omega^i_{skl}\wedge\omega^s_j - \omega^i_{sk}\wedge\omega^s_{jl} - \omega^i_{sjk}\wedge\omega^s_l - \omega^i_{sjkl}\wedge\omega^s.
\end{aligned}$$

Forms ω^i_{jk}, ω^i_{jkl}, ω^i_{jkls} are symmetric with respect to lower indices.
We compute exterior differentials of the expression of the object $\bar{\Gamma}$ of the prolonged connection (2′):

$$d\bar{\Gamma}^i_{jkl} = d\Gamma^i_{jl;k} + (d\Gamma^i_{jh})\Gamma^h_{kl} + \Gamma^i_{jh}(d\Gamma^h_{kl}), \tag{6}$$

where: $\Gamma^i_{jl;k} = \partial_k \Gamma^i_{jl}$.
After substituting (4) to (6) and some easy calculations we obtain:

$$\begin{aligned} d\bar{\Gamma}^i_{jkl} - \bar{\Gamma}^i_{skl}\omega^s_j - \bar{\Gamma}^i_{jsl}\omega^s_k - \bar{\Gamma}^i_{jks}\omega^s_l + \bar{\Gamma}^s_{jkl}\omega^i_s + \\ - \Gamma^i_{sl}\omega^s_{jk} - 2\Gamma^i_{js}\omega^s_{kl} + \Gamma^s_{jl}\omega^i_{sk} - \Gamma^s_{kl}\omega^i_{sj} - \omega^i_{jkl} = \Gamma^i_{jkl,s}\omega^s. \end{aligned} \tag{7}$$

Applying contraction to (7) with respect to the indices: i, k we obtain:

$$d\Gamma^0_{jl} - \Gamma^0_{sl}\omega^s_j - \Gamma^0_{js}\omega^s_l + \Gamma^s_{jl}\omega^0_s - \tilde{\omega}^0_{jl} = \Gamma^0_{jl,s}\omega^s \tag{8}$$

where: $\tilde{\omega}^0_{jl} = \omega^0_{jl} + 2\Gamma^s_{jr}\omega^r_{ls} + 2\Gamma^s_{rl}\omega^r_{js}$.
Equations (7), (8) are infinitesimal equations for the object $\bar{\Gamma}$ of prolonged connection and for the object Γ^0 of center-projective connection respectively. Thus we have obtained the following:

Theorem 1.: *The components $[\Gamma^i_{jl}, \Gamma^0_{jl}]$ of center-projective connection may be obtained from a semiholonomic prolongation of $[\Gamma^i_{jl}]$ by a contraction*:

$$[\Gamma^i_{jl}, \bar{\Gamma}^i_{jkl}] \mapsto [\Gamma^i_{jl}, \bar{\Gamma}^s_{jsl}],$$

where: $\bar{\Gamma}^i_{jkl} = \partial_k \Gamma^i_{jl} + \Gamma^i_{jh}\Gamma^h_{kl}$,

and
$$\Gamma^0_{jl} = \bar{\Gamma}^s_{jsl} = \partial_s \Gamma^s_{jl} + \Gamma^s_{jh}\Gamma^h_{sl}.$$

3. Forms $\omega^{(3)} = \sum_{k=1}^{3} \omega^a_{a_1\ldots a_k} \otimes \varepsilon^{a_1\ldots a_k}_a$, where $\varepsilon^{a_1\ldots a_k}_a$ are basic vectors in the Lie algebra of the group L^n_3, are not connection forms, and together with forms $\omega^1, \ldots, \omega^n$ they form a base of linear forms in the principal bundle $H^3(M, L^n_3, \pi^{(3)})$ of the frames of the 3rd order, [1].
The connection form, $\bar{\omega}$, may be expressed in the following way, [1,3]:

$$\bar{\omega} = \omega^{(3)} + \chi, \tag{9}$$

where: $\chi = \sum_{k=1}^{3} C^a_{a_1\ldots a_k i}\omega^i \otimes \varepsilon^{a_1\ldots a_k}_a$
In our case we have:

$$\begin{aligned} \bar{\omega}^i_j &= \omega^i_j + C^i_{js}\omega^s, \\ \bar{\omega}^i_{jk} &= \omega^i_{jk} + C^i_{jks}\omega^s, \\ \bar{\omega}^i_{jkl} &= \omega^i_{jkl} + C^i_{jkls}\omega^s. \end{aligned} \tag{10}$$

Functions C^i_{js}, C^i_{jks}, C^i_{jkls} are so-called non-holonomic components of the object of holonomic connection in $H^3(M)$.

If we apply contraction to expression (10) for connection forms with respect to indices i, k we'll get:

$$\overline{\omega}_j^0 = \omega_j^0 + C_{js}^0\,\omega^s, \qquad \overline{\omega}_{jl}^0 = \omega_{jl}^0 + C_{jls}^0\,\omega^s. \tag{11}$$

Now we have to prove the following:

Theorem 2: *The object C^0 with components: $[C_{jl}^i, C_{jl}^0]$, where $C_{jl}^0 = C_{j;l}^s$ is just the object of center-projective connection on M.*

Proof: We want to obtain infinitesimal equations of C^0 by means of the formula:

$$d\overline{\omega}_j^0 + \overline{\omega}_k^0 \wedge \overline{\omega}_j^k = R_{jqp}^0\,\omega^q \wedge \omega^p, \tag{12}$$

where $[R_{jqp}^0]$ is a matrix of non-holonomic components of the object of curvature of the center-projectiwe connection.
After some calculations we obtain:

$$d\overline{\omega}_j^0 + \overline{\omega}_k^0 \wedge \overline{\omega}_j^k = (dC_{jp}^0 - C_{sp}^0\,\omega_j^s - C_{js}^0\,\omega_p^s + C_{jp}^s\,\omega_s^0 - \omega_{jp}^0 + C_{kq}^0 C_{jp}^k\,\omega^q) \wedge \omega^p. \tag{13}$$

Formula (12) will be satisfied if we set:

$$dC_{jp}^0 - C_{sp}^0\,\omega_j^s - C_{js}^0\,\omega_p^s + C_{jp}^s\,\omega_s^0 - \omega_{jp}^0 = C_{jp,q}^0\,\omega^q \tag{14}$$

where $C_{jp,q}^0$ are just components of the first prolongation of C_{jp}^0, [3].
From the expressions (13), (14) we have:

$$d\overline{\omega}_j^0 + \overline{\omega}_k^0 \wedge \overline{\omega}_j^k = [C_{jp,q}^0 + C_{kq}^0 C_{jp}^k]\,\omega^q \wedge \omega^p \tag{15}$$

Therefore, the formula for the curvature object will take a form:

$$R_{jqp}^0 = C_{jp,q}^0 - C_{jq,p}^0 + C_{kq}^0 C_{jp}^k - C_{kp}^0 C_{jq}^k. \tag{16}$$

It still remains to be proved that the object with components $[R_{jqp}^i, R_{jqp}^0]$ satisfies infinitesimal equations of a tensor type:

$$dR_{jqp}^0 - R_{sqp}^0\,\omega_j^s - R_{jsp}^0\,\omega_q^s - R_{jqs}^0\,\omega_p^s + R_{jqp}^s\,\omega_s^0 = R_{jqps}^0\,\omega^s. \tag{17}$$

To this end, we differentiate (14) and we obtain the infinitesimal equations of $C_{jp,q}^0$ by a method of G. F. Laptiev. Thus we have:

$$dC_{jp,q}^0 - C_{sp,q}^0\,\omega_j^s - C_{js,q}^0\,\omega_p^s - C_{jp,s}^0\,\omega_q^s + C_{jp,q}^s\,\omega_s^0 - \tilde{\omega}_{jp,q}^0 = C_{jp,qr}^0\,\omega^r \tag{18}$$

where: $\tilde{\omega}_{jp,q}^0 = \omega_{jpq}^0 - C_{jp}^s\,\omega_{sq}^0 + C_{sp}^0\,\omega_{jq}^s + C_{js}^0\,\omega_{pq}^s$.

Differentiating (16) and making use (18), (14) we have (17), where:

$$R_{jqp,s}^0 = C_{jp,qs}^0 - C_{jq,ps}^0 + C_{rp,s}^0\,C_{jq}^r + C_{rp}^0\,C_{jq,s}^r - C_{rq,s}^0\,C_{jp}^r - C_{rq}^0\,C_{jp,s}^r. \tag{19}$$

From this, it follows that R_{jqp}^0 are the components of curvature in bundle with center-projective structure group.

Remark: Equations (17) may be obtained in another way. Namely if we look for a connection object in $H^2(M)$ (i.e. in a bundle of holonomic frames of the second order) then we'll obtain the corresponding components of curvature $[R^i_{jqp}, R^i_{jkqp}]$ and their infinitesimal equations. A contraction $[R^i_{jqp}, R^s_{jsqp}]$ yields the same object of curvature of center-projective connection as in the above theorem and its infinitesimal equations (17)

4. The object of curvature for center-projective connection may be obtained in another way. Curvature tensors R^i_{jkl}, R^i_{jkls} for the linear connection $[\Gamma^i_{jk}]$ and prolonged connection $[\Gamma^i_{jk}, \overline{\Gamma}^i_{jkl}]$ may be writen respectively:

$$R^i_{jkl} = \partial_l \Gamma^i_{jk} - \partial_k \Gamma^i_{jl} + \Gamma^i_{sk}\Gamma^s_{jl} - \Gamma^i_{sl}\Gamma^s_{jk}, \tag{20}$$

$$R^i_{jkls} = \partial_s \overline{\Gamma}^i_{jkl} - \partial_l \overline{\Gamma}^i_{jks} + \overline{\Gamma}^i_{rkl}\Gamma^r_{js} - \overline{\Gamma}^i_{rks}\Gamma^r_{jl} + \overline{\Gamma}^i_{jrl}\Gamma^r_{ks} - \overline{\Gamma}^i_{jrs}\Gamma^r_{kl} - \Gamma^i_{rs}\overline{\Gamma}^r_{jkl} + \Gamma^i_{rl}\overline{\Gamma}^r_{jks} \tag{21}$$

Next, we calculate the object of curvature for center-projective connection $[\Gamma^i_{jl}, \Gamma^0_{jl}]$. If we apply contraction with respect indices i, k to curvature tensor (21) of the prolonged connection $[\Gamma^i_{jk}, \overline{\Gamma}^i_{jkl}]$, we'll have:

$$R^0_{jls} = \partial_s \Gamma^0_{jl} - \partial_l \Gamma^0_{js} + \Gamma^0_{rl}\Gamma^r_{js} - \Gamma^0_{rs}\Gamma^r_{jl}, \tag{22}$$

where: $\Gamma^0_{jl} = \overline{\Gamma}^s_{jsl}$.

From formulas (21) (20) it follows that:

$$R^i_{jkls} = \partial_k R^i_{jls} + \Gamma^i_{jh} R^h_{kls} + R^i_{jhs}\Gamma^h_{kl} + R^i_{jlh}\Gamma^h_{ks}. \tag{23}$$

We now apply contraction to (23) with respect to the indices i, k. We obtain:

$$R^0_{jls} = \partial_h R^h_{jls} + \Gamma^h_{jr} R^r_{hls} + R^h_{jrs}\Gamma^r_{hl} + R^h_{jlr}\Gamma^r_{hs} \tag{24}$$

Remarks: 1. From formula (23) it follows that if the linear connection is flat ($R^i_{jkl} = 0$), then the prolonged connection is also flat ($R^i_{jkls} = 0$).

2. The formula (24) means that if the former linear connection is flat ($R^i_{jkl} = 0$), then the obtained centre-projective connection is also flat ($R^0_{jls} = 0$).

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STRESZCZENIE

W pracy przedstawiamy metodę półholonomicznego przedłużenia koneksji liniowej w głównej wiązce $H(M)$ reperów liniowych pierwszego rzędu do koneksji w wiązce $\bar{H}^2(M)$ półholonomicznych reperów drugiego rzędu. Następnie dokonując odpowiedniej kontraktacji dla obiektu koneksji drugiego rzędu (przedłużenia półholonomicznego koneksji liniowej) otrzymujemy obiekt koneksji środkowo-rzutowej, [4].

Wyprowadzamy równania infinitesymalne dla obiektów: koneksji przedłużonej i koneksji środkowo-rzutowej.

Obliczamy obiekt krzywizny dla koneksji przedłużonej i koneksji środkowo-rzutowej.

РЕЗЮМЕ

В работе представлен метод полуголономного продолжения линейной связности в расслоении $H(M)$ линейных реперов до связности в расслоении $\bar{H}^2(M)$ полуголономных реперов второго ранга.

Потом проводя соответствующее свертывание для объекта связности второго ранга (полуголономного продолжения линейной связности), мы получаем объект центропроективной связности [4].

Мы получаем инфинитесимальные уравнения для объектов: продолженной связности и центропроективной связности.

Потом вычисляем объект кривизны для продолженной связности и центропроективной связности.