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Instytut Matematyki, Uniwersytet Marii Curie-Skłodowskiej, Lublin

WOJCIECH ZYGMUNT

On the Full Solution of the Paratingent Equations

O pełnym rozwiązaniu równania paratyngensowego

О полном решении паратингенсного уравнения

In this paper we shall deal with the problem of existence of the solution of a paratingent equation

$$(Ptx)(t) = F(t, x(t)).$$

The classical theory of paratingent equations considers the weaker problem

$$(Ptx)(t) \subset F(t, x(t)),$$

(cf [1], [4]). We shall prove that with somewhat stronger assumptions than the classical ones we obtain the existence of the solution in our stronger sense.

I. We shall use the following notations: Let $|x|$ denote the Euclidian norm of $x = (x^1, \ldots, x^m) \in R^m$, $\mathscr{M}$ – the family of all convex compact and non-empty subsets of R^m, $(t, x) \in R^{1+m} = R \times R^m$, $D = \langle 0, 1\rangle \times R^m \subset R^{1+m}$.

Having a continuous function $g: \langle 0, 1\rangle \to R^m$ and $t \in \langle 0, 1\rangle$ the set of all limit points

$$x = \frac{g(t_i) - g(s_i)}{t_i - s_i}$$

where $s_i, t_i \in \langle 0, 1\rangle$, $s_i \to t$, $t_i \to t$ and $t_i \neq s_i$, will be called paratingent of g at the point t and denoted by $(Ptg)(t)$. It is easy to see that $Ptg: \langle 0, 1\rangle \to R^m$ maps the interval $\langle 0, 1\rangle$ into the family of the non-empty closed subsets of R^m (cf [1], [4]). Taking only the limit points for which $s_i = t$ and $t_i \to t$ one obtains the subset $(Ctg)(t) \subset (Ptg)(t)$ – contingent of g at the point t.

Let F be a continuous mapping: $D \to \mathscr{M}$, the distance between the two sets in $\mathscr{M}$ being understood in the Hausdorff sense. We shall put $\|F(t, x)\| = \sup\{|y|: y \in F(t, x), (t, x) \in D\}$.

II. Theorem. *If the continuous mapping $F\colon D \to M$ satisfies the condition*

(1) $$\|F(t, x)\| \leqslant m(t)|x| + n(t) \text{ for } (t, x) \in D$$

where the functions m and n are non-negative and integrable in $\langle 0, 1\rangle$ and if $x_0 \in R^m$ then there exists an absolutely continuous function $g\colon \langle 0, 1\rangle \to R^m$ such that

(2) $$(Ptg)(t) = F(t, g(t)),\ t \in \langle 0, 1\rangle$$

and

(3) $$g(0) = x_0.$$

The proof of this theorem will be based on the following lemmas.

III. Lemma 1. *There exists a sequence of sets $A_n \subset \langle 0, 1\rangle$, $n = 0, 1, \ldots$ such that*

(*i*) $$A_i \cap A_j = \emptyset \text{ if } i \neq j$$

(*ii*) $$\bigcup_{n=0}^{\infty} A_n = \langle 0, 1\rangle$$

(*iii*) $$\bigwedge_{(a,b) \subset \langle 0,1\rangle} \mu((a, b) \cap A_n) > 0, \text{ for } n = 0, 1, \ldots$$

μ being the Lebesque measure.

Proof. Having an interval Δ and a positive number d we denote by $C(\Delta, d)$ a Cantor set of the measure d contained in Δ and such that the length of $(\alpha, \beta) \subset \Delta \setminus C(\Delta, d)$ does not extend beyond a half of the length of Δ. Now let $C_1, C_2, \ldots$ be a sequence of sets defined as follows:

$$C_1 = C(\langle 0, 1\rangle, 3^{-1}),\ 0 \in C_1,\ 1 \in C_1$$

$$C_{n+1} = \bigcup_{(\alpha,\beta) \in K_n} C((\alpha, \beta), 3^{-n-1}(\beta - \alpha))$$

where K_n denotes the set of all intervals (α, β) contained in $\langle 0, 1\rangle \setminus C_n^*$, $C_n^* = C_1 \cup C_2 \cup \ldots \cup C_n$ such that $\alpha \in C_n^*$ and $\beta \in C_n^*$. It is easy to verify that

(w_1) $$C_i \cap C_j = \emptyset \text{ if } i \neq j$$

(w_2) $$\mu(C_{n+1}) = 3^{-n-1}\mu(\langle 0, 1\rangle \setminus C_n^*) < 3^{-n-1}$$

(w_3) $$\mu(C_n^*) < \sum_{i=1}^{n} 3^{-i} < \tfrac{1}{2}$$

(w_4) $$(\beta - \alpha) < 2^{-n} \text{ for } (\alpha, \beta) \in K_n$$

In view of the fact that the formula

$$n = 2^r(2s-1),\ r = 0, 1, \ldots, s = 1, 2, \ldots$$

establishes the one to one correspondence $T: N_1 \to N_0 \times N_1$ where $N_k = \{k, k+1, \ldots\}$ and bearing in mind (w_1), the sets

$$A_j = \bigcup_{r=0}^{\infty} C_{2^r(2j-1)},\ j = 1, 2, \ldots$$

$$A_0 = \langle 0, 1\rangle \setminus \bigcup_{j=1}^{\infty} A_j = \langle 0, 1\rangle \setminus \bigcup_{i=1}^{\infty} C_i$$

satisfy (i) and (ii). To verify (iii) we take an arbitrary interval $(a, b) \subset \langle 0, 1\rangle$ and some $j \geqslant 1$. Then if $r_0 \in N_0$ and

$$\frac{1}{2^{r_0}(2j-1)} < \frac{b-a}{3}$$

it follows from (w_1) that there exists an interval $\Delta \in K_{2^{r_0}(2j-1)-1}$ contained in (a, b). Thus

(4) $$\Delta \cap C_i = \emptyset \text{ for } i = 1, 2, \ldots, 2^{r_0}(2j-1)-1$$

(5) $$\mu(\Delta \cap C_{2^{r_0}(2j-1)}) = \mu(\Delta)\cdot 3^{-2^{r_0}(2j-1)}$$

(6) $$\mu(\Delta \cap C_i) \leqslant \mu(\Delta)\cdot 3^{-i} \text{ for } i > 2^{r_0}(2j-1)$$

Since $C_{2^{r_0}(2j-1)} \subset A_j$ and $\Delta \subset (a, b)$ in view of (5) we obtain

$$\mu\big((a, b) \cap A_j\big) \geqslant \mu(\Delta \cap C_{2^{r_0}(2j-1)}) = \mu(\Delta)\cdot 3^{-2^{r_0}(2j-1)} > 0.$$

In order to complete the proof in the case $j = 0$ let us notice that

$$\mu(\Delta) = \mu\Big(\Delta \cap \bigcup_{i=1}^{\infty} C_i\Big) + \mu(\Delta \cap A_0)$$

and

$$\mu\Big(\Delta \cap \bigcup_{i=1}^{\infty} C_i\Big) = \sum_{i=1}^{\infty} \mu(\Delta \cap C_i) < \mu(\Delta)$$

thence

$$\mu\big((a, b) \cap A_0\big) \geqslant \mu(\Delta \cap A_0) = \mu(\Delta) - \mu\Big(\Delta \cap \bigcup_{i=1}^{\infty} C_i\Big) > 0.$$

Lemma 2. *If the absolutely continuous function $g\colon \langle 0, 1\rangle \to R^m$ satisfies the condition*

(7) $$g'(t) \in F\big(t, g(t)\big) \text{ a.e. almost everywhere in } \langle 0, 1\rangle$$

then

(8) $$(Ptg)(t) \subset F(t, g(t)) \text{ everywhere in } \langle 0, 1\rangle.$$

Proof. Let $t \in \langle 0, 1\rangle$, $\varepsilon > 0$ and $F_\varepsilon = \{y \in R^m: \bigvee_x [x \in F(t, g(t)) \wedge |y-x| \leqslant \varepsilon]\}$. Since the function F is continuous, there exists an interval (a, β) such that $t \in (a, \beta)$ and $F(s, g(s)) \subset F_\varepsilon$ if $s \in (a, \beta) \cap \langle 0, 1\rangle$. Thence, by (7) $g'(s) \in F_\varepsilon$ a.e. in $\Delta = (a, \beta) \cap \langle 0, 1\rangle$ and in view of the lemma 1 [3] we have

$$\frac{g(t_1)-g(t_2)}{t_1-t_2} \in F_\varepsilon \text{ for } t_i \in \Delta,\ i = 1, 2,\ t_1 \neq t_2.$$

It follows that $(Ptg)(t) \subset F_\varepsilon$ for any $\varepsilon > 0$ and thus condition (8) is fulfilled (owing to the optionality of ε).

IV. The proof of the theorem. Let A_n, $n = 0, 1, \ldots$ be a sequence of sets satisfying (i) – (iii). By a lemma 5.2 in [2], there exists a sequence of continuous selections $f_n: D \to R^m$, $n = 0, 1, \ldots$ such that $f_n(t, x) \in F(t, x)$ for every $(t, x) \in D$, $n = 0, 1, \ldots$ and the set $\{f_n(t, x)\}_{n=0,1,\ldots}$ is dense in $F(t, x)$ for each $(t, x) \in D$. Let us put

$$f(t, x) = f_n(t, x) \text{ if } (t, x) \in A_n \times R^m,\ n = 0, 1, \ldots$$

The function f is continuous on R^m for every fixed $t \in \langle 0, 1\rangle$. Putting

$$h_n(t, x) = \begin{cases} f_n(t, x) & \text{if } t \in A_n \\ 0 & \text{if } t \notin A_n \end{cases}$$

and $\sup_n h_n(t, x) = (\sup_n h_n^1(t, x), \ldots, \sup_n h_n^m(t, x))$ (analogically $\inf_n h_n(t, x)$) the function $f(t, x) = \sup_n h_n(t, x) + \inf_n h_n(t, x)$ is measurable on $\langle 0, 1\rangle$ for any fixed $x \in R^m$. In view of (1) $|f(t, x)| \leqslant m(t)|x| + n(t)$. Thus the function f fulfills all the hypotheses of the well known theorem by Caratheodory concerning the generalised solutions of ordinary differential equations. Therefore, there exists on absolutely continuous function g such that

(9) $$g'(t) = f(t, g(t)) \text{ a.e. in } \langle 0, 1\rangle$$

and

(10) $$g(0) = x_0.$$

By the lemma 2 we have

(11) $$(Ptg)(t) \subset F(t, g(t)) \text{ for every } t \in \langle 0, 1\rangle.$$

Now suppose there exists a $t \in \langle 0, 1\rangle$ such that

(12) $$(Ptg)(t) \neq F(t, g(t)).$$

Therefore in $F(t, g(t))$ there must exists a point x not belonging to $(Ptg)(t)$.

As the set $\{f_n(t, g(t))\}_{n=0,1,\dots}$ is dense on $F(t, g(t))$ one can choose a sequence f_{n_k} such that

(13) $$|f_{n_k}(t, g(t)) - x| < 2^{-k}.$$

On the other hand, from the continuity of the functions f_n and measurable density of the sets A_n [cf (iii)] it follows that there exists a sequence $t_k \in \langle 0, 1\rangle$, $k = 1, 2, \dots$ satisfying the following conditions

$$t_k \in A_{n_k}, \quad \lim_{k\to+\infty} t_k = t$$

(14) $$g'(t_k) = f_{n_k}(t_k, g(t_k))$$

and

(15) $$|f_{n_k}(t_k, g(t_k)) - f_{n_k}(t, g(t))| < 2^{-k}.$$

Now in view of (14) we can choose another sequence s_k, $k = 1, 2, \dots$ such that $|s_k - t_k| < 2^{-k}$, $s_k \neq t_k$, $s_k \in \langle 0, 1\rangle$ and

$$\left|\frac{g(s_k) - g(t_k)}{s_k - t_k} - f_{n_k}(t_k, g(t_k))\right| < 2^{-k}.$$

From (13) and (15) we shall have

$$\left|\frac{g(s_k) - g(t_k)}{s_k - t_k} - x\right| < 3\cdot 2^{-k}$$

and by the same $-x \in (Ptg)(t)$ despite the assumption (12). Finally there must be $(Ptg)(t) = F(t, g(t))$ for every $t \in \langle 0, 1\rangle$ which completes the proof of our theorem.

REFERENCES

[1] Bielecki, A., *Sur certaines conditions necessaires et suffisantes pour l'unicite des solutions des systemes d'equations differentielles ordinaires et des equations au paratingent*, Ann. Univ. M. Curie-Skłodowska, Sectio A, 2, 2 (1948), 49-106.

[2] Michael, E., *Continuous Selections* I, Ann. Math. ser. 2, 63, Nr 2 (1956), 361-382.

[3] Ważewski, T., *Sur une condition equivalente a l'equation au contingent*, Bull. Acad. Polon. Sci., Ser. sci. math. astr. et phys., 9 (1961), 865-867.

[4] Zaremba, S. K., *O równaniach paratyngensowych*, Ann. Soc. Polon. Math., Suppl. 9 (1935).

STRESZCZENIE

W pracy tej rozważany jest problem istnienia rozwiązania równania paratyngensowego $(Ptx)(t) = F(t, x(t))$. Zakładając, że funkcja wieloznaczna F jest ciągła i spełnia (1) udowodniono, że istnieje co najmniej jedna funkcja $\varphi\colon \langle 0, 1\rangle \to R^m$, która spełnia (2) i (3).

РЕЗЮМЕ

В работе рассматривается проблема существования решения паратингенсного уравнения $(Ptx)(t) = F(t, x(t))$. Предполагая, что непрерывная многозначная функция F удовлетворяет условию (1), доказывается, что существует по крайней мере одна функция $\varphi\colon \langle 0, 1\rangle \to R^m$, для которой выполнены условия (2) и (3).