ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN — POLONIA

VOL. XVIII, 4 SECTIO A 1964

Z Zakładu Matematycznych Metod Fizyki Zespołowej Katedry Matematyki
Wydziału Mat.-Fiz.-Chem. UMCS
Kierownik Zakładu: doc. dr Zdzisław Lewandowski

ZDZISŁAW LEWANDOWSKI

Modular and Domain Majorants of Regular Functions

Majoranty modułowe i obszarowe funkcji holomorficznych

Мажоранты по модулю и области голоморфных функций

1. Introduction

Put $C_r = \{z: |z| < r\}$ for $r > 0$ and suppose $f(z)$, $F(z)$ are regular in C_1 and satisfy $f(0) = F(0) = 0$. We say that $f(z)$ is subordinate to $F(z)$ in C_r, $r \epsilon (0, 1\rangle$ and write (f, F, r) if there exists a function $w(z)$ regular in C_r and such that $w(0) = 0$, $|w(z)| < r$, $f(z) \equiv F(w(z))$ in C_r. We also say that F is a domain majorant of f in C_r. On the other hand, if the inequality $|f(z)| \leqslant |F(z)|$ holds in C_r, we say that f is subordinate to F in modluus and write $|f, F, r|$. In this case we call F a modular majorant of f in C_r. Obviously (f, F, r) (resp. $|f, F, r|$) implies (f, F, r') (resp. $|f, F, r'|$) for any $0 < r' < r$. According to the Schwarz lemma (f, F, r) is equivalent to $|f, F, r|$ in case $F(z) = z$ but this does not hold for arbitrary F. M. Biernacki initiated in 1935 the investigations concerning relations of the following form: $(f, F, 1)$ implies $|f, F, r|$, under various assumptions on f and F, with $r \epsilon (0, 1)$ depending only on classes in which f and F are admitted to vary.

In [1] the author initiated the converse problem and obtained an estimation for r_0 such that $|f, F, 1|$ implies (f, F, r_0) with F being univalent.

Biernacki considered two cases. First he assumed that both f and F belong to the same class of functions and later on he put less restrictions on f, e.g. he supposed F to be univalent and f to be arbitrary.

The aim of this paper is to present a general method (Theorem 1) which enables us to tackle problems concerned with the converse problem of Biernacki for the different range of f and F. As an application of this method the Theorem 2 is given.

2. Notations and auxiliary lemmas.

Let S be the class of functions $F(z) = z + a_2 z^2 + \ldots$ regular and univalent in C_1 and let S_0 be the subclass of functions starshaped w.r.t. the origin. Let further N be the class of functions $w(z)$ regular in C_1 and such that $|w(z)| \leqslant 1$ in C_1.

If ∂C_r is the boundary of C_r, we call the Rogosinski domain an open and bounded set O_r^0 with the boundary consisting of the left half of ∂C_r and two circular arcs symmetric w.r.t. the real axis, passing through $z = 1$ and touching ∂C_r at $z = \mp ir$. In [2] the following lemma was of basic importance.

Lemma 1. Suppose that z ranges over the closed disc $\overline{C_r}$, $(0 < r < 1)$, and $w(z)$ ranges over N so that $w'(0) \geqslant 0$. Then the set of all posible values of $w(z)$ is identical with $\overline{O_r^0}$.

Let now O_r^n be the bounded domain with the boundary consisting of the left half of $|z| = r^{n+1}$ and two circular arcs symmetric w.r.t. the real axis, passing through $z = r^n$ and touching $\partial C_{r^{n+1}}$ at $z = \mp ir^{n+1}$, where n is a positive integer.

Lemma 2. Suppose that $w(z)$ has a zero of order at least $n (n \geqslant 0)$ at the origin and $w^{(n)}(0) \geqslant 0$. Suppose moreover that the function $w(z)$ ranges over N and z over $\overline{C_r}$. Then the set of all possible values $w(z)$ is identical with $\overline{O_r^n}$.

Proof. The case $n = 0$ is included in Lemma 1. Suppose $n \geqslant 1$. The function $w_1(z) = z^{-n} w(z)$ is regular in C_1 and obviously belongs to N. In view of Lemma 1, applied to $w_1(z)$, we see that all possible values $w(z)$ cover the set obtained from O_r^0 by a homothety with the ratio r^n, and this proves the lemma.

3. Main result

Suppose that K is a compact subclass of S and let $D(K, r)$ denote the closed set of variability of the ratio $g(z_1)/g(z_2)$, where z_1, z_2 vary indepedently over ∂C_r and g ranges over the class K.

Theorem 1. *Suppose that $f(z)$ is regular in C_1 and satisfies $f(0) = f'(0) = \ldots = f^{(n-1)}(0) = 0$, $f^{(n)}(0) \geqslant 0$ $(n \geqslant 1)$. Then $|f, F, 1|$ implies (f, F, r_0) for any f satisfying the above stated conditions and for any $F \in K$ if and only if for any $r \in (0, r_0\rangle$ the sets $D(K, r)$, $\overline{O_r^{n-1}}$ are disjoint $(n \geqslant 2)$, resp. have $z = 1$ as their unique common point $(n = 1)$.*

Proof. Suppose on the contrary that $|f, F, 1|$ and (f, F, r_0) with $0 < r_0 < 1$ hold, whereas $u \in D(K, r) \cap \overline{O_r^{n-1}}$ with $0 < r < r_0$ $(u \neq 1)$. There exist $F \in K$ and z_1, z_2, $z_1 \neq z_2$ on ∂C_r such that $u = F(z_1)/F(z_2)$. Besides, $u \in \overline{O_r^{n-1}}$.

Hence for a certain $w(z) \in N$ we have $w^{(n)}(0) \geqslant 0$, $w(z) \neq 1$, $w(z_2) = u = F(z_1)/F(z_2)$ in view of Lemma 2. Put $f_1(z) = w(z)F(z)$. The function $f_1(z)$ satisfies the assumptions of the Theorem 1. We have obviously $f_1(z_2) = F(z_1)$ and this contradicts the assumption (f_1, F, r_0) in view of Lindelöf's principle.

Suppose now, conversely, that the sets $D(K, r)$, $\overline{O_r^{n-1}}$ are either disjoint for all $r \in (0, r_0)$, or have $z = 1$ as their only common point in case $n = 1$. Suppose, moreover, that there exist the functions $f(z)$, $F(z)$ satisfying the conditions of the Theorem 1 as well as $|f, F, 1|$, whereas (f, F, r_0) does not hold.
Hence, there exists $z_1 \in C_r$, $0 < r < r_0$, such that $f(z_1)$ lies outside $F(C_r)$. However, $f(z) = w(z)\ F(z)$ with $w(z) \in N$ and $w^{(n)}(0) \geqslant 0$. Thus, we can find z_2 with $|z_2| = |z_1| = r_1 < r$ such that $F(z_1)w(z_1)/F(z_2) > 1$. This implies $w(z_1) = (1+\eta)\ F(z_2)/F(z_1)$, $\eta > 0$. However, $D(K, r)$, $\overline{O_r^{n-1}}$ are disjoint, hence $w(z_1) \in \overline{O_{r_1}^{n-1}}$ in view of the structure of O_r^{n-1}.

This contradicts Lemma 2 and our theorem is proved.

4. Applications.

Theorem 2. *Suppose R_n, $n = 1, 2, \ldots$ is the least positive root of the equation $x^n = (1-x)^2(1+x)^{-2}$. Suppose $F \in S_0$ and $f(z) = a_n z^n + \ldots$ with $a_n \geqslant 0$ is regular in C_1. If $|f, F, 1|$, then (f, F, R_n).*

The number R_n cannot be replaced by any greater number in case $F(z) = z(1+z)^{-2} = F_1(z), f(z) = (-1)^{n+1} z^n F(z) = f_n(z)$.

Proof. Put $D_0(r) = D(S_0, r)$ according to the notation of sections 2,3. In [2] we have proved that the set O_b^0 lies outside $D_0(r)$ and their closures have only two common points $-b, 1$, where $b = (1-r)^2(1+r)^{-2}$. Suppose $0 < r < R_n$. If $r^n < (1-r)^2(1+r)^{-2}$, then $\overline{O_r^{n-1}} \subset O_b^0$ and therefore $\overline{O_r^{n-1}} \cap D_0(r)$ is empty. Now Theorem 2 follows in view of Theorem 1.

On the other hand, taking $r \in (R_n, 1)$ we easily see that $f_n(-x) > F_1(x)$ for $R_n < x < r$. Since $F_1 \in S_0$, the domain $F_1(C_x)$ is starshaped w.r.t. the origin and therefore it does not contain the point $f_n(-x)$, resp. the domain $f_n(C_x)$. Hence (f_n, F_1, r) does not hold for $r > R_n$ and this proves that R_n cannot be replaced by any greater number. In case $n = 1$ Theorem 2 is identical with Theorem *B* which was the main result of [2].

REFERENCES

[1] Lewandowski, Z., *Sur les majorantes des fonctions holomorphes dans le cercle* $|z| < 1$, Annales Universitatis Mariae Curie-Sklodowska, Sectio A, **15** (1961), p. 5-11.

[2] Lewandowski, Z., *Starlike Majorants and Subordination*, Annales Universitatis Mariae Curie-Skłodowska, Sectio A, **15** (1961), p. 79-84.

Streszczenie

Niech S oznacza klasę funkcji $f(z) = z + a_2 z^2 + \ldots$, holomorficznych i jednolistnych w kole $|z| < 1$, natomiast $K \subset S$ niech będzie klasą zwartą. W pracy tej podaję warunki konieczne i dostateczne na to by nierówność $|f(z)| \leqslant |F(z)|$ dla $|z| < 1$ pociągała za sobą relację podporządkowania $f(z) \prec F(z)$ dla $|z| < r_0$, gdzie $F \in K$, f jest funkcją holomorficzną dla $|z| < 1$, $f(0) = 0, f'(0) \geqslant 0$ i $r_0 \in (0, 1)$ jest stałą absolutną niezależną od szczególnego doboru funkcji $f(z)$ i $F(z)$ (twierdzenie 1). W twierdzeniu 2 podaję zastosowania powyższego twierdzenia.

Резюме

Пусть S будет классом голоморфных и однолистных функций $f(z) = z + a_2 z^2 + \ldots$ в круге $|z| < 1$, а K-какой-нибудь компактный подкласс S. В работе даны необходимые и достаточные условия для того, чтобы неравенство $|f(z)| \leqslant |F(z)|$ в целом единичном круге повлекло за собой $f(z) \prec F(z)$ в круге $|z| < r_0$ (где $r_0 \in (0, 1)$ — некоторая константа, зависимая только от класса K).