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On Properties of the Lévy, Prokhorov and Sibley Type Metrics

O własnościach metryk typu Lévy'ego, Prochorowa i Sibley'a

1. Let $\mathbf{R}$ be the Euclidean space and let $\mathcal{F}_d$ denote the class of all distribution functions on $\mathbf{R}$.

Definition 1. The Lévy distance is the function $L : \mathcal{F}_d \times \mathcal{F}_d \to [0,1]$ such that $L(F,G) = \inf\{h > 0 : F(x-h) - h \le G(x) \le F(x+h) + h \ , \ x \in \mathbf{R}\}$.

In [4] the following concept was introduced.

Definition 2. The generalized Lévy distance is the function $L^\theta : \mathcal{F}_d \times \mathcal{F}_d \to [0, (\sqrt{2}\sin\theta)^{-1}]$, $0 < \theta < \pi/2$, such that

$$L^\theta(F,G) = \inf\{h > 0 : F(x - \sqrt{2}\,h\cos\theta) - \sqrt{2}\,h\sin\theta \le G(x) \le \\ \le F(x + \sqrt{2}\,h\cos\theta) + \sqrt{2}\,h\sin\theta, \ x \in \mathbf{R}\} \ .$$

Note that $L^{\pi/4} = L$. Moreover, one can see that

$$(1) \quad (\sqrt{2}\cos\theta)^{-1}L(F,G) \le L^\theta(F,G) \le (\sqrt{2}\sin\theta)^{-1}L(F,G) \quad , \quad \text{if } 0 < \theta < \pi/4$$

and

$$(2) \quad (\sqrt{2}\sin\theta)^{-1}L(F,G) \le L^\theta(F,G) \le (\sqrt{2}\cos\theta)^{-1}L(F,G) \quad , \quad \text{if } \pi/4 < \theta < \pi/2 \ .$$

These inequalities lead to the following theorem.

Theorem 1([4]). *Let $F_a(x) = F(x/a)$. If $a < 1$, then*

$$(3) \qquad aL(F_1, G_1) \le L(F_a, G_a) \le L(F_1, G_1) \ ,$$

and if $a > 1$, then

$$L(F_1, G_1) \le L(F_a, G_a) \le aL(F_1, G_1) . \tag{4}$$

From the inequalities (1) and (2) we conclude that the convergence in the L^θ–distance is equivalent to the convergence in the L–distance and at the same time equivalent to the weak convergence.

Let now $\mathcal{F}$ be the family of the functions $F : \mathbb{R} \to [0,1]$, nondecreasing and left continuous. Obviously $\mathcal{F}_d \subset \mathcal{F}$. It is known that the weak convergence of a sequence $\{F_n \ , \ n \ge 1\}$ functions of the class $\mathcal{F}$ is not equivalent the convergence in the Lévy distance in this class ($\mathcal{F}$).

Example 1. Let $\{F_n \ , \ n \ge 1\}$ be a sequence of functions belonging to the class $\mathcal{F}$ such that $F_n(x) = 0$ for $x \le n$ and $F_n(x) = 1$ for $x > n$. Then $F_n(x) \to F(x) \equiv 0$, $n \to \infty$, for every $x \in \mathbb{R}$ but $L(F_n, F) = 1$ for every n.

It was shown by D.A.Sibley [3] that the weak convergence of a sequence $\{F_n \ , \ n \ge 1\}$ of functions of the class $\mathcal{F}$ can be also metrized.

Definition 3. The Sibley distance is the function $L_S : \mathcal{F} \times \mathcal{F} \to [0,1]$ such that

$$\begin{aligned} L_S(F,G) = \inf\{h > 0 : & F(x-h) - h \le G(x) \le F(x+h) + h, \\ & G(x-h) - h \le F(x) \le G(x+h) + h, \ x \in (-1/h, 1/h)\} . \end{aligned}$$

Definition 2 suggests introducing the following concept.

Definition 4. The generalized Sibley function is the function $L_S^\theta : \mathcal{F} \times \mathcal{F} \to [0, (\sqrt{2}\sin\theta)^{-1}]$, $0 < \theta < \pi/2$, such that

$$\begin{aligned} L_S^\theta(F,G) = \inf\{h > 0 : & F(x-\sqrt{2}h\cos\theta) - \sqrt{2}h\sin\theta \le G(x) \le F(x+\sqrt{2}h\cos\theta) + \sqrt{2}h\sin\theta, \\ & G(x-\sqrt{2}h\cos\theta) - \sqrt{2}h\sin\theta \le F(x) \le G(x+\sqrt{2}h\cos\theta) + \sqrt{2}h\sin\theta, \ x \in (-1/h, 1/h)\} . \end{aligned}$$

Note that the function L_S^θ is not in general the metric. It does not satisfy the triangle inequality.

Example 2. Let

$$F(x) = \begin{cases} 0 & , x \le 0, \\ 0.7\sqrt{2} & , 0 < x \le 1-0.2\sqrt{6}, \\ 1 & , x > 1-0.2\sqrt{6}, \end{cases} \quad G(x) = \begin{cases} 0 & , x \le 0, \\ 0.5\sqrt{2} & , 0 < x \le 1. \\ 1 & , x > 1, \end{cases} \quad H(x) = \begin{cases} 0 \ , x \le 5, \\ 1 \ , x > 5, \end{cases}$$

and let $\theta = \pi/6$. Then for $h = 0.4$ we have

$F(x - \sqrt{2}h\cos\theta) - \sqrt{2}h\sin\theta = F(x - 0.2\sqrt{6}) - 0.2\sqrt{2} \le G(x) \le F(x + 0.2\sqrt{6}) + 0.2\sqrt{2} =$
$= F(x + \sqrt{2}h\cos\theta) + \sqrt{2}h\sin\theta$, $x \in \mathbb{R}$,
and
$G(x - \sqrt{2}h\cos\theta) - \sqrt{2}h\sin\theta = G(x - 0.2\sqrt{6}) - 0.2\sqrt{2} \le F(x) \le G(x + 0.2\sqrt{6}) + 0.2\sqrt{2} =$

$=G(x+\sqrt{2}\,h\cos\theta)+\sqrt{2}\,h\sin\theta\,,\quad x\in\mathbb{R}.$

Moreover, $F(0.5)=0.7\sqrt{2}=G(0.5+0.2\sqrt{6})+0.2\sqrt{2}=G(0.5+\sqrt{2}\,h\cos\theta)+\sqrt{2}\,h\sin\theta$, $h=0.4$. Hence, $L_S^\theta(F,G)=0.4$.

Now, for $h=1$ we have

$G(x-\sqrt{2}\,h\cos\theta)-\sqrt{2}\,h\sin\theta=G(x-0.5\sqrt{6})-0.5\sqrt{2}\le H(x)\le G(x+0.5\sqrt{6})+0.5\sqrt{2}=$
$=G(x+\sqrt{2}\,h\cos\theta)+\sqrt{2}\,h\sin\theta\,,\quad x\in(-1,1),$

and

$H(x-\sqrt{2}\,h\cos\theta)-\sqrt{2}\,h\sin\theta=H(x-0.5\sqrt{6})-0.5\sqrt{2}\le G(x)\le H(x+0.5\sqrt{6})+0.5\sqrt{2}=$
$=H(x+\sqrt{2}\,h\cos\theta)+\sqrt{2}\sin\theta\,,\quad x\in(-1,1).$

Moreover, $G(0.5)=0.5\sqrt{2}=H(0.5+0.5\sqrt{6})+0.5\sqrt{2}=H(0.5+\sqrt{2}\,h\cos\theta)+\sqrt{2}\,h\sin\theta$, $h=1$. Thus $L_S^\theta(G,H)=1$.

We now count $L_S^\theta(F,H)$. Choose any $h\in(0,\sqrt{2})$. Then $0.7\in(-1/h,1/h)$ as $(-1/\sqrt{2},1/\sqrt{2})\subset(-1/h,1/h)$. Moreover, $F(0.7)=1$ and $H(0.7+\sqrt{2}\,h\cos\theta)=0$ since $h<\sqrt{2}$, $\theta=\pi/6$. Hence $F(0.7)>0+0.5\sqrt{2}\,h=H(0.7+\sqrt{2}\,h\cos\theta)+\sqrt{2}\,h\sin\theta$. Therefore, $L_S^\theta(F,H)\ge h$, so $L_S^\theta(F,H)\ge\sqrt{2}$. But, by Definition 4, for any $F_1,F_2\in\mathcal{F}$, $L_S^\theta(F_1,F_2)\le\sqrt{2}$. Hence, we see that $L_S^\theta(F,H)=\sqrt{2}$. Those facts prove that L_S^θ does not satisfy the triangle inequality.

We now give relations between L_S and L_S^θ.

Theorem 2. *If $0<\theta<\pi/4$, then*

$$(\sqrt{2}\cos\theta)^{-1}L_S(F,G)\le L_S^\theta(F,G)\le(\sqrt{2}\sin\theta)^{-1}L_S(F,G)\,, \tag{5}$$

and if $\pi/4<\theta<\pi/2$, then

$$(\sqrt{2}\sin\theta)^{-1}L_S(F,G)\le L_S^\theta(F,G)\le(\sqrt{2}\cos\theta)^{-1}L_S(F,G)\,. \tag{6}$$

Proof. Let $h_0=L_S^\theta(F,G)$. Then for $x\in(-1/h_0,1/h_0)$

$$F(x-\sqrt{2}\,h_0\cos\theta)-\sqrt{2}\,h_0\sin\theta\le G(x)\le F(x+\sqrt{2}\,h_0\cos\theta)+\sqrt{2}\,h_0\sin\theta\,.$$

If $0<\theta<\pi/4$ then $\cos\theta>\sin\theta$, so for $x\in(-1/h_0,1/h_0)$

$$F(x-\sqrt{2}\,h_0\cos\theta)-\sqrt{2}\,h_0\cos\theta\le G(x)\le F(x+\sqrt{2}\,h_0\cos\theta)+\sqrt{2}\,h_0\cos\theta\,. \tag{7}$$

Taking into account that for $0<\theta<\pi/4$ $\quad 1/(\sqrt{2}\,h_0\cos\theta)<1/h_0$, we conclude that (7) holds for $x\in\left(-1/(\sqrt{2}\,h_0\cos\theta),1/(\sqrt{2}\,h_0\cos\theta)\right)$ which proves that $F(x-h_1)-h_1\le G(x)\le F(x+h_1)+h_1$ for $x\in(-1/h_1,1/h_1)$ with $h_1=\sqrt{2}\,h_0\cos\theta$.

Similarly, we state that $G(x-h_1)-h_1\le F(x)\le G(x+h_1)+h_1$ for $x\in(-1/h_1,1/h_1)$.

Now the definition of L_S implies that

$$L_S(F,G)\le h_1=\sqrt{2}\,h_0\cos\theta=\sqrt{2}\cos\theta L_S^\theta(F,G)\text{ or }(\sqrt{2}\cos\theta)^{-1}L_S(F,G)\le L_S^\theta(F,G)\,,$$

which proves the left hand side of the inequality (5).

Let now $h_0^*=L_S(F,G)$. By Definition 3, for $x\in(-1/h_0^*,1/h_0^*)$ $\quad F(x-h_0^*)-h_0^*\le$ $\le G(x)\le F(x+h_0^*)+h_0^*$. Since $\cos\theta/\sin\theta>1$ then for $x\in(-1/h_0^*,1/h_0^*)$

$$\begin{aligned}&F(x-\sqrt{2}\,h_0^*(\sqrt{2}\sin\theta)^{-1}\cos\theta)-\sqrt{2}\,h_0^*(\sqrt{2}\sin\theta)^{-1}\sin\theta\le G(x)\le\\&\le F(x+\sqrt{2}\,h_0^*(\sqrt{2}\sin\theta)^{-1}\cos\theta)+\sqrt{2}\,h_0^*(\sqrt{2}\sin\theta)^{-1}\sin\theta\,.\end{aligned} \tag{8}$$

The inequality (8) also holds for $x \in (-\sqrt{2}\sin\theta/h_0^*, \sqrt{2}\sin\theta/h_0^*)$ as $(-\sqrt{2}\sin\theta/h_0^*, \sqrt{2}\sin\theta/h_0^*) \subset (-1/h_0^*, 1/h_0^*)$. Thus we see that $F(x-\sqrt{2}h_1^*\cos\theta) - \sqrt{2}h_1^*\sin\theta \leq G(x) \leq F(x+\sqrt{2}h_1^*\cos\theta)+\sqrt{2}h_1^*\sin\theta$ for $x \in (-1/h_1^*, 1/h_1^*)$ where $h_1^* = h_0^*(\sqrt{2}\sin\theta)^{-1}$

Similarly we state that for $x \in (-1/h_1^*, 1/h_1^*)$

$$G(x-\sqrt{2}h_1^*\cos\theta) - \sqrt{2}h_1^*\sin\theta \leq F(x) \leq G(x+\sqrt{2}h_1^*\cos\theta) + \sqrt{2}h_1^*\sin\theta .$$

Therefore, by the Definition 4,

$$L_S^\theta(F,G) \leq h_1^* = h_0^*(\sqrt{2}\sin\theta)^{-1} = (\sqrt{2}\sin\theta)^{-1} L_S(F,G)$$

which proves the right hand side inequality of (5).

The proof of (6) is similar.

The following example shows that Theorem 1 is not still true for the Sibley distance.

Example 3. Let

$$F(x) = \begin{cases} 0 & , x \leq 5 , \\ 1 & , x > 5 , \end{cases} \qquad G(x) = \begin{cases} 0 & , x \leq 20 , \\ 1 & , x > 20 , \end{cases}$$

and let $a = 1/10$. Then $L_S(F_1, G_1) = L_S(F,G) = 1/5$, and $L_S(F_{1/10}, G_{1/10}) = 1$, where

$$F_{1/10}(x) = \begin{cases} 0 & , x \leq 0.5 , \\ 1 & , x > 0.5 , \end{cases} \qquad G_{1/10}(x) = \begin{cases} 0 & , x \leq 2 , \\ 1 & , x > 2 . \end{cases}$$

Therefore, $L_S(F_{1/10}, G_{1/10}) > L_S(F_1, G_1)$, which proves that the inequalities

$$\begin{aligned} a L_S(F_1,G_1) \leq L_S(F_a,G_a) \leq L_S(F_1,G_1) , \quad & a < 1 ; \\ L_S(F_1,G_1) \leq L_S(F_a,G_a) \leq a L_S(F_1,G_1) , \quad & a > 1 \end{aligned}$$

are not in general true.

2. Let $\mathcal{X}$ be a normed linear space with the norm $\|\cdot\|$. Denote by $\mathcal{P}_d$ the space of all probability measures on $(\mathcal{X}, \mathcal{B}_\mathcal{X})$, $\mathcal{B}_\mathcal{X}$ — σ-field of subsets of $\mathcal{X}$. Let $K(r)$ stands for the ball $K(r) = \{x \in \mathcal{X} : \|x\| < r\}$.

Definition 5. The Prokhorov distance is the function $\Pi : \mathcal{P}_d \times \mathcal{P}_d \to [0,1]$ such that $\Pi(P,Q) = \inf\{h > 0 : P(A) \leq Q(A^h) + h, Q(A) \leq P(A^h) + h, \ A \in \mathcal{B}_\mathcal{X}\}$, where $A^h = \{x \in \mathcal{X} : \text{dist}\,(x,A) < h\}$.

Let us introduce the following generalization of the Prokhorov distance.

Definition 6. The generalized Prokhorov distance is the function $\Pi^\theta : \mathcal{P}_d \times \mathcal{P}_d \to [0, (\sqrt{2}\sin\theta)^{-1}]$, $0 < \theta < \pi/2$, such that

$$\begin{aligned} \Pi^\theta(P,Q) = \inf\{h > 0 : & P(A) \leq Q(A^{\sqrt{2}h\cos\theta}) + \sqrt{2}h\sin\theta, \\ & Q(A) \leq P(A^{\sqrt{2}h\cos\theta}) + \sqrt{2}h\sin\theta, \ A \in \mathcal{B}_\mathcal{X}\} . \end{aligned}$$

It is obvious that $\Pi^{\pi/4} = \Pi$. We see that the function Π^θ is the metric.

Lemma 1. *The function Π^θ satisfies the axioms of the metric.*

Proof. By Definition 6 we have

$$\text{(i)} \qquad \Pi^\theta(P,Q) = 0 \quad \text{iff } P = Q\,,$$
$$\text{(ii)} \qquad \Pi^\theta(P,Q) = \Pi^\theta(Q,P)$$

for all $P, Q \in \mathcal{P}_d$.

Let now $P, Q, R \in \mathcal{P}_d$. If $\Pi^\theta(P,Q) < x$ and $\Pi^\theta(Q,R) < y$ then for any set $A \in \mathcal{B}_\mathcal{X}$ we have

$$P(A) \le Q(A^{\sqrt{2}x\cos\theta}) + \sqrt{2}\,x\sin\theta \le R\big((A^{\sqrt{2}x\cos\theta})^{\sqrt{2}y\cos\theta}\big) + \sqrt{2}\,y\sin\theta + \sqrt{2}\,x\sin\theta =$$
$$= R\big(A^{\sqrt{2}(x+y)\cos\theta}\big) + \sqrt{2}\,(x+y)\sin\theta$$

and

$$R(A) \le Q(A^{\sqrt{2}y\cos\theta}) + \sqrt{2}\,y\sin\theta \le P\big((A^{\sqrt{2}y\cos\theta})^{\sqrt{2}x\cos\theta}\big) + \sqrt{2}\,x\sin\theta + \sqrt{2}\,y\sin\theta =$$
$$= P\big(A^{\sqrt{2}(x+y)\cos\theta}\big) + \sqrt{2}\,(x+y)\sin\theta\,.$$

Thus $\Pi^\theta(P,R) \le x + y$. Taking infimum over all x and y we get

$$\text{(iii)} \qquad \Pi^\theta(P,R) \le \Pi^\theta(P,Q) + \Pi^\theta(Q,R)$$

which completes the proof of Lemma 1.

Let now $\mathcal{P}$ denote the space of probability measures on $(\mathcal{X}, \mathcal{B}_\mathcal{X})$ together with defective probability measures, i.e. $P \in \mathcal{P}$ iff $P(\mathcal{X}) \le 1$.

Define the Prokhorov–Sibley distance as follows.

Definition 7. The Prokhorov–Sibley distance is the function $\Pi_S : \mathcal{P} \times \mathcal{P} \to [0,1]$ such that

$$\Pi_S(P,Q) = \inf\{h > 0 : P(A) \le Q(A^h) + h,\ Q(A) \le P(A^h) + h,\ A \in \mathcal{B}_\mathcal{X},\ A \subset K(1/h)\}.$$

Lemma 2. *The function Π_S satisfies the axioms of the metric.*

Proof. By Definition 7 we have

$$\text{(i)} \qquad \Pi_S(P,Q) = 0 \quad \text{iff } P = Q\,,$$
$$\text{(ii)} \qquad \Pi_S(P,Q) = \Pi_S(Q,P)$$

for all $P, Q \in \mathcal{P}$.

Let now $P, Q, R \in \mathcal{P}$. If $\Pi_S(P,Q) + \Pi_S(Q,R) \ge 1$ then the triangle inequality is obvious. Assume now that $\Pi_S(P,Q) + \Pi_S(Q,R) < 1$, and let $\Pi_S(P,Q) < x$, $\Pi_S(Q,R) < y$ with $x + y < 1$. Then we have $P(A) \le Q(A^x) + x$, $A \subset K(1/x)$, and $Q(A) \le R(A^y) + y$, $A \subset K(1/y)$. Hence, $P(A) \le R(A^{x+y}) + x + y$, $A \subset$

$K(1/x) \cap K(1/y-x)$. Moreover, we see that $1/(x+y) < 1/x$ and $1/(x+y) < 1/y-x$, so $K\big(1/(x+y)\big) \subset K(1/x) \cap K\big(1/y-x\big)$. Therefore, $P(A) \le R(A^{x+y}) + x + y$, $A \subset K\big(1/(x+y)\big)$.

Similarly one can prove that $R(A) \le P(A^{x+y}) + x + y$, $A \subset K\big(1/(x+y)\big)$.

Thus

$$\Pi_S(P,R) = \inf\{h > 0 : P(A) \le R(A^h)+h, R(A) \le P(A^h)+h,\ A \subset K(1/h)\} \le x+y.$$

Taking infimum over all x, y we get

$$\text{(iii)} \qquad \Pi_S(P,R) \le \Pi_S(P,Q) + \Pi_S(Q,R)$$

which completes the proof.

Note that the convergences in the metrics Π and Π_S for sequences measures of $\mathcal{P}$ are not equivalent.

Example 4. Let $\{x_n, n \ge 1\}$ be any sequence of $x_n \in \mathcal{X}$, $n \ge 1$, such that $\|x_n\| = n$. Define the sequence $\{P_n, n \ge 1\}$ of measures $P_n \in \mathcal{P}$ as follows

$$P_n(A) = \begin{cases} 0 & \text{if } x_n \notin A\,, \\ 1 & \text{if } x_n \in A\,. \end{cases}$$

Let P be a measure with $P(\mathcal{X}) = 0$. Then $\Pi(P_n, P) = 1$, $n \ge 1$. The inequality $P(A) \le P_n(A^h) + h$ is true for every $h > 0$. The inequality $P_n(A) \le P(A^h) + h$, $A \subset K(1/h)$, is satisfied for $h \ge 1/n$ and it does not hold for $h < 1/n$. Thus $\Pi_S(P_n, P) = 1/n$. Hence, $\Pi_S(P_n, P) \to 0$, while $\Pi(P_n, P) \not\to 0$, $n \to \infty$.

The definitions of Π and Π_S imply that $\Pi_S(P,Q) \le \Pi(P,Q)$ for any $P, Q \in \mathcal{P}$.

Following Definition 4 we can introduce the following concept.

Definition 8. The generalized Prokhorov–Sibley function is the function $\Pi_S^\theta : \mathcal{P} \times \mathcal{P} \to [0, (\sqrt{2}\sin\theta)^{-1}]$, $0 < \theta < \pi/2$, such that

$$\Pi_S^\theta(P,Q) = \inf\{h > 0 : P(A) \le Q\big(A^{\sqrt{2}h\cos\theta}\big) + \sqrt{2}h\sin\theta,$$
$$Q(A) \le P\big(A^{\sqrt{2}h\cos\theta}\big) + \sqrt{2}h\sin\theta,\ A \in \mathcal{B}_{\mathcal{X}},\ A \subset K(1/h)\}\,.$$

The following example shows that the function Π_S^θ is not in general the metric.

Example 5. Let $y \in \mathcal{X}$ be such that $\|y\| = 1$, let $x = (1 - 0.2\sqrt{6})y$, and let P, Q, R be any probability measures from $\mathcal{P}_d$ such that

$$P(\{0\}) = 0.7\sqrt{2} \quad , \qquad P(\{x\}) = 1 - 0.7\sqrt{2}\,,$$
$$Q(\{0\}) = 0.5\sqrt{2} \quad , \qquad Q(\{y\}) = 1 - 0.5\sqrt{2}\,,$$
$$R(\{5x\}) = 1\,.$$

The considerations similar to the considerations in the Example 2 show that

$$\Pi_S^\theta(P,Q) + \Pi_S^\theta(Q,R) = 0.4 + 1 = 1.4 < \sqrt{2} = \Pi_S^\theta(P,R)\,,$$

which contradicts the triangle inequality.

Relations between Π and Π^θ and between Π_S and Π_S^θ contain the following theorems.

Theorem 3. *If $0 < \theta < \pi/4$, then*

$$(\sqrt{2}\cos\theta)^{-1}\Pi(P,Q) \le \Pi^\theta(P,Q) \le (\sqrt{2}\sin\theta)^{-1}\Pi(P,Q)\ ;$$

if $\pi/4 < \theta < \pi/2$, then

$$(\sqrt{2}\sin\theta)^{-1}\Pi(P,Q) \le \Pi^\theta(P,Q) \le (\sqrt{2}\cos\theta)^{-1}\Pi(P,Q)\ .$$

Theorem 4. *If $0 < \theta < \pi/4$, then*

$$(\sqrt{2}\cos\theta)^{-1}\Pi_S(P,Q) \le \Pi_S^\theta(P,Q) \le (\sqrt{2}\sin\theta)^{-1}\Pi_S(P,Q)\ ; \tag{9}$$

if $\pi/4 < \theta < \pi/2$, then

$$(\sqrt{2}\sin\theta)^{-1}\Pi_S(P,Q) \le \Pi_S^\theta(P,Q) \le (\sqrt{2}\cos\theta)^{-1}\Pi_S(P,Q)\ . \tag{10}$$

It is enough to prove Theorem 4. The proof of Theorem 3 is similar.

Proof of Theorem 4. Let $h_0 > \Pi_S^\theta(P,Q)$. Then $P(A) \le Q(A^{\sqrt{2}h_0\cos\theta}) + \sqrt{2}\,h_0\sin\theta$ for $A \subset K(1/h_0)$. If $0 < \theta < \pi/4$, then $\cos\theta > \sin\theta$, and for $A \subset K(1/h_0)$

$$P(A) \le Q(A^{\sqrt{2}h_0\cos\theta}) + \sqrt{2}h_0\cos\theta\ . \tag{11}$$

Moreover, for $0 < \theta < \pi/4$ $\quad 1/(\sqrt{2}\,h_0\cos\theta) < 1/h_0$ and so (11) holds for $A \subset K(1/(\sqrt{2}\,h_0\cos\theta))$ which proves that $P(A) \le Q(A^{h_1}) + h_1$ for $A \subset K(1/h_1)$ where $h_1 = \sqrt{2}\,h_0\cos\theta$. Similarly, we get $Q(A) \le P(A^{h_1}) + h_1$ for $A \subset K(1/h_1)$, $h_1 = \sqrt{2}\,h_0\cos\theta$. Hence, by Definition 7, we see that $\Pi_S(P,Q) \le h_1 = \sqrt{2}\,h_0\cos\theta$. Taking infimum over all h_0 we get $\Pi_S(P,Q) \le \sqrt{2}\cos\theta\,\Pi_S^\theta(P,Q)$, i.e. $(\sqrt{2}\cos\theta)^{-1}\Pi_S(P,Q) \le \Pi_S^\theta(P,Q)$ which proves the left hand side of the inequality (9).

Let now $h_0^* > \Pi_S(P,Q)$. Then, by Definition 7, we have $P(A) \le Q(A^{h_0^*}) + h_0^*$, $A \subset K(1/h_0^*)$. Since $\cos\theta/\sin\theta > 1$ for $0 < \theta < \pi/4$, then

$$P(A) \le Q\left(A^{\sqrt{2}h_0^*(\sqrt{2}\sin\theta)^{-1}\cos\theta}\right) + \sqrt{2}\,h_0^*(\sqrt{2}\sin\theta)^{-1}\sin\theta \tag{12}$$

$A \subset K(1/h_0^*)$. Moreover, $\sqrt{2}\sin\theta/h_0^* < 1/h_0^*$, so (12) holds for $A \subset K(\sqrt{2}\sin\theta/h_0^*)$, and also $P(A) \le Q(A^{\sqrt{2}h_1^*\cos\theta}) + \sqrt{2}\,h_1^*\sin\theta$ for $A \subset K(1/h_1^*)$, where $h_1^* = h_0^*(\sqrt{2}\sin\theta)^{-1}$. Similarly, $Q(A) \le P(A^{\sqrt{2}h_1^*\cos\theta}) + \sqrt{2}\,h_1^*\sin\theta$ for $A \subset K(1/h_1^*)$, $h_1^* = h_0^*(\sqrt{2}\sin\theta)^{-1}$. Therefore, by Definition 8, $\Pi_S^\theta(P,Q) \le h_1^* = h_0^*(\sqrt{2}\sin\theta)^{-1}$. Taking infimum over all h_0^* we get $\Pi_S^\theta(P,Q) \le (\sqrt{2}\sin\theta)^{-1}\Pi_S(P,Q)$ which proves the right hand side of the inequality (9).

The proof of (10) is similar.

The following theorems show the difference between Π and Π_S. Let $P_a(A) = P(a^{-1}A)$ where $a^{-1}A = \{a^{-1}x : x \in A\}$, $a > 0$.

Theorem 5. *If $a < 1$, then*

$$a\,\Pi(P_1,Q_1) \le \Pi(P_a,Q_a) \le \Pi(P_1,Q_1)\;; \tag{13}$$

if $a > 1$, then

$$\Pi(P_1,Q_1) \le \Pi(P_a,Q_a) \le a\,\Pi(P_1,Q_1)\;. \tag{14}$$

Proof. Let $\mathrm{tg}\,\theta = a$, where $0 < \theta < \pi/4$. By Definition 5, we have

$$\begin{aligned}\Pi(P_a,Q_a) &= \inf\{h>0 : P_a(A) \le Q_a(A^h) + h, Q_a(A) \le P_a(A^h) + h\} = \\ &= \inf\{h>0 : P(a^{-1}A) \le Q(a^{-1}A^h) + h, Q(a^{-1}A) \le P(a^{-1}A^h) + h\}\;.\end{aligned}$$

Taking into account that $\{x : x \in a^{-1}A^h\} = \{x : x \in (a^{-1}A)^{h/a}\}$ as $\{x : x \in a^{-1}A^h\} = \{x : ax \in A^h\} = \{x : \mathrm{dist}\,(ax, A) < h\} = \{x : \mathrm{dist}\,(ax, aa^{-1}A) < h\} = \{x : a\,\mathrm{dist}\,(x, a^{-1}A) < h\} = \{x : \mathrm{dist}\,(x, a^{-1}A) < h/a\} = \{x : x \in (a^{-1}A)^{h/a}\}$, we obtain

$$\begin{aligned}\Pi(P_a,Q_a) = \inf\{h>0 : P_1(a^{-1}A) &\le Q_1\big((a^{-1}A)^{h/a}\big) + h, \\ Q_1(a^{-1}A) &\le P_1\big((a^{-1}A)^{h/a}\big) + h\}\;.\end{aligned}$$

Putting $h' = h/(\sqrt{2}\sin\theta)$ and $a = \sin\theta/\cos\theta$, we get

$$\begin{aligned}\Pi(P_a,Q_a) &= \inf\{\sqrt{2}\sin\theta h' > 0 : P_1(a^{-1}A) \le Q_1\big((a^{-1}A)^{\sqrt{2}\cos\theta h'}\big) + \sqrt{2}\sin\theta h', \\ &Q_1(a^{-1}) \le P_1\big((a^{-1}A)^{\sqrt{2}\cos\theta h'}\big) + \sqrt{2}\sin\theta h'\} = \sqrt{2}\sin\theta\,\Pi^\theta(P_1,Q_1)\;.\end{aligned}$$

Thus, by Theorem 3, we get $\mathrm{tg}\,\theta\,\Pi(P_1,Q_1) \le \sqrt{2}\sin\theta\,\Pi^\theta(P_1,Q_1) = \Pi(P_a,Q_a)$ and $\sqrt{2}\sin\theta\,\Pi^\theta(P_1,Q_1) \le \Pi(P_1,Q_1)$. Thus

$$a\,\Pi(P_1,Q_1) \le \Pi(P_a,Q_a) \le \Pi(P_1,Q_1)\;.$$

The proof of (14) is similar.

Theorem 5 is not in general true for the distance Π_S.

Example 6. Let $x, y \in X$ be such that $\|x\| = 5$, $\|y\| = 20$ and let $a = 1/10$. Define the measures P and Q as follows:

$$P(A) = \begin{cases} 0 & \text{for } x \notin A\,, \\ 1 & \text{for } x \in A\,, \end{cases} \qquad Q(A) = \begin{cases} 0 & \text{for } y \notin A\,, \\ 1 & \text{for } y \in A\,. \end{cases}$$

Then $\Pi_S(P_1,Q_1) = 1/5$. Moreover, the measures $P_{1/10}$ and $Q_{1/10}$ are defined as follows:

$$P_{1/10}(A) = \begin{cases} 0 & \text{for } x/10 \notin A\,, \\ 1 & \text{for } x/10 \in A\,, \end{cases} \qquad Q_{1/10}(A) = \begin{cases} 0 & \text{for } y/10 \notin A\,, \\ 1 & \text{for } y/10 \in A\,. \end{cases}$$

Hence $\Pi_S(P_{1/10}, Q_{1/10}) = 1$. Thus $\Pi_S(P_{1/10}, Q_{1/10}) > \Pi_S(P_1, Q_1)$.

To show that (14) is not in general true it is enough to take $a = 10$ and to replace in Example 6 $P_{1/10}$ by P, and $Q_{1/10}$ by Q, respectively.

REFERENCES

[1] Constantin, G., Istrăţescu, I., *Elements of probabilistic analysis and applications*, Edit. Academiei R.S.R., Bucharest 1981.

[2] Dudley, R. M., *Probabilities and metrics*, Aarhus Universitet, Aarhus, 1976.

[3] Sibley, D.A., *A metric for weak convergence of distribution functions*, Rocky Mountain J. Math., 1(1971), p. 427–430.

[4] Thompson, J. W., *A note on the Lévy distance*, J. Appl. Probab., 12(1975), p. 412–414

STRESZCZENIE

Niniejsza praca wprowadza funkcje podobne do metryk Lévy'ego, Prochorowa i Sibley'a oraz podaje relacje między nimi.

SUMMARY

In this paper some functions resembling the Lévy, Prokhorov and Sibley metrics are introduced and the relations between them are given.