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Instytut Matematyki
Uniwersytet Marii Curie-Skłodowskiej

W. CIEŚLAK, W. MOZGAWA

Euclidean Plane Foliations

Płaskie foliacje euklidesowe

1. Throughout, the adjective "smooth" will refer to the class C^∞.

Let a n-dimensional smooth manifold V be given and let us fix a natural number $p \leq n$. By a p-element of contact at the point $x \in V$ we mean a p-dimensional subspace M_x of the tangent space T_xV. The mapping $M : x \to M_x$ is called a field of p-elements of contact or p-field on V. A smooth vector field X on U will be called tangent to M, if for each point $x \in V$, $X_x \in M_x$. The set $\mathfrak{X}_M$ of all such vector fields forms a submodule of the module $\mathfrak{X}(V)$ of all smooth vector fields on V over the ring of smooth functions on V.

Definition 1. A smooth p-field M on V such that

$$X, Y \in \mathfrak{X}_M \Longrightarrow [X, Y] \in \mathfrak{X}_M \tag{1}$$

will be called a p-foliation.

A smooth field of 1-elements of contact is a 1-foliation, of course.

Definition 2. The set of all points of V which can be joined to a point $x_0 \in V$ by a piecewise smooth curve having tangent vector at each point tangent to M will be called a leaf M_{x_0} of the foliation M passing through the point x_0.

In this paper we will consider only 1-dimensional oriented riemannian foliations. Such foliations are formed by the integral curves of certain smooth vector fields without singularities.

We give here a definition of 1-dimensional riemannian foliation according to [1]. There are several equivalent definitions [3], [2].

Let M be a 1-foliation on V, given by the vector field X. The foliation M is said to be riemannian if there exists a riemannian metric γ on V, such that for the vector field X and for each local unit vector field Y orthogonal to M to denote a function we have

$$\gamma(Y, [X, Y]) = 0 \tag{2}$$

In this case γ is called a bundle-like metric.

Now we can give a non-trivial example of a riemannian foliation and its bundle-like metric.

Let (V,γ) be a riemannian manifold which admits a global Killing field X without singularities. Thus X is a nowhere zero smooth vector field which is integrated to a 1-parameter transformation group of isometries with respect to the metric γ. Infinitesimally this condition can be formulated as follows [4] :

$$X\gamma(Y,Z) - \gamma(Y[X,Z]) - \gamma(Z,[X,Y]) = 0 \tag{3}$$

for arbitrary smooth vector fields Y, Z on V.

Let φ_t denote the flow given by the vector field X. Since φ_t are isometries then we have

$$\gamma((\varphi_t)_*(Y),\ (\varphi_t)_*(Y)) - \gamma(Y,Y) = 0 \qquad \text{for all } t, \tag{4}$$

where $(\varphi_t)_*$ is the tangent mapping of φ_t.
Hence

$$\gamma\left(\frac{1}{t}\Big((\varphi_t)_*(Y) - Y\Big),\ Y\right) + \gamma\left((\varphi_t)_*(Y),\ \frac{1}{t}\Big((\varphi_t)_*(Y) - Y\Big)\right) = 0 \tag{5}$$

for each $t \neq 0$. When t tends to zero we obtain

$$\gamma([X,Y],Y) = 0$$

and the orbits of the Killing vector field X form a riemannian 1-foliation.

We describe all riemannian foliations in E^2 (with maximal domain) for which the canonical metric is bundle-like, and discuss with Killing fields of this metric.

If $X = \xi^1 \dfrac{\partial}{\partial x^1} + \xi^2 \dfrac{\partial}{\partial x^2}$, then equations (5) take the form

$$\begin{cases} \dfrac{\partial \xi^i}{\partial x^j} + \dfrac{\partial \xi^j}{\partial x^i} = 0, \\ \dfrac{\partial^2 \xi^i}{\partial x^j \partial x^k} = 0, \qquad\qquad i,j,k = 1,2. \end{cases} \tag{6}$$

Integrating this system of equations we get

$$\begin{cases} \xi^1 = ax^2 + a^1 \\ \xi^2 = -ax^1 + a^2 \qquad \text{where } a, a^1, a^2 \in \mathbb{R}. \end{cases} \tag{7}$$

We now use the following property of riemannian foliations [3]. Let $\mathcal{M}$ be a riemannian foliation and let G be a geodesic curve (with respect to its bundle-like metric) orthogonal at some point to a leaf of $\mathcal{M}$. Then G is orthogonal to the leaves of $\mathcal{M}$ at each point of its domain. Next the Reinhart Lemma implies the following fact. If p, q are sufficiently close in one leaf and orthogonal geodesic curves passing through p, q respectively intersect a neighbouring leaf in p', q' then arcs of these geodesic lines between p, p' and q, q' have the same length. It follows that two leaves of a riemannian foliation are locally at constant distance one from another.

2. Let $\mathcal{M}$ denote a smooth 1-foliation defined on an open non-empty subset of E^2. By X we denote a non-singular vector field whose orbits are leaves of the foliation. From this moment we assume the maximality (with respect to the inclusion) of the domain of X.

Definition 3. A foliation $\mathcal{M}$ such that

$$< Y, [X, Y] >= 0 \tag{8}$$

for each local unit vector field Y orthogonal to X will be called an euclidean foliation, where $< , >$ denotes the canonical metric on E^2.

Such an euclidean foliation is a particular case of a riemannian foliation. The condition (8) is equivalent with

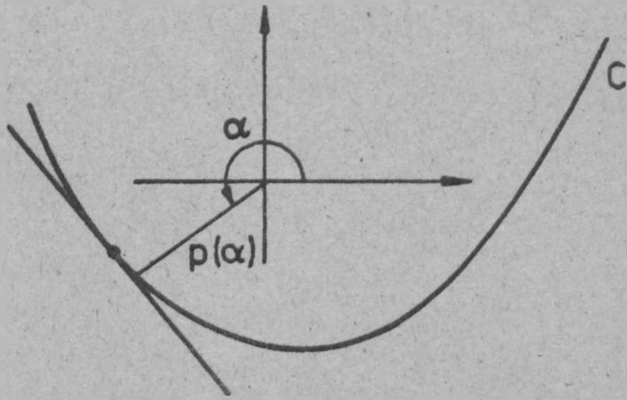
$$f\left(\frac{\partial f}{\partial y}g - f\frac{\partial g}{\partial y}\right) = g\left(\frac{\partial f}{\partial x}g - f\frac{\partial g}{\partial x}\right), \tag{9}$$

if $X = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$ in its domain.

Let us consider a smooth curve C without self-intersections in E^2. We assume that C (open or closed) bounds a convex region in E^2. By the exterior of C we mean the connected component of $\mathbb{R}^2 \setminus C$, which contains each tangent line to C. The other component of $\mathbb{R}^2 \setminus C$ is called the interior of C. Such a curve C will be called convex.

Theorem 1. *Let C be a convex curve on E^2. Then the vector field X whose orbits are involutes of C satisfies the condition* (9).

Proof. Let C be described by its support function $p(\alpha)$, $a < \alpha < b$, and let the coordinate system be chosen so that its origin lies in the interior of C.



Let us consider a point (x, y) which lies in the exterior of C. The tangent line to C passing through the point (x, y) will be described by the angle $\alpha(x, y)$ which satisfies the implicit equation

$$x\cos\alpha + y\sin\alpha - p(\alpha) = 0\,. \tag{10}$$

Thus the normal vector field to the family of tangent lines of C has the form

$$X(x,y) = \cos\alpha(x,y)\frac{\partial}{\partial x} + \sin\alpha(x,y)\frac{\partial}{\partial y}\,. \tag{11}$$

Making use of the implicit function theorem we can verify that the equality (9) is satisfied for $f(x,y) = \cos\alpha(x,y)$, $g(x,y) = \sin\alpha(x,y)$.

3. In this paragraph we will consider the inverse problem to that occurring in the previous paragraph. Let Y be a smooth vector field whose orbits are leaves of a riemannian foliation $\mathcal{M}$ defined on some open subset A of E^2 and let $Y = h\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$ on A.

By (8) we see that for any smooth nowhere-zero function f on A, the vector field fY defines the same riemannian foliation $\mathcal{M}$. For this reason we can take $X = \frac{1}{g}Y = f\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ in the domain where g is nowhere-zero and $f = h/g$. The case $h \neq 0$ is considered similarly and since Y is nowhere-zero the functions f, g do not vanish simultaneously. Let us consider the envelope of the fixed orbit $\Phi_t(p_0)$, $p_0 = (x_0, y_0)$, of the field X. The orbits of X satisfy the system of differential equations

$$\begin{cases} \dot{\Phi}_t^1 = f(\Phi_t^1, \Phi_t^2) \\ \dot{\Phi}_t^2 = 1\,. \end{cases} \tag{12}$$

It is easy to calculate that the curvature of the orbit $\Phi_t(p_0)$ is equal to

$$\kappa_\Phi = \frac{-f_y}{\sqrt{1+f^2}}\,, \qquad \text{where } f_y = \frac{\partial f}{\partial y}\,. \tag{13}$$

The evolute of the orbit $\Phi_t(p_0)$ will be the envelope of its family of normal lines.

$$F(x,y,t) = (x - \Phi_t^1)f(\Phi_t^1, \Phi_t^2) + y - \Phi_t^2\,. \tag{14}$$

The conditions

$$\frac{\partial^2 F}{\partial t \partial t} \neq 0 \qquad \text{and} \qquad \frac{D(F, F_t')}{D(x,y)} \neq 0$$

are sufficient for the existence of a smooth envelope. In our case they are satisfied if f_y and f_{yy} never vanish. Furthermore the evolute is of the following form:

$$\begin{aligned} x(t) &= \Phi_t^1(p_0) + \frac{1}{f_y(\Phi_t^1(p_0), \Phi_t^2(p_0))}\,, \\ y(t) &= \Phi_t^2(p_0) - \frac{f(\Phi_t^1(p_0), \Phi_t^2(p_0))}{f_y(\Phi_t^1(p_0), \Phi_t^2(p_0))}\,. \end{aligned} \tag{15}$$

The curvature of this evolute is equal to

$$\kappa_{ev} = \frac{\kappa^3}{|f_{yy}|} > 0\,.$$

Thus our evolute is convex. Since the leaves of riemannian foliations are of constant distance from each other the evolutes of different orbits coincide and do not depend on the choice of the point p_0.

We will now consider a number of singular cases:

a) Let us assume $f_y \equiv 0$. Then we have $f_x \equiv 0$ and $f(x,y) \equiv$ const. In this case the evolute is empty, the leaves of the foliation are parallel straight lines, and the domain can be taken to be the whole of E^2.

b) Let us assume $f_y = 0$ at some points. Then $f_x = 0$ at the same points if $f_{xy} \neq 0$ (or equivalently $f_{yy} \neq 0$) and

$$\text{Hess } f = -f_y{}^4 = 0 \, .$$

The set of zeros of f_y is a smooth curve. Each orbit of X and the curve have exactly one common point. Hence the envelope will consist of two branches.

c) Let us assume $f_{yy} = 0$. The condition (10) implies that

$$f(x,y) = \frac{-y+a}{x-b} \, .$$

Thus the foliation is given by the orbits of the Killing field with a singularity at the point (a,b), the leaves being just concentric circles with centre (a,b).

In this way we have shown

Theorem 2. *Let X be a smooth vector field whose orbits are leaves of a riemannian foliation on an open subset of the plane. Then the boundary of the maximal domain of X is a regular convex curve (or two such curves, or a point, or the empty set) and the leaves of the foliation are involutes of this boundary (or the foliation consists of orbits of a Killing field with a singularity, or the foliation is a graph of orbits of the Killing field without a singularity).*

4. We now consider an arbitrary smooth closed, convex curve C. We know by Theorem 1 that its involutes form a riemannian foliation. Let us choose the coordinate system so that the origin lies inside the curve. Let us take a smooth homotopy $F_t(-)$, $0 \leq t \leq 1$, such that $F_0 = (0,0)$, $F_1 = C$ and F_t, $0 < t < 1$, is a convex and compact curve inside of C.

Theorem 3. *Let X_t be the vector field whose orbits are involutes of the convex curve F_t. Then we have*

$$\lim_{t\to 0} X_t = f \cdot \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) \qquad f \neq 0 \quad \text{in} \quad \mathbf{R}^2 \setminus (0,0)$$

(or equivalently the riemannian foliations given by X_t tend to the foliation given by the orbits of the Killing field with the singularity, independently of the chosen homotopy).

Proof. For the given homotopy $F_t(-)$ the curves F_t are described by their support functions $p_t(\alpha)$, $0 \leq \alpha \leq 2\pi$ where $\lim_{t\to 0} p_t(\alpha) = 0$. The tangent line to F_t passing through the point (x,y) which lies in the exterior of F_t will be determined by the angle $\alpha_t(x,y)$ satisfying the equation

$$x \cos \alpha_t(x,y) + y \sin \alpha_t(x,y) - p_t(\alpha_t(x,y)) = 0 \, . \tag{16}$$

The field $X_t(x,y) = \cos\alpha_t(x,y)\frac{\partial}{\partial x} + \sin\alpha_t(x,y)\frac{\partial}{\partial y}$ is determined by the equation (16). Let t tends to 0. Then (16) implies

$$x\cos\alpha_0(x,y) + y\sin\alpha_0(x,y) = 0 .$$

Hence

$$\cos\alpha_0(x,y)\frac{\partial}{\partial x} + \sin\alpha_0(x,y)\frac{\partial}{\partial y} = f\cdot\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) ,$$

as claimed.

Now let C denote a smooth convex open curve. The origin of the coordinate system will be chosen in the exterior of C. We prove an analogue of Theorem 3 in this case. In this case we have to choose deformations of the curve C in such a way that intermediate curves depending on t tend to infinity in some sense as $t \to \infty$. Let us choose the homotopy $F_t(-)$, $0 \le t \le \infty$, such that $F_0 = C$ and F_t, $0 < t < \infty$, is a convex curve contained inside of C and the exteriors of F_t form an increasing family of sets whose union is E^2.

Theorem 4. *Let X_t be a vector field whose orbits are involutes of the curve F_t. Then*

$$\lim_{t\to\infty} X_t = f\frac{\partial}{\partial x} \qquad\qquad f \neq 0 \quad in \quad E^2$$

(the riemannian foliations tend to a Killing field without singularities, independently of the chosen homotopy).

Proof. For the given homotopy $F_t(-)$ the curves F_t are described by their support functions $p_t(\alpha)$, $a < \alpha < b$, where $\lim_{t\to\infty} p_t(\alpha) = \infty$. The tangent line to F_t passing through the point (x,y) lying in the exterior of F_t is determined by an angle satisfying the condition

$$x\cos\alpha_t(x,y) + y\sin\alpha_t(x,y) = p_t(\alpha_t(x,y)) .$$

Taking the limit when t tends to infinity and next determining $\frac{x}{y}$ and $\frac{y}{x}$ we obtain $\frac{x}{y} = \infty$, $\frac{y}{x} = 0$. Hence we have

$$\lim_{t\to\infty} X_t(x,y) = f\cdot\left(1\frac{\partial}{\partial x} + 0\frac{\partial}{\partial y}\right) = f\frac{\partial}{\partial x} .$$

The above results suggest that it might be useful to formmlate a notion of riemannian foliation with singularities. So far we have not found a natural way of doing this.

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STRESZCZENIE

W pracy tej opisujemy wszystkie jednowymiarowe zorientowane foliacje riemannowskie z maksymalną dziedziną zawartą w E^2, dla których kanoniczna metryka euklidesowa jest metryką bundle-like. Podajemy związki tych foliacji z polami Killinga na E^2.

SUMMARY

In this paper all one-dimensional oriented riemannian foliations whose maximal domain is contained in E^2 and canonical euclidean metric is bundle-like are described. A relation of Killing fields on E^2 to these foliations is exhibited.