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On the monotonity of certain functionals in the theory of analytic functions [1])

O monotoniczności pewnych funkcjonałów w teorii funkcyj analitycznych

О монотонности некоторых функционалов в теории аналитических функций

This paper deals with certain functionals defined for functions regular in the circle $|z| < R$ which are, the function $f(z)$ being fixed, real and monotonic functions of the real variable $r = |z|$ in the open interval $(0, R)$. Some theorems are proved and some conjectures are announced.

The results of the part I, are due to the former of both authors, those of the part II, to the latter.

I. Let $f(z)$ be a function regular for $|z| < R$. Put

$$M_p(r, f) = M_p(r) = \left[\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\Theta})|^p \, d\Theta\right]^{\frac{1}{p}}$$

$$I_p(r, f) = I_p(r) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\Theta})|^p \, d\Theta = |M_p(r)|^p.$$

It was proved by G. H. Hardy [3] that $M_p(r)$ is an increasing function of r and that $\log M_p(r)$ is a convex function of $\log r$ $(0 < r < R)$. In other words, $\frac{r M_p'(r)}{M_p(r)}$ is an increasing function of r. For $I_p(r)$ we can make evi-

[1]) The principal results of this paper have been presented to the IV Congress of Roumanian Mathematicians at Bucarest, May 28, 1956.

dently analogous statements. In particular, $\frac{rM_2'(r)}{M_2(r)}$ increases and this suggests that also the ratio

$$(1.1) \qquad \frac{\int_0^{2\pi} |rf'(re^{i\Theta})|^2 d\Theta}{\int_0^{2\pi} |f(re^{i\Theta})|^2 d\Theta} = \frac{r^2 I_2(r, f')}{I_2(r, f)}$$

increases with r. In fact, this conjecture holds and we shall prove it. We first prove a lemma (due to J. Krzyż).

Lemma 1. *Suppose both series with real coefficients $\sum_n \alpha_n z^n$ and $\sum_n \beta_n z^n$ converge for $|z| < R$ and β_i being non-negative not all vanish. If $\begin{vmatrix} \alpha_k & \alpha_i \\ \beta_k & \beta_i \end{vmatrix} \geqslant 0$ for all $k > i$, then the quotient $\varphi(x) = \frac{\sum_{n=0}^{\infty} \alpha_n x^n}{\sum_{n=0}^{\infty} \beta_n x^n}$ is either a strictly increasing function of a real variable $x \epsilon (0, R)$, or it is constant. In the latter case all the determinants $\begin{vmatrix} \alpha_k & \alpha_i \\ \beta_k & \beta_i \end{vmatrix}$ vanish. In particular, $\varphi(x)$ increases strictly if the sequence $\left\{\frac{\alpha_k}{\beta_k}\right\}$ increases and not all its terms are equal.*

P r o o f. In order to prove the lemma it is sufficient to observe that the numerator of $\varphi'(x)$ is equal to

$$(1.2) \qquad \begin{vmatrix} \alpha_1 & \alpha_0 \\ \beta_1 & \beta_0 \end{vmatrix} + 2 \begin{vmatrix} \alpha_2 & \alpha_0 \\ \beta_2 & \beta_0 \end{vmatrix} x + \left(3 \begin{vmatrix} \alpha_3 & \alpha_0 \\ \beta_3 & \beta_0 \end{vmatrix} + \begin{vmatrix} \alpha_2 & \alpha_1 \\ \beta_2 & \beta_1 \end{vmatrix}\right) x^2 +$$
$$+ \left(4 \begin{vmatrix} \alpha_4 & \alpha_0 \\ \beta_4 & \beta_0 \end{vmatrix} + 2 \begin{vmatrix} \alpha_3 & \alpha_1 \\ \beta_3 & \beta_1 \end{vmatrix}\right) x^3 + \ldots + \left[n \begin{vmatrix} \alpha_n & \alpha_0 \\ \beta_n & \beta_0 \end{vmatrix} + (n-2) \begin{vmatrix} \alpha_{n-1} & \alpha_1 \\ \beta_{n-1} & \beta_1 \end{vmatrix} + \right.$$
$$\left. + (n-4) \begin{vmatrix} \alpha_{n-2} & \alpha_2 \\ \beta_{n-2} & \beta_2 \end{vmatrix} + \ldots \right] x^{n-1} + \ldots$$

and that the denominator of $\varphi'(x)$ is positive for $x \epsilon (0, R)$ and therefore $\varphi'(x) \geqslant 0$ and $\varphi(x)$ increases for $x \epsilon (0, R)$. Being an analytic function of $|z| < R$ the numerator of $\varphi'(x)$ either vanishes at isolated points, or it vanishes identically. In the former case $\varphi'(x) > 0$ except at isolated points of the interval $(0, R)$ and then $\varphi(x)$ increases strictly, in the latter case all the coefficients in (12) and therefore all the determinants $\begin{vmatrix} \alpha_k & \alpha_i \\ \beta_k & \beta_i \end{vmatrix}$ must vanish and then $\varphi(x)$ is constant.

We now prove

Theorem 1. *If $f(z)$ is regular for $|z| < R$, $f(z) \not\equiv 0$, then the quotient*

$$(1.1)\qquad \frac{r^2 I_2(r, f')}{I_2(r, f)} = \frac{\int_0^{2\pi} |r f'(r e^{i\Theta})|^2 \, d\Theta}{\int_0^{2\pi} |f(r e^{i\Theta})|^2 \, d\Theta}$$

is a strictly increasing function of $r \in (0, R)$, unless $f(z) = a_n z^n$ ($a_n \neq 0$, n is a non-negative integer), when the quotient (1.1) *is constant.*

Proof. Put $f(x) = \sum_{n=0}^{\infty} a_n z^n$. Then

$$\frac{r^2 I_2(r, f')}{I_2(r, f)} = \frac{\sum_{n=0}^{\infty} n^2 |a_n|^2 r^{2n}}{\sum_{n=0}^{\infty} |a_n|^2 r^{2n}}.$$

Putting $r^2 = x$ we bring (1.1) to the form of quotient considered in Lemma 1, with $\alpha_k = k^2 |a_k|^2$, $\beta_k = |a_k|^2$. We have for $k > i$

$$\begin{vmatrix} \alpha_k & \alpha_i \\ \beta_k & \beta_i \end{vmatrix} = \begin{vmatrix} k^2 |a_k|^2, & i^2 |a_i|^2 \\ |a_k|^2, & |a_i|^2 \end{vmatrix} = (k^2 - i^2) |a_i|^2 |a_k|^2 \geqslant 0.$$

This means that (1.1) increases strictly or is constant. If the latter case occurs, then $(k^2 - i^2) |a_i|^2 |a_k|^2 = 0$ for all k, i. Since $a_n \neq 0$ for some n ($f(z) \not\equiv 0$), therefore $(n^2 - i^2) |a_i|^2 = 0$ i. e. $a_i = 0$ for all $i \neq n$. This means that $f(z) = a_n z^n$ and this is the desired result.

The theorem just proved suggests that also the ratio

$$(1.3)\qquad \frac{r^p I_p(r, f')}{I_p(r, f)} = \frac{\int_0^{2\pi} |r f'(r e^{i\Theta})|^p \, d\Theta}{\int_0^{2\pi} |f(r e^{i\Theta})|^p \, d\Theta}$$

is an increasing function of r for all $p \geqslant 1$. In this case also the ratio

$$\frac{r M_p(r, f')}{M_p(r, f)} = \frac{\left[\frac{1}{2\pi} \int_0^{2\pi} |r f'(r e^{i\Theta})|^p \, d\Theta\right]^{1/p}}{\left[\frac{1}{2\pi} \int_0^{2\pi} |f(r e^{i\Theta})|^p \, d\Theta\right]^{1/p}}$$

would be an increasing function of $r \in (0, R)$. Making $p \to +\infty$ we could then prove that

$$\frac{\sup_{|z|=r} |z f'(z)|}{\sup_{|z|=r} |f(z)|} = \frac{r M(r, f')}{M(r, f)}$$

is an increasing function of $r \in (0, R)$, too.

It is, however, not true that (1.3) increases for arbitrary $f(z)$ and $p > 0$, Making $p \to 0$, we should obtain ([5] pp. 98, 99) that the quotient

$$\text{(1.4)} \qquad \frac{\exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log|r f'(re^{i\Theta})|\, d\Theta\right\}}{\exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\Theta})|\, d\Theta\right\}}$$

increases with r. If $f(z)$ is an integral function with an infinite number of zeros and a non-vanishing derivative (e. g. $f(z) = e^z + 1$), then the Jensen formula shows that the numerator of (1.4) has the form Ar (A = const) and the denominator increases more rapidly than any positive power of r. Therefore the quotient (1.4) cannot be an increasing function of r for p small enough.

II. The above mentioned property of $\log M_p(r)$ to be a convex function of $\log r$ is equivalent with the analogous property of

$$I_p(r) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\Theta})|^p\, d\Theta \quad (p > 0), \quad \text{i. e.} \quad \frac{r I_p'(r)}{I_p(r)}$$

also increases (see [8], p. 174). We can combine this property with an identity due to S. Mandelbrojt and so we obtain

Theorem 2. *Suppose the function $f(z)$ regular for $|z| < R$ and non-vanishing identically, maps the circle $|z| = r < R$ into the curve C_r. If $\Phi = \arg f(re^{i\Theta}) = \Phi(\Theta)$ on C_r, then the quotient*

$$\frac{\int_{C_r} |f(re^{i\Theta})|^p\, d\Phi(\Theta)}{\int_0^{2\pi} |f(re^{i\Theta})|^p\, d\Theta}$$

is an increasing function of $r \in (0, R)$ for any $p > 0$.

Proof. If p is real and $f(z) \neq 0$ for $|z| = r$, then

$$\frac{d}{dr}\int_0^{2\pi} |f(re^{i\Theta})|^p \, d\Theta = \frac{p}{r}\int_{C_r} |f(re^{i\Theta})|^p \, d\Phi(\Theta).$$

This formula is due to S. Mandelbrojt, (see [6]). Thus

$$\frac{\int_{C_r} |f(re^{i\Theta})|^p \, d\Phi}{\int_0^{2\pi} |f(re^{i\Theta})|^p \, d\Theta} = \frac{\frac{r}{p}\frac{d}{dr} I_p(r)}{I_p(r)}$$

for $r \in (r', r'')$, (r', r'') being the interval such that $f(z) \neq 0$ for $r' < |z| < r''$. Dividing up the interval $(0, R)$ into partial intervals (r_i, r_{i+1}) such that $f(z) \neq 0$ for $r_i < |z| < r_{i+1}$ we obtain in view of continuity the desired result.

In particular, if $p = 2$, we see that the quotient $\frac{S(r)}{I_2(r)}$ increases, $S(r)$ being the area of the map of $|z| < r$ by $f(z)$. This may be also easily proved by using the lemma 1 and the well known representation of $S(r)$ and $I_2(r)$ by means of coefficients of $f(z)$.

We shall now prove a result somewhat connected with a conjecture announced above, which enables us to give to the Hadamard's three circles theorem a very simple geometrical interpretation. This is the

Theorem 3. *Let $f(z)$ be regular for $|z| < R$ and let Γ denote the locus of points ξ such that $|f(\xi)| = M(|\xi|)$. If the derivative $M'(r)$ exists for a given value r, then $M'(r) = |f'(\xi)|$, where ξ is an arbitrary point of Γ lying on the circle $|z| = r$. If $M'(r)$ does not exist, then the left-hand (right-hand) derivative of $M(r)$ is equal to $|f'(\xi)|$, ξ being the end-point of an arc of Γ lying locally inside (outside) of the circle $|z| = r$.*

Proof. O. Blumenthal [1], [2] proved that $M(r)$ is an analytic function of r, except at isolated points $r_1 < r_2 < \ldots$ so that $M(r)$ is represented by distinct analytic functions in the intervals $r_i \leqslant r \leqslant r_{i+1}$ $(r_i < R,$ $i = 1, 2, \ldots)$. This implies the existence of the one-sided derivatives of $M(r)$ and their one-sided continuity at $r = r_i$. Besides, he showed that the locus Γ consists in $|z| \leqslant r < R$ of a finite system of analytic arcs, unless $f(z) = a\, z^n$.

Suppose first that $M'(r_0)$ exists and Γ is not tangential to the circle $|z| = r_0$. Put $\xi_0 = r_0 e^{i\Theta_0}$, $\xi_1 = (r_0 + \Delta r)\, e^{i(\Theta_0 + \Delta\Theta)}$, $\Delta r \neq 0$, $\xi_0, \xi_1 \in \Gamma$. Then, by the mean value theorem.

$$M(r_0+\Delta r)-M(r_0)=|f(\xi_1)|-|f(\xi_0)|=\Delta r\frac{\partial|f|}{\partial r}+\Delta\Theta\frac{\partial|f|}{\partial\Theta}$$

both partial derivatives being taken at $r=r_0+\vartheta\Delta r$, $\Theta=\Theta_0+\vartheta\Delta\Theta$, $0<\vartheta<1$. Therefore $\frac{M(r_0+\Delta r)-M(r_0)}{\Delta r}=\frac{\partial|f|}{\partial r}+\frac{\Delta\Theta}{\Delta r}\frac{\partial|f|}{\partial\Theta}$ and making $\Delta r\to 0$ we obtain $M'(r_0)=\frac{\partial|f|}{\partial r}(z=\xi_0)$, since $\frac{\partial|f|}{\partial\Theta}=0$ at ξ_0 and $\frac{\Delta\Theta}{\Delta r}$ is bounded. We have also

$$\frac{\partial}{\partial r}\log f=e^{i\Theta_0}\frac{f'(\xi_0)}{f(\xi_0)}=\frac{\partial}{\partial r}(\log|f|+i\arg f)=\frac{\partial}{\partial r}\log|f|,$$

since

$$\frac{\partial}{\partial r}\arg f=-\frac{1}{r}\frac{\partial}{\partial\Theta}\log|f|=0 \text{ at } \xi_0.$$

(Riemann-Cauchy equation).
Therefore

$$\frac{M'(r_0)}{M(r_0)}=\frac{e^{i\Theta_0}f'(\xi_0)}{f(\xi_0)}=\frac{\frac{\partial}{\partial r}|f|}{|f|}\qquad(z=\xi_0)$$

and this implies

$$|f'(\xi_0)|=\frac{\partial}{\partial r}|f|,\quad\text{or}\quad M'(r_0)=|f'(\xi_0)|.$$

If Γ is tangential to $|z|=r_0$ and $M'(r_0)$ exists, we take slight greater (or less) values of r such that Γ and the corresponding circle $|z|=r$ intersect at a non-zero angle.

Then we have $M'(r)=|f'(\xi_0)|$. Let us now suppose that $r\to r_0$. Since $M'(r_0)$ exists, so it must be continuous. $|f'(\xi_0)|$ is obviously continuous, too. Therefore $M'(r)=|f'(\xi_0)|$.

If the one-sided derivative of $M(r)$ exists, we keep in view its one-sided continuity. The result then follows by passing to the limit.

We next give an alternative proof of Theorem 3, due to the former of both authors. It is based on the Lemma 2. concerning real functions which may be of independent interest.

Lemma 2. *Suppose $f(x,a)$, $\varphi(x,a)$, $\psi(x,a)$ are real functions of two real variables (x,a) continuous in the rectangle D: $a\leqslant x\leqslant b$, $a_1\leqslant a\leqslant a_2$* and $f(x,a)\geqslant 0$ for $(x,a)\in D$. If $f(x,a)$ *attains for each value $a\in[a_1,a_2]$*

its least upper bound $M(a)>0$ *at just one point* $\xi(a)\in|a,b|$ and $\varphi(\xi(a),a)\neq$ $\neq 0, \psi(\xi(a),a)\neq 0$, *then*

$$\lim_{p\to+\infty}\frac{\int_a^b |f(x,a)|^p\,\varphi(x,a)\,dx}{\int_a^b |f(x,a)|^p\,\psi(x,a)\,dx}=\frac{\varphi(\xi(a),a)}{\psi(\xi(a),a)}, \tag{3.2}$$

the convergence being uniform over the interval $|a_1, a_2|$.

(If the functions considered do not depend on a and $\varphi = f\psi$, the lemma is due to P. Csillag and Pólya-Szegö (see [7] 1 Band, p. 78. Aufg. 199 and 201).

We omit the proof of this lemma since it can be easily obtained by evident modification of the Pólya-Szegö proof for the particular case mentioned above.

We now give the alternative proof of Theorem 3.

Suppose $|f(z)|$ attains at the point P of the circle $|z| = r_0$ the maximum and P is lying on a regular arc of Γ which may be represented by the equation $\Theta=\Theta_0(r)$ in the neighbourhood of P. If $\eta>0$ is small enough, there exists a neighbourhood of P: $r_1\leqslant r\leqslant r_2$, $\Theta_0(r_0)-\eta\leqslant\Theta\leqslant\Theta_0(r_0)+\eta$ such that for each r we have just one $\Theta=\Theta_0(r)$ for which $|f(re^{i\Theta_0(r)})|=M(r)$. We now apply the Lemma 2, with $a=r$. Putting

$$\varphi(r,p)=\left[\frac{1}{2\eta}\int_{\Theta_0-\eta}^{\Theta_0+\eta}|f|^p\,d\Theta\right]^{1/p}$$

and

$$g(r)=\frac{d|f|}{dr}\cdot\frac{1}{|f|}\qquad (z=re^{i\Theta_0(r)})$$

we see that

$$\frac{\varphi_r'(r,p)}{\varphi(r,p)}=\frac{\int_{\Theta_0-\eta}^{\Theta_0+\eta}|f|^p\,g\,d\Theta}{\int_{\Theta_0-\eta}^{\Theta_0+\eta}|f|^p\,d\Theta}\xrightarrow[p\to\infty]{} g(r)$$

uniformly over the interval $[r_1, r_2]$.

Integrating from r_0 to r, we obtain

$$\log\varphi(r,p)-\log\varphi(r_0,p)\to\int_{r_0}^{r} g(r)\,dr.$$

It is well known that $\lim_{p\to+\infty} \varphi(r, p) = M(r)$ [7, p. 78] and therefore

$$\int_{r_0}^{r} g(r)\,dr = \log M(r) - \log M(r_0).$$

By continuity of $g(r)$ the derivative $M'(r)$ exists and we have

$$\frac{M'(r)}{M(r)} = g(r) = \frac{\frac{d|f|}{dr}}{M(r)} = \frac{|f'|}{M(r)}.$$

This implies $M'(r) = |f'(\xi_0)|$. The rest of proof is the same as above.

When the curve Γ is discontinuous for a given value r_0 the similar considerations are valid for both intervals (r_1, r_0), (r_0, r_2) and we obtain an analogous result with left-hand and right-hand derivatives of $M(r)$ instead of $M'(r)$. When P is a point of ramification of Γ, we may consider slightly greater (or less) values of r and then suppose that r tends to the limit r_0, the result being analogous.

Corollaries.

1. $M'(r)$ may not exist for such r only for which the circle $|z| = r$ contains the discontinuity points of Γ.

2. When the circle $|z| = r$ intersects Γ at several points ξ_i so that at all such points ξ_i Γ surpasses the circle from the inside to the outside, or all arcs of Γ terminating at ξ_i approach $|z| = r$ from the inside (resp. from the outside) of $|z| = r$, then the values of $|f'(z)|$ at all such points ξ_i are necessarily equal.

3. In all the intervals $(0, r_1)$, (r_1, r_2), ... (in which $M(r)$ is analytic) we have $\frac{rM'(r)}{M(r)} = \frac{\xi f'(\xi)}{f(\xi)}$, for all $\xi \in \Gamma$ such that $|\xi| = r$. (It is well known that $\frac{\xi f'(\xi)}{f(\xi)}$ is real on Γ). By the Hadamard's three circles theorem the left-hand side increases with r and the right-hand side does so, too. Since $\frac{\partial \arg f}{\partial \Theta} = \mathcal{R}\frac{\xi f'(\xi)}{f(\xi)} = \frac{\xi f'(\xi)}{f(\xi)}$ for $\xi \in \Gamma$, our result means that the angular velocity of the point $f(z)$, as z is moving steadily on the circle $|z| = r$ and surpasses the point ξ, increases with $|\xi| = r$. This fact is equivalent with the Hadamard's three circles theorem.

The equality $\frac{rM'(r)}{M(r)} = \frac{\xi f'(\xi)}{f(\xi)}$ for ξ such that Γ and $|z| = |\xi|$ are not tangential was also used by W. K. Hayman ([4], Lemma 6, p. 141).

The former of us conjectured, in connexion with the above considerations, that also the quotient

$$\frac{L^2(r)}{S(r)} = \frac{r^2\left[\int\limits_0^{2\pi} |f'(r\,e^{i\Theta})|\,d\Theta\right]^2}{\int\limits_0^r \varrho\left\{\int\limits_0^{2\pi} |f'(\varrho\,e^{i\Theta})|^2\,d\Theta\right\}d\varrho}$$

increases with r, $L(r)$ denotes the length of the curve being the map of the circle $|z| = r$ by $f(r)$ and $S(r)$ the area of the corresponding region of the Riemann surface of $f(z)$. This conjecture means that the shape of the maps of the circles $|z| = r$ deviates monotonically from that of a circle with increasing r. We could not prove this conjecture but we give a proof of a similar statement being a conclusion of this conjecture.

If $\dfrac{L^2(r)}{4\pi S(r)} = 1 + h(r)$ and $h(r)$ increases, then also $L^2(r) - 4\pi S(r)$ does so. This difference can be also considered as a measure of deviation from the circular shape. We now prove the

Theorem 4. *If $f(z)$ is a function regular for $|z| < R$ and $f'(z) \neq 0$ for $|z| < R$, then*

$$\delta(r) = L^2(r) - 4\pi S(r)$$

increases strictly for $r \in (0, R)$, unless $f(z) = \dfrac{az+b}{cz+d}$ $(ad - bc \neq 0)$, *when* $\delta(r) = 0$.

Proof. If $f'(z) \neq 0$ for $|z| < R$, then a branch of $\sqrt{f'(z)}$, say $\varphi(z)$, is regular for $|z| < R$.

Let

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{for } |z| < R.$$

Then

$$\int\limits_0^{2\pi} |f'(r e^{i\Theta})|\,d\Theta = \int\limits_0^{2\pi} |\varphi(r\,e^{i\Theta})|^2\,d\Theta = 2\pi \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

Hence

$$L^2(r) = r^2\left[\int\limits_0^{2\pi} |f'(r\,e^{i\Theta})|\,d\Theta\right]^2 = 4\pi^2 r^2\Big\{|a_0|^4 + \tag{4.1}$$

$$+ (|a_0 a_1|^2 + |a_1 a_0|^2)\,r^2 + \ldots + \left(\sum_{\nu=0}^{k} |a_\nu a_{k-\nu}|^2\right) r^{2k} + \ldots\Big\}.$$

In order to calculate the area $S(r)$ we must obtain the development of $f'(z)$:

$$f'(z) = a_0^2 + (a_0 a_1 + a_1 a_0)\,z + (a_0 a_2 + a_1 a_1 + a_2 a_0)\,z^2 +$$

$$+ \ldots + \left(\sum_{\nu=0}^{k} a_\nu a_{k-\nu}\right) z^k + \ldots$$

and hence

$$\int_0^{2\pi} |f'(re^{i\Theta})|^2 d\Theta = 2\pi \Big\{ |a_0|^4 + |a_0 a_1 + a_1 a_0|^2 r^2 + \\ + \dots + \Big| \sum_{\nu=0}^{k} a_\nu a_{k-\nu} \Big|^2 r^{2k} + \dots \Big\}.$$

Now

$$S(r) = \int_0^r \varrho\, d\varrho \int_0^{2\pi} |f'(\varrho e^{i\Theta})|^2 d\Theta = 2\pi \Big[\frac{1}{2} |a_0|^4 r^2 + \\ + \frac{1}{4} |a_0 a_1 + a_1 a_0|^2 r^4 + \dots + \frac{1}{2k+2} \Big| \sum_{\nu=0}^{k} a_\nu a_{k-\nu} \Big|^2 r^{2k+2} + \dots \Big]$$

and

$$(4.2) \qquad 4\pi S(r) = 4\pi^2 r^2 \Big\{ |a_0|^4 + \frac{1}{2} |a_0 a_1 + a_1 a_0|^2 r^2 + \\ + \dots + \frac{1}{k+1} \Big| \sum_{\nu=0}^{k} a_\nu a_{k-\nu} \Big|^2 r^{2k} + \dots \Big\}.$$

If we compare the developments (4.1) and (4.2), we can observe that the coefficients of r^{2n} in brackets in (4.1) exceed the corresponding coefficients in (4.2). Or, in other words,

$$(4.3) \qquad \sum_{\nu=0}^{n} |a_\nu a_{n-\nu}|^2 \geqslant \frac{1}{n+1} \Big| \sum_{\nu=0}^{n} a_\nu a_{n-\nu} \Big|^2.$$

This is the immediate consequence of the following statement:

If $z_0, \dots, z_n$ are arbitrary complex numbers, then

$$(4.4) \qquad \frac{|z_0|^2 + |z_1|^2 + \dots + |z_n|^2}{n+1} \geqslant \Big| \frac{z_0 + z_1 + \dots + z_n}{n+1} \Big|^2$$

with the sign of equality for $z_0 = z_1 = \dots = z_n$ only.

To prove this statement observe that

$$\sum_{0 \leqslant i < k \leqslant n} (z_i - z_k)(\bar{z}_i - \bar{z}_k) \geqslant 0$$

with the sign of equality for $z_0 = z_1 = \dots = z_n$ only.

After multiplication we obtain

$$n \sum_{k=0}^{n} |z_k|^2 - \sum_{\substack{i \neq k \\ 0 \leqslant i, k \leqslant n}} z_i \bar{z}_k = (n+1) \sum_{k=0}^{n} |z_k|^2 - \sum_{i,k=0}^{n} z_i \bar{z}_k \geqslant 0,$$

resp.

$$(n+1)\sum_{k=0}^{n}|z_k|^2-\left(\sum_{i=0}^{n} z_i\right)\left(\sum_{k=0}^{n}\bar{z}_k\right)\geqslant 0$$

which is equivalent with (4.4). Putting $z_\nu = a_\nu a_{n-\nu}$ in (4.4) we obtain (4.3).

We have proved somewhat more: *the ratio* $\dfrac{L^2(r)-4\pi S(r)}{r^6}$ *increases with* r.

Suppose now that $\delta(r_1)=\delta(r_2)$ for $r_1<r_2$. Since $\delta(r)$ increases and is analytic as a function of $r\in(0, R)$, so $\delta(r)=$ const. $\equiv 0$. We have for any n the sign of equality in (4.3). Therefore $a_0 a_n = a_1 a_{n-1} = \ldots = a_n a_0$ for any n. Thus

$$f'(z)=\varphi^2(z)=a_0^2+(a_0 a_1+a_1 a_0)z+(a_0 a_2+a_1 a_1+a_2 a_0)z^2+$$
$$+\ldots = a_0^2+2a_0 a_1 z+3a_0 a_1 z^2+\ldots$$

Hence

$$f(z)=A+a_0 z(a_0+a_1 z+a_2 z^2+\ldots),$$

or

$$f(z)=A+a_0 z\sqrt{f'(z)},$$

This implies

$$\frac{f'(z)}{[f(z)-A]^2}=\frac{1}{a_0^2 z^2}, \quad \text{resp.} \quad f(z)=\frac{az+b}{cz+d}.$$

If $f(z)=\dfrac{az+b}{cz+d}$ then the maps of $|z|=r$ are circles and really $\delta(r)\equiv 0$.

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Streszczenie

W pracy tej zajmujemy się pewnymi funkcjonałami, określonymi dla funkcyj regularnych w kole $|z| \leqslant r < R$, które przy ustalonej funkcji $f(z)$ są monotonicznymi funkcjami zmiennej r w przedziale $(0, R)$.

W części I., napisanej przez pierwszego z nas, wykazane jest następujące twierdzenie:

$$\frac{r^2 I_2(r, f')}{I_2(r, f)}$$

jest bądź funkcją ściśle rosnącą od r, $r \in (0, R)$, bądź też stałą. Ten ostatni przypadek ma miejsce jedynie dla $f(z) = a_n z^n$ $(n = 0, 1, 2, \ldots)$. $I_p(r, f)$ oznacza, jak zwykle, średnią całkową p-tej potęgi modułu $f(z)$, wziętą po kole $|z| = r$.

W części II., napisanej przez drugiego z nas, wykazane jest twierdzenie następujące:

$$\frac{\int\limits_{C(r)} |f(z)|^p \, d\Phi}{\int\limits_0^{2\pi} |f(z)|^p \, d\Theta}$$

rośnie wraz z r, $C(r)$ jest tu obrazem okręgu $|z| = r$ poprzez $f(z)$, $\Phi = \arg f(z)$.

Ponadto wykazane jest na dwa sposoby (2-gi dowód jest podany przez pierwszego z nas), że w punktach ξ, gdzie $|f(z)|$ osiąga maksimum w kole $|z| \leqslant r$ mamy $M'(r) = |f'(\xi)|$ (oraz $M(r) = |f(\xi)|$), z wyjątkiem r tworzących zbiór izolowany.

W związku z wysuniętą przez pierwszego z nas hipotezą, że obrazy okręgów $|z| = r$ poprzez $f(z)$ coraz bardziej odbiegają od kształtu kołowego, tzn. że stosunek $\dfrac{L^2(r)}{4\pi S(r)}$ rośnie wraz z r ($L(r)$ = długość obrazu $|z| = r$; $S(r)$ pole ograniczone przez ten obraz), drugi z nas wykazał wniosek wypływający z tej hipotezy: $\delta(r) = L^2(r) - 4\pi S(r)$ bądź rośnie ściśle wraz z r, bądź też $\delta(r) \equiv 0$ (w przypadku funkcji ułamkowo-liniowej).

Резюме

В предлагаемом труде мы занимаемся некоторыми функционалами, определёнными для функций, регулярных в круге $|z| \leqslant r < R$; эти функционалы при установленной функции $f(z)$ являются монотонными функциями переменной r в интервале $(0, R)$.

В части I, написанной первым из нас, доказана следующая теорема:

$$\frac{r^2 I_2(r, f')}{I_2(r, f)}$$

или строго возрастающая функция от r, $r \in (0, R)$, или же постоянная. Этот последний случай имеет место исключительно для $f(z) = a_n z^n$ $(n = 0, 1, 2, \dots)$ $I_p(r, f)$ обозначает, как обыкновенно, интегральную среднюю p-ой степени модуля $f(z)$, взятую по окружности $|z| = r$.

В части II, написанной вторым из нас, доказана следующая теорема:

$$\frac{\int\limits_{C(r)} |f(z)|^p \, d\Phi}{\int\limits_0^{2\pi} |f(z)|^p \, d\Theta}$$

растёт вместе с r; здесь $C(r)$ представляет образ окружности $|z| = r$ посредством $f(z)$, $\Phi = \arg f(z)$.

Сверх того показано двумя способами (2-е доказательство дано первым из нас), что в точках ξ, где $f(z)$ достигает максимум в круге $|z| \leqslant r$, имеем $M'(r) = |f'(\xi)|$ $(M(r) = |f(\xi)|)$ за исключением r, образующих изолированное множество.

В связи с выдвинутой первым из нас гипотезой, что образы окружностей $|z| = r$ посредством $f(z)$ всё более отходят от формы круга, то-есть что отношение $\frac{L^2(r)}{4\pi S(r)}$ растёт вместе с r ($L(r) =$ длина образа окружности $|z| = r$, $S(r)$ площадь, ограниченная этим образом), второй из нас доказал вытекающее из этой гипотезы следствие: $\delta(r) = L^2(r) - 4\pi S(r)$ или строго растёт вместе с r, или же $\delta(r) \equiv 0$ (в случае дробно-линейной функции).