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Tables of Expected Values and Variances of Numbers of Runs in Random Sequences with Probabilities of Exceeding Expected Values

Tablice wartości oczekiwanych i dyspersji liczby ogniw w sekwencjach losowych z podaniem prawdopodobieństw przekroczenia wartości oczekiwanych.

Таблицы математических ожиданий и дисперсий числа звеньей в случайных последовательностях альтернатив с приведением вероятности превышения математических ожиданий.

When there are a number of short two-valued sequences, the test of randomness, such as e. g. a test of the number of runs, may not be decisive for separate sequences, while the excesses or deficiencies of the number of runs over the expected numbers may be quite consistent in most sequences. In such cases the sign test may be indicative to bring out the systematic character of sequences. For that purpose there will be needed probabilities of the number of runs to exceed the expected values.

With a view to facilitating the carrying out of sign test, as well as for other uses, the following tables were prepared. The entry f_{min} refers to the smaller of the two numbers: number of zeros or number of ones in a sequence composed of $n = f_{min} + f_{max}$ elements. The column $E(R)$ gives expected numbers of runs

$$E(R) = \frac{2 f_{min} \cdot f_{max}}{f_{min} + f_{max}} + 1$$

The colum $P(R > E(R))$ gives probabilities for the numbers of runs in random sequences to exceed the expected numbers given in the preceding column. These probabilities were determined on the basis

of the tables prepared by Swed, Freda S., and Eisenhart, C. [1]) The column $D^2(R)$ gives sampling variances for the numbers of runs in random sequences.

Number of sequence	f_1	f_0	r	Sign of $r - E(R)$	$P\ [R > E\ (R)]$	PQ
1	4	3	4	—	0.457143	0.248163
2	4	1	2	—	0.600000	0.240000
3	10	10	8	—	0.414070	0.242616
4	11	4	6	—	0.626374	0.234030
5	2	1	3	+	0.333333	0.222222
6	3	3	2	—	0.300000	0.210000
7	1	5	2	—	0.666667	0.222222
8	1	6	2	—	0.714286	0.204082
9	2	14	3	—	0.650000	0.227500
10	9	6	7	—	0.433566	0.245587
11	1	5	2	—	0.666667	0.222222
12	1	2	2	—	0.333333	0.222222
13	1	4	2	—	0.600000	0.240000
14	1	7	2	—	0.750000	0.187500
15	3	1	2	—	0.500000	0.250000
16	1	7	3	+	0.750000	0.187500
17	2	6	4	0	0.357143	0.229592
18	2	2	2	—	0.333333	0.222222
19	3	3	2	—	0.300000	0.210000
20	5	1	2	—	0.666667	0.222222
21	1	5	2	—	0.666667	0.222222
22	5	2	2	—	0.666667	0.222222
23	2	4	2	—	0.600000	0.240000
24	5	1	2	—	0.666667	0.222222
25	4	2	3	—	0.600000	0.240000
26	2	5	2	—	0.666667	0.222222
27	5	1	2	—	0.666667	0.222222
28	1	4	2	—	0.600000	0.240000
29	2	5	4	+	0.666667	0.222222
30	5	1	3	+	0.666667	0.222222
31	2	3	4	+	0.500000	0.250000
32	4	1	2	—	0.600000	0.240000
33	1	6	2	—	0.714286	0.204082
34	3	3	4	0	0.300000	0.210000
				Total	19.033537	7.669538

[1]) Swed, F. S., and Eisenhart, C. Tables for Testing Randomness of Grouping in a Sequence of Alternatives. Annals of Math. Statistics, v. XIV, 1943.

$$D^2(R) = \frac{[E(R) - 1]\,[E(R) - 2]}{f_{min} + f_{max} - 1}$$

And finally, column $D^2\left(\frac{R}{n}\right)$ gives sampling variances for the numbers of runs per element.

The average probality for the whole series of independent sequences of the numbers of runs to exceed the expected numbers gives the probability of plus sign of differences between the observed and expected numbers of runs in the whole series.

An example. In a biological experiment there were obtained 34 sequences none of which could be considered significantly non-random by the test of number of runs. The preceding table brings about various features such as length and composition of sequences (f_1, f_0), numbers of runs (r), signs of the differences $r - E(R)$, probabilities of exceeding expected numbers of runs (i. e. probabilities of plus sign) taken from the appropriate sections of our tables $P(R > E(R))$, and finally, sampling variances of plus sign (PQ).

The expected number of occurences of plus sign in the whole series is 19.033537. Te actual number is 5. The normalized deviate test gives:

$$u = \frac{5 - 19.033537}{\sqrt{7.669538}} = -5{,}067$$

Conclusion: the sequences cannot be taken as random, for there is a highly significant deficiency in the number of runs for the whole series.

Streszczenie

Podane są tablice wartości oczekiwanych i dyspersji liczby ogniw w sekwencjach losowych alternatyw z podaniem prawdopodobieństw przekroczenia wartości oczekiwanych.

Резюме

Даются таблицы математических ожиданий и дисперсий числа звеньей в случайных последовательностях альтернатив с приведением вероятностей превышения математических ожиданий.

TABLES OF EXPECTED VALUES AND VARIANCES OF NUMBERS OF RUNS IN RANDOM SEQUENCES WITH PROBABILITIES OF EXCEEDING EXPECTED VALUES

Tablice wartości oczekiwanych i dyspersji liczby ogniw w sekwencjach losowych z podaniem prawdopodobieństw przekroczenia wartości oczekiwanych

Таблицы математических ожиданий и дисперсий числа звеньей в случайных последовательностях альтернатив с приведением вероятностей превышения математических ожиданий.

$f_{\min} = 1$

$f_{\max}$	$E(R)$	$P[R > E(R)]$	$D^2(R)$	$D^2\left(\frac{R}{n}\right)$
1	2.000000	0.000000	0.000000	0.000000
2	2.333333	0.333333	0.222222	0.024691
3	2.500000	0.500000	0.250000	0.015625
4	2.600000	0.600000	0.240000	0.009600
5	2.666667	0.666667	0.222222	0.006173
6	2.714286	0.714286	0.204082	0.004165
7	2.750000	0.750000	0.187500	0.002930
8	2.777778	0.777778	0.172840	0.002134
9	2.800000	0.800000	0.160000	0.001600
10	2.818182	0.818182	0.148760	0.001229
11	2.833333	0.833333	0.138889	0.0009645
12	2.846154	0.846154	0.130178	0.0007702
13	2.857143	0.857143	0.122449	0.0006247
14	2.866667	0.866667	0.115556	0.0005136
15	2.875000	0.875000	0.109375	0.0004272
16	2,882353	0.882353	0.103806	0.0003592
17	2.888889	0.888889	0.098765	0.0003048
18	2.894737	0.894737	0.094183	0.0002609
19	2.900000	0.900000	0.090000	0.0002250
20	2.904762	0.904762	0.086168	0.0001954

$f_{\min} = 2$

$f_{\max}$	$E(R)$	$P[R > E(R)]$	$D^2(R)$	$D^2\left(\frac{R}{n}\right)$
2	3.000000	0.333333	0.666667	0.041667
3	3.400000	0.500000	0.840000	0.033600
4	3.666667	0.600000	0.888889	0.024691
5	3.857143	0.666667	0.884354	0.018048
6	4.000000	0.357143	0.857143	0.013393
7	4.111111	0.416667	0.820988	0.010136
8	4.200000	0.466667	0.782222	0.0078222
9	4.272727	0.509091	0.743802	0.0061471
10	4.333333	0.545454	0.707071	0.0049102
11	4.384615	0.576923	0.663041	0.0039233
12	4.428571	0.604396	0.640502	0.0032677
13	4.466667	0.628571	0.610794	0.0027146
14	4.500000	0.650000	0.583333	0.0022786
15	4.529412	0.669118	0.557959	0.0019307
16	4.555555	0.686274	0.534495	0.0016497
17	4.578947	0.701754	0.512773	0.0014204
18	4.600000	0.715790	0.492632	0.0012316
19	4.619048	0.728571	0.473923	0.0010747
20	4.636364	0.740260	0.456513	0.0009432

$f_{min} = 3$

f_{max}	$E(R)$	$P[R > E(R)]$	$D^2(R)$	$D^2\left(\frac{R}{n}\right)$
3	4.000000	0.300000	1.200000	0.033333
4	4.428571	0.457143	1.387750	0.028321
5	4.750000	0.571429	1.473210	0.023019
6	5.000000	0.357143	1.500000	0.018519
7	5.200000	0.416667	1.493333	0.014933
8	5.363636	0.466667	1.467768	0.012130
9	5.500000	0.509091	1.431818	0.0099431
10	5.615385	0.545454	1.390533	0.0082280
11	5.714286	0.576923	1.346939	0.0068721
12	5.800000	0.604396	1.302857	0.0057905
13	5.875000	0.628571	1.259750	0.0049209
14	5.941176	0.650000	1.217128	0.0042115
15	6.000000	0.446078	1.176471	0.0036311
16	6.052636	0.469556	1.137583	0.0031512
17	6.100000	0.491228	1.100526	0.0027513
18	6.142857	0.511278	1.065306	0,0024154
19	6.181818	0.529870	1.031877	0.0021320
20	6.212121	0.547148	0.997913	0.0018864

$f_{min} = 4$

f_{max}	$E(R)$	$P[R > E(R)]$	$D^2(R)$	$D^2\left(\frac{R}{n}\right)$
4	5.000000	0.371429	1.714286	0.026786
5	5.444444	0.500000	1.913580	0.023624
6	5.800000	0.595238	2.026667	0.020267
7	6.090909	0.393939	2.082645	0.017212
8	6.333333	0.466667	2.101010	0.014590
9	6.538462	0.528671	2.094675	0.012394
10	6.714286	0.581419	2.072214	0.010573
11	6.866667	0.626374	2.039365	0.0090638
12	7.000000	0.362637	2.000000	0.0078125
13	7.117647	0.392857	1,956747	0.0067706
14	7.222222	0.420588	1.911401	0.0058994
15	7.313789	0.446078	1.865189	0.0051667
16	7.400000	0.469556	1.818947	0.0045474
17	7.476190	0.491228	1.773242	0.0040210
18	7.545455	0.511278	1.728455	0.0035712
19	7.608696	0.529870	1.684826	0.0031849
20	7.666667	0.547148	1.642512	0.0028516

$f_{min} = 5$

f_{max}	$E(R)$	$P[R > E(R)]$	$D^2(R)$	$D^2\left(\frac{R}{n}\right)$
5	6.000000	0.357143	2.222222	0.022222
6	6.454545	0.478355	2.429752	0.020081
7	6.833333	0.575758	2.563131	0.017799
8	7.153846	0.424242	2.642998	0.015639
9	7.428571	0.489510	2.684458	0.013696
10	7.666667	0.545455	2.698413	0.011993
11	7.875000	0.593407	2.692708	0.010518
12	8.058824	0.421299	2.673011	0.0092492
13	8.222222	0.464753	2.643428	0.0081587
14	8.368421	0.504214	2.606956	0.0072215
15	8.500000	0.539990	2.565789	0.0064145
16	8.619048	0.572411	2.518862	0.0057117
17	8.727273	0.601808	2.475403	0.0051145
18	8.826087	0.628488	2.428252	0.0045903
19	8.916667	0.652739	2.380737	0.0041332
20	9.000000	0.364766	2.333333	0.0037333

$f_{min} = 6$

f_{max}	$E(R)$	$P[R > E(R)]$	$D^2(R)$	$D^2\left(\frac{R}{n}\right)$
6	7.000000	0.391775	2.727273	0,018939
7	7.461538	0.500000	2.940828	0.017401
8	7.857143	0.587413	3.089482	0.015763
9	8.200000	0.433566	3.188571	0.014171
10	8.500000	0.503496	3.250000	0.012695
11	8.705882	0.564318	3.229671	0.011175
12	9.000000	0.394635	3.294118	0.010167
13	9.210526	0.439076	3.289012	0.0091108
14	9.400000	0.479618	3.271579	0.0081789
15	9.571429	0.516512	3.244898	0.0073580
16	9.727273	0.550052	3.211334	0.0066350
17	9.869565	0.580542	3.172710	0.0059976
18	10.000000	0.431454	3.130435	0.0054348
19	10.120000	0.460757	3.085600	0.0049370
20	10.230769	0.488225	3.039053	0.0044956

$f_{min} = 7$

f_{max}	$E(R)$	$P[R > E(R)]$	$D^2(R)$	$D^2\left(\frac{R}{n}\right)$
7	8.000000	0.383450	3.230769	0.016484
8	8.466667	0.486402	3.448889	0.015328
9	8.875000	0.573427	3.609750	0.014101
10	9.235294	0.451049	3.724048	0.012886
11	9.555556	0.515837	3.802470	0.011736
12	9.942105	0.572398	3.945508	0.010929
13	10.100000	0.429954	3.879475	0.0096987
14	10.333333	0.479618	3.888889	0.0088183
15	10.545455	0.524897	3.884299	0.0080254
16	10.739130	0.565960	3.868706	0.0073132
17	10.916667	0.603079	3.844505	0.0066745
18	11.008000	0.433772	3.756336	0.0060101
19	11.230769	0.443478	3.777515	0.0055880
20	11.370370	0.471390	3.737469	0.0051268

$f_{min} = 8$

f_{max}	$E(R)$	$P[R > E(R)]$	$D^2(R)$	$D^2\left(\frac{R}{n}\right)$
8	9.000000	0.404817	3.733338	0.014583
9	9.470588	0.500000	3.955017	0.013685
10	9.888889	0.580625	4.142909	0.012787
11	10.263158	0.453322	4.252385	0.011779
12	10.600000	0.520005	4.345263	0.010863
13	10.990476	0.578947	4.490957	0.010184
14	11.181818	0.442724	4.451790	0.0091979
15	11.434782	0.489680	4.474995	0.0084593
16	11.666667	0.532553	4.483092	0.0077831
17	11.880000	0.571528	4.478933	0.0071663
18	12.076923	0.440508	4.464852	0.0066048
19	12.259259	0.576765	4.442756	0.0060934
20	12.428571	0.510672	4.414210	0.0056304

$f_{min} = 9$

f_{max}	$E(R)$	$P[R > E(R)]$	$D^2(R)$	$D^2\left(\frac{R}{n}\right)$
9	10.000000	0.399218	4.235294	0.013072
10	10.473684	0.490452	4.459834	0.012354
11	10.900000	0.570017	4.637368	0.011593
12	11.285714	0.464992	4.775510	0.010829
13	11.636364	0.527864	4.880756	0.010084
14	11.956522	0.583591	4.958584	0.009374
15	12.250000	0.461098	5.013587	0.008704
16	12.520000	0.510830	5.049750	0.008080
17	12.769231	0.556305	5.069823	0.0074997
18	13.000000	0.412781	5.076923	0.0069642
19	13.214285	0.449750	5.073129	0.0064708
20	13.413793	0.484484	5.060302	0.0060170

$f_{min} = 10$

f_{max}	$E(R)$	$P[R > E(R)]$	$D^2(R)$	$D^2\left(\frac{R}{n}\right)$
10	11.000000	0.414070	4.736842	0.011842
11	11.476190	0.500000	4.963718	0.011256
12	11.909091	0.575018	5.147580	0.010635
13	12.304348	0.414867	5.294724	0.010009
14	12.666667	0.528470	5.410628	0.0093935
15	13.000000	0.418226	5.500000	0.0088000
16	13.307692	0.469673	5.566864	0.0082350
17	13.592593	0.517162	5.614925	0.0077022
18	13.857143	0.560655	5.646259	0.0072019
19	14.103448	0.444560	5.664175	0.0067350
20	14.333333	0.484484	5.670498	0.0063055

$f_{min} = 11$

f_{max}	$E(R)$	$P[R > E(R)]$	$D^2(R)$	$D^2\left(\frac{R}{n}\right)$
11	12.000000	0,409979	5.238095	0.010823
12	12.478261	0.492824	5.466919	0.010334
13	12.916667	0.566531	5.656099	0.0098196
14	13.320000	0.473348	5.810933	0.0092975
15	13.692308	0.533982	5,936095	0.0087812
16	14.037037	0.427167	6.035665	0.0082794
17	14.357143	0.480142	6.113190	0.0077974
18	14.655172	0.529241	6.171734	0.0073386
19	14.933333	0.574313	6.213946	0.0069044
20	15.193548	0.451974	6.242109	0.0064954

$f_{min} = 12$

f_{max}	$E(R)$	$P[R > E(R)]$	$D^2(R)$	$D^2\left(\frac{R}{n}\right)$
12	13.000000	0.421068	5.739130	0.0099638
13	13.480000	0.500000	5.969750	0.0095516
14	13.923077	0.570362	6.163314	0.0091173
15	14.333333	0.472311	6.324786	0.0086760
16	14.714286	0.533118	6.458050	0.0082373
17	15.068966	0.435393	6.566673	0.0078082
18	15.400000	0.486315	6.653793	0.0073931
19	15.709677	0.533533	6.722164	0.0069950
20	16.000000	0.429658	6.774194	0.0066154

$f_{min} = 13$

f_{max}	$E(R)$	$P[R > E(R)]$	$D^2(R)$	$D^2\left(\frac{R}{n}\right)$
13	14.000000	0.417911	6.240000	0.0092308
14	14.481481	0.494354	6.471879	0.0088777
15	14.928571	0.563521	6.669501	0.0085070
16	15.344828	0.478826	6.836759	0.0081293
17	15.733333	0.537232	6.977164	0.0077524
18	16.096774	0.440837	7.093860	0.0073817
19	16.437500	0.492603	7.189642	0.0070211
20	16.757576	0.540792	7.266988	0.0066731

$f_{min} = 14$

f_{max}	$E(R)$	$P[R > E(R)]$	$D^2(R)$	$D^2\left(\frac{R}{n}\right)$
14	15.000000	0.426598	6.740741	0.0085979
15	15.482759	0.500000	6.973841	0.0082923
16	15.933333	0.566442	7.174865	0.0079721
17	16.354839	0.477444	7.347208	0.0076454
18	16.750000	0.535742	7.493952	0.0073183
19	17.121212	0.447255	7.617883	0.0069946
20	17.470588	0.497276	7.721505	0.0066795

$f_{min} = 15$

f_{max}	$E(R)$	$P[R > E(R)]$	$D^2(R)$	$D^2\left(\frac{R}{n}\right)$
15	16.000000	0.424066	7.241379	0.0080460
16	16.483870	0.495407	7.475545	0.0077789
17	16.937500	0.560726	7.679561	0.0074996
18	17.363636	0.482650	7.856405	0.0072143
19	17.764706	0.538960	8.008808	0.0069280
20	18.142857	0.450688	8.139256	0.0066443

$f_{min} = 16$

f_{max}	$E(R)$	$P[R > E(R)]$	$D^2(R)$	$D^2\left(\frac{R}{n}\right)$
16	17.000000	0.431109	7.741935	0.0075605
17	17.484848	0.500000	7.977043	0.0073251
18	17.941176	0.563092	8.183705	0.0070793
19	18.371429	0.481162	8.364562	0.0068282
20	18.777778	0.537208	8.522046	0.0065757

$f_{min} = 17$

f_{max}	$E(R)$	$P[R > E(R)]$	$D^2(R)$	$D^2\left(\frac{R}{n}\right)$
17	18.000000	0.429021	8.242424	0.0071301
18	18.485714	0.496169	8.478367	0.0069211
19	18.944444	0.558225	8.687389	0.0067032
20	19.378378	0.485444	8.871844	0.0064805

$f_{min} = 18$

f_{max}	$E(R)$	$P[R > E(R)]$	$D^2(R)$	$D^2\left(\frac{R}{n}\right)$
18	19.000000	0.434880	8.742854	0.0067460
19	19.486486	0.500000	8.979547	0.0065592
20	19.947368	0.560192	9.190686	0.0063647

$f_{min} = 19$

f_{max}	$E(R)$	$P[R > E(R)]$	$D^2(R)$	$D^2\left(\frac{R}{n}\right)$
19	20.000000	0.433120	9.243243	0.0063994
20	20.487179	0.496742	9.480604	0.0062367

$f_{min} = 20$

f_{max}	$E(R)$	$P[R > E(R)]$	$D^2(R)$	$D^2\left(\frac{R}{n}\right)$
20	21.000000	0.438093	9.743590	0.0060897