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On Some Classes of Functions Regular in a Half Plane

Abstract. The object of this paper is to present some results concerning functions regular in a half plane and having a special normalization at infinity. The conditions for starlikeness, convexity and convexity in the direction of the real axis are given. Some extremal problems for such classes are investigated.

1. Introduction

Let D denote the right half plane

$$D = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$$

and let $\widetilde{H} = \widetilde{H}(D)$ denote the class of functions f which are regular in D and have the so-called hydrodynamic normalization (see e.g. [1-5])

$$\lim_{z\to\infty} (f(z) - z) = 0 \ , \quad z \in D \ .$$

We denote by $H = H(D)$ the class of functions F which are regular in D and have the following normalization

$$\lim_{z\to\infty} (f(z) - z) = a \ , \quad z \in D \ ,$$

where a is an arbitrary fixed complex number such that $\operatorname{Re} a \geq 0$.

Next, we denote by $\widetilde{P} = \widetilde{P}(D)$ and $P = P(D)$ the subclasses of $\widetilde{H}$ and H, respectively, such that

$$\operatorname{Re} p(z) > 0 \ \text{ for } \ z \in D \ .$$

The functions (not necessary univalent) the classes $\widetilde{P}$ and P map the half plane D into itself and have the corresponding normalizations (near the point at infinity they are close to identity, $p(z) \cong z$ or $p(z) \cong z + a$). The class P is an analogoue of the familiar Carathéodory class of functions with positive real part. This class has also analogous properties.

Theorem 1. *If* $p \in P$ *then*

$$\text{Re}\, p(z) \geq \text{Re}\, z \quad \text{for} \quad z \in D. \tag{1.1}$$

Remark 1. It is easy to check that the estimate (1.1) is sharp. The extremal functions have the form

$$p_t(z) = z + it\ , \ t \in \mathbb{R}\ .$$

2. The class of functions convex in the direction of the real axis

Definition 1. A domain $B \subset \mathbb{C}$ is called convex in the direction of the real axis if the intersection of B and any straight line parallel to the real axis is connected.

Definition 2. A function $f \in H$ is called convex in the direction of the real axis if it maps the half plane D conformally onto a domain $f(D)$ which is convex in the direction of the real axis. The set of all such functions is denoted by $R(D)$.

Theorem 2. *If* $f \in R(D)$ *then for every* $s \geq 0$ *the domain* $F(D_s)$ *is convex in the direction of the real axis.*

Now we give an analytic condition for the convexity in the direction of the real axis:

Theorem 3. *A function* $f \in H$ *belongs to the class* $R(D)$ *if and only if*

$$\text{Re}\, f'(z) > 0 \quad \text{for} \quad z \in D\ .$$

Remark 2. The class of functions of bounded rotation ($\mathrm{Re} f'(z) > 0$) in the half plane D and the class $R(D)$ of functions convex in the direction of the real axis coincide.

3. The class of convex functions in a half plane

Definition 3. A function $f \in H$ is called convex if it is univalent in D and maps D conformally onto a convex domain $f(D)$. Such a class of functions is denoted by C or $C(D)$. .

It is easy to observe that $f(D)$ is convex domain if and only if for every $z \in D$, $x \in (-\infty, +\infty)$ we have

$$[f(z - ix) + f(z + ix)]/2 \in f(D) .$$

Analogously, $f(D_s)$ will be convex domain if and only if

$$[f(z - ix) + f(z + ix)]/2 \in f(D_s) \text{ for } z \in D_s,\ x \in (-\infty, +\infty) .$$

Theorem 4. *If $f \in C(D)$ then for every $s > 0$ the domain $f(D_s)$ is convex.*

Theorem 5. *A function $f \in H$ belongs to the class $C(D)$ if and only if*

$$\mathrm{Re}\,\frac{f''(z)}{f'(z)} < 0 \quad \textit{for} \quad z \in D .$$

4. The class of functions starlike in a half plane

For every $f \in H$ and $s \geq 0$ the point ∞ is a boundary point of $f(D_s)$ which means that the domains $f(D_s)$ are unbounded.

Definition 4. A function $f \in H$ will be called starlike (with respect to the origin) if f is univalent in D and maps D onto a domain $f(D)$, $0 \notin f(D)$ which is starlike with respect to the origin. The class of starlike functions is denoted by $S^*(D)$ or S^*.

Theorem 6. *If $f \in S^*(D)$ then for every $s > 0$ the domain $f(D_s)$ is starlike (with respect to the origin).*

Remark 3. It is easy to observe that a Jordan domain G ($0 \notin G$, $\infty \in \partial G$) is starlike (with respect to the origin) if and only if the argument of $w \in \partial G$ changes monotonically, as w moves on ∂G. Such curves ∂G will be called starlike.

Theorem 7. *A function $f \in H$, $f(z) \neq 0$ for $z \in D$, belongs to the class $S^*(D)$ if and only if*

$$\operatorname{Re} \frac{f'(z)}{f(z)} > 0 \quad \textit{for } z \in D.$$

5. Some integral formulae for the class of starlike functions in a half plane

Let $Q = Q(D)$ denote the class of functions $q(z)$ which are regular in D and satisfy the condition

$$\lim_{z \to \infty} z(q(z) - 1/z) = 0\,, \quad \operatorname{Re} q(z) > 0 \text{ for } z \in D.$$

It is easy to observe that

$$q(z) \in Q(D) \Longleftrightarrow p(z) := \frac{1}{q(z)} \in P(D)$$

$$f \in S^*(D) \Longleftrightarrow q(z) = \frac{f'(z)}{f(z)} \in Q(D).$$

Using these relations we obtain

Theorem 8. *If $q \in Q$, then*

$$|q(z) - 1/(2\operatorname{Re} z)| \leq 1/(2\operatorname{Re} z) \quad \textit{for } z \in D$$

and in particular

$$|\operatorname{Im} q(z)| \leq 1/(2\operatorname{Re} z) \quad \textit{for } z \in D$$

$$\operatorname{Re} q(z) \leq |q(z)| \leq 1/\operatorname{Re} z \quad \textit{for } z \in D.$$

These results are sharp. The extremal functions have the form

$$q_t(z) = \frac{1}{z+it}, \quad t \in \mathbb{R}.$$

Theorem 9. *A function f of the class $H(D)$ non–vanishing in D, belongs to the class $S^*(D)$ if and only if it may be written in the form*

$$f(z) = z \exp\left\{\int_{\infty}^{z} (q(\zeta) - 1/\zeta)\, d\zeta\right\}, \quad q \in Q(D).$$

Theorem 10. *A function f of the class $H(D)$ non–vanishing in D, belongs to the class $S^*(D)$ if and only if it may be written in the form*

$$f(z) = f(z_0) \int_{z_0}^{z} q(\zeta)\, d\zeta$$

for some arbitrary $z_0 \in D$ and $q \in Q(D)$.

6. Some estimates for the class of starlike functions

Using the integral formula and the above estimates we can obtain some estimates for the class $S^*(D)$ (see [3]).

Theorem 11. *Let $z_0 \in D$ be fixed and let $f \in S^*(D)$. Then for every $z \in D$*

$$\left(\frac{\operatorname{Re} z}{\operatorname{Re} z_0}\right)^{(1-|z-z_0|/\operatorname{Re}(z-z_0))/2} \le \left|\frac{f(z)}{f(z_0)}\right| \le \left(\frac{\operatorname{Re} z}{\operatorname{Re} z_0}\right)^{(1+|z-z_0|/\operatorname{Re}(z-z_0))/2},$$

if $\operatorname{Re}(z - z_0) \neq 0$,

$$\exp\frac{-|\operatorname{Im}(z-z_0)|}{2\operatorname{Re} z_0} \le \left|\frac{f(z)}{f(z_0)}\right| \le \exp\frac{|\operatorname{Im}(z-z_0)|}{2\operatorname{Re} z_0},$$

if $\operatorname{Re}(z-z_0)=0$).

To prove this theorem we need the following lemma

Lemma 1. *Let* $q\in Q(D)$. *Then for every* $z,\ z_0,\ \zeta\in D$

$$\frac{\operatorname{Re}(z-z_0)-|z-z_0|}{2\operatorname{Re}\zeta}\le \operatorname{Re}((z-z_0)q(\zeta))\le\frac{\operatorname{Re}(z-z_0)+|z-z_0|}{2\operatorname{Re}\zeta},$$

$$\frac{\operatorname{Im}(z-z_0)-|z-z_0|}{2\operatorname{Re}\zeta}\le \operatorname{Im}((z-z_0)q(\zeta))\le\frac{\operatorname{Im}(z-z_0)+|z-z_0|}{2\operatorname{Re}\zeta}.$$

This Lemma is an immediate consequence of Theorem 8.

Remark 4. If $\operatorname{Im} z=\operatorname{Im} z_0$ and $\operatorname{Re} z>\operatorname{Re} z_0$, then for $f\in S^*(D)$ we have

$$1\le\left|\frac{f(z)}{f(z_0)}\right|\le\frac{\operatorname{Re} z}{\operatorname{Re} z_0}.$$

This estimate is sharp for $f(z)\equiv z,\ \operatorname{Im} z=\operatorname{Im} z_0=0$.

Theorem 12. *Let* $z_0\in D$ *be fixed and let* $f\in S^*(D)$. *Then for every* $z\in D$ *we have*

$$\arg\frac{f(z)}{f(z_0)}\ge\begin{cases}\frac{\operatorname{Im}(z-z_0)-|z-z_0|}{2\operatorname{Re}(z-z_0)}\log\frac{\operatorname{Re} z}{\operatorname{Re} z_0}, & \text{if } \operatorname{Re}(z-z_0)\ne 0\\ \frac{\operatorname{Im}(z-z_0)-|z-z_0|}{2\operatorname{Re} z_0}, & \text{if } \operatorname{Re}(z-z_0)=0,\end{cases}$$

and

$$\arg\frac{f(z)}{f(z_0)}\le\begin{cases}\frac{\operatorname{Im}(z-z_0)+|z-z_0|}{2\operatorname{Re}(z-z_0)}\log\frac{\operatorname{Re} z}{\operatorname{Re} z_0}, & \text{if } \operatorname{Re}(z-z_0)\ne 0\\ \frac{\operatorname{Im}(z-z_0)+|z-z_0|}{2\operatorname{Re} z_0}, & \text{if } \operatorname{Re}(z-z_0)=0,\end{cases}$$

where we choose the branch of $\log(f(z)/f(z_0))$ *which is zero for* $z=z_0$.

Remark 5. For $\operatorname{Re}(z-z_0)=0$, $\operatorname{Im} z>\operatorname{Im} z_0$ we have $\arg[f(z)/f(z_0)]>0$ which means that $\arg f(z_0+it)$ is increasing with respect to t. This agrees with the definition of a starlike function in the half plane D .

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