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# Examples of a Pointwise Convergence of Semigroups**

ABSTRACT. This article is a continuation of the paper [2]. It contains further examples of families of semigroups approximating semigroups strongly continuous only on a subspace of the original Banach space.

**1. Introduction - the theorem of Trotter and Kato.** Let $\{T_\epsilon(t), t \geq 0\}$, $0 < \epsilon < 1$, be a family of strongly continuous semigroups acting in a Banach space $L$ and let us assume that there exists a positive constant $M$ such that $\|T_\epsilon(t)\|_{\mathcal{L}(L,L)} \leq M$. The following theorem establishes relation between convergence of the semigroups and resolvents of their generators.

**Theorem 1 (Trotter - Kato).** *The limit*

$$\lim_{\epsilon \to 0} T_\epsilon(t) f \tag{1.1}$$

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*exists almost uniformly in* $[0,\infty)$ *for all* $f \in L$ *iff there exists* $\lambda > 0$ *such that the following two conditions hold simultaneously*

(a) $\lim_{\epsilon\to 0} R_\lambda(A_\epsilon)f$ *exists for all* $f \in L$,

(b) *the set* $\{g : \exists f, g = \lim_{\epsilon\to 0} R_\lambda(A_\epsilon)f\}$ *is dense in* $L$,

*where* $R_\lambda(A_\epsilon) = (\lambda - A_\epsilon)^{-1}$ *is the resolvent of the infinitesimal generator* $A_\epsilon$ *of the semigroup* $\{T_\epsilon(t), t \geq 0\}$, $0 < \epsilon < 1$.

A natural question arises of what can be said if the condition (b) is removed. The following theorem that is similar to that presented in [6; p.80, Th. 3.17] gives an answer to this question.

**Theorem 2.** *Let, as before,* $\{T_\epsilon(t), t \geq 0\}$, $0 < \epsilon < 1$, *be a family of equibounded semigroups. If the condition (a) of the previous theorem is satisfied then the convergence* (1.1) *takes place for all* $f \in L_0$, *where* $L_0$ *is the closure of the set* $\{g : \exists f, g = \lim_{\epsilon\to 0} R_\lambda(A_\epsilon)f\}$.

Obviously, (1.1) can be applied to define the semigroup $\{T_0(t), t \geq 0\}$ of bounded operators acting in $L_0$.

The above theorem can be easily proved by using well known arguments, for example those presented in [10], [13], [15]. In what follows, however, we would like to present the proof that exhibits the relationship with the Ray version of the Hille - Yosida theorem. This idea is due to J.Kisyński.

Let us first recall necessary definitions:

**Definition 1.** Let $E$ be a Banach space. A map $\mathbf{R}^+ \ni \lambda \to R_\lambda \in \mathcal{L}(E,E)$ from $\mathbf{R}^+$ into an algebra of bounded operators $\mathcal{L}(E,E)$, which satisfies the Hilbert equation

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu, \quad \text{for all} \quad \mu, \lambda \in \mathbf{R}^+ ,$$

is called a resolvent. Moreover, if the operators $\lambda R_\lambda$ are contractions i.e. $\|\lambda R_\lambda\|_{\mathcal{L}(E,E)} \leq 1$ then it is called contraction resolvent.

Let us note that, by the Hilbert equation, the set Range $(R_\lambda)$ is independent of $\lambda$.

**Definition 2.** The regularity space of the contraction resolvent $(R_\lambda)$ is the subspace $R = \{f \in E : \lim_{\lambda\to\infty} \lambda R_\lambda f = f\}$.

The condition $\lim_{\lambda\to\infty}\|R_\lambda\| = 0$ together with the Hilbert equation implies that for $f \in E, \lambda, \mu > 0$

$$\lim_{\lambda\to\infty} \lambda R_\lambda R_\mu f = \lim_{\lambda\to\infty}((\lambda-\mu)R_\lambda R_\mu f + \mu R_\lambda R_\mu f)$$
$$= \lim_{\lambda\to\infty}(R_\mu f - R_\lambda f + \mu R_\lambda R_\mu f) = R_\mu f.$$

This means that $R_\mu E \subset R$ and that $R$ is an invariant set for the operators $R_\mu, \mu > 0$. Furthermore it is easy to see that Definition 2 implies $R \subset \mathrm{cl}\,(\mathrm{Range}(R_\lambda))$. Thus $R = \mathrm{cl}\,(\mathrm{Range}(R_\lambda))$.

The following is a generalization of the Hille-Yosida-type theorem presented in [7; p.311]. Although it is a very simple proposition, it turns out to be a useful tool in dealing with the problem of convergence of semigroups.

**Theorem 3.** *For any contraction resolvent $(R_\lambda)_{\lambda>0}$ in a Banach space E there exists the unique strongly continuous contraction semigroup $\{S(t):\ t \geq 0\}$ acting in a regularity space $R$ such that*

$$R_\lambda f = \int_0^\infty e^{-\lambda t} S(t) f dt \quad \textit{for every} \quad \lambda > 0\,,\ f \in R.$$

**Proof.** The mapping $\mathbf{R}^+ \ni \lambda \to R'_\lambda = (R_\lambda)_{|R}$ from $\mathbf{R}^+$ to $\mathcal{L}(R,R)$ is a regular contraction resolvent in $R$, i.e., $\lim_{\lambda\to\infty}\lambda R'_\lambda f = f$ for all $f \in R$. Thus, in order to complete the proof it is enough to apply the theorem presented in [7; p.311], which states that every regular contraction resolvent in a Banach space is the Laplace transform of a unique contraction semigroup. ∎

**Proof of Theorem 2.** Let $(\epsilon_n)_{n\geq 1}$ be a sequence of positive numbers and let $\lim_{n\to\infty}\epsilon_n = 0$. Let us define the Banach space $\mathcal{E} = \{(f_n)_{n\geq 1} : f_n \in L, (f_n)_{n\geq 1}$ is convergent$\}$, with the norm $|||(f_n)_{n\geq 1}||| = \sup_{n\geq 1}\|f_n\|$. Let us note that the Hilbert equation implies that, if the condition (a) is satisfied for one chosen $\lambda$, then it holds for any $\lambda > 0$, too. Thus, if the condition (a) of Theorem 1 is satisfied, then the formula

$$\lambda \to \mathcal{R}_\lambda[(f_n)_{n\geq 1}] = (R_\lambda(A_{\epsilon_n})f_n)_{n\geq 1}$$

defines a resolvent in the space $\mathcal{E}$. Let us define another norm in this space, by setting

$$\|(f_n)_{n\geq 1}\|_* = \sup_{n\geq 1}\left(\sup_{t\geq 0}\|T_{\epsilon_n}(t)f_n\|\right).$$

Since the semigroups $\{T_\epsilon(t), t\geq 0\}$, $0<\epsilon<1$, are equibounded, the norms $\|\cdot\|_*$, $|||\cdot|||$ are equivalent:

$$|||(f_n)_{n\geq 1}||| \leq \|(f_n)_{n\geq 1}\|_* \leq \sup_{n\geq 1}(M\|f_n\|) = M|||(f_n)_{n\geq 1}|||.$$

Furthermore, the map $\lambda \to \mathcal{R}_\lambda$ is a contraction resolvent:

$$\begin{aligned}\|\lambda R_\lambda\left[(f_n)_{n\geq 1}\right]\|_* &= \|\left(\lambda R_\lambda(A_{\epsilon_n})f_n\right)_{n\geq 1}\|_*\\ &= \sup_{n\geq 1}\left(\sup_{t\geq 0}\|T_{\epsilon_n}(t)\lambda R_\lambda(A_{\epsilon_n})f_n\|\right)\\ &= \sup_{n\geq 1}\left(\sup_{t\geq 0}\|T_{\epsilon_n}(t)\lambda\int_0^\infty e^{-\lambda s}T_{\epsilon_n}(s)f_n ds\|\right)\\ &= \sup_{n\geq 1}\left(\sup_{t\geq 0}\|\lambda\int_0^\infty e^{-\lambda s}T_{\epsilon_n}(s+t)f_n ds\|\right)\\ &\leq \sup_{n\geq 1}\left(\sup_{t\geq 0}\|T_{\epsilon_n}(t)f_n\|\lambda\int_0^\infty e^{-\lambda s}ds\right)\\ &= \|(f_n)_{n\geq 1}\|_*.\end{aligned}$$

Let us set $L_0 = \mathrm{cl}\{g : \exists f, g = \lim_{\epsilon\to 0} R_\lambda(A_\epsilon)f\}$. The closure of the set $\mathrm{Range}(\mathcal{R}_\lambda)$ (in the norm $|||\cdot|||$, and thus in the norm $\|\cdot\|_*$,) is equal to

$$\{(f_n)_{n\geq 1} : \exists f\in L_0, \lim_{n\to\infty} f_n = f\}.$$

By Theorem 3, there exists the unique semigroup $\{S(t) : \ t\geq 0\}$ acting in this subspace, such that

$$\mathcal{R}_\lambda\left[(f_n)_{n\geq 1}\right] = \int_0^\infty e^{-\lambda t}S(t)\left[(f_n)_{n\geq 1}\right]dt$$

for all $(f_n)_{n\geq 1}\in\{(f_n)_{n\geq 1} : \exists f\in L_0, \lim_{n\to\infty} f_n = f\}$. By using the very definition of $R_\lambda$ and the invertibility of the Laplace transform for continuous functions one can easily prove that $S(t)[(f_n)_{n\geq 1}] = (T_{\epsilon_n}(t)f_n)_{n\geq 1}$. Thus, in particular, for any $f \in L_0$ the sequence $(T_{\epsilon_n}(t)f)_{n\geq 1}$ is convergent. Since the semigroup $\{S(t) : t \geq 0\}$ is strongly continuous, convergence is almost uniform and the theorem is proved. ■

Another interesting question arises of what can be said of convergence for $f \notin L_0$. In the paper [2] suitable examples were presented showing that in general the limit in (1.1) may not exist in a strong or even in a weak topology. However, it was proved by T.G.Kurtz [15; Prop.(2-22), p.29], that the limit $\lim_{\epsilon\to 0} \int_0^t T_\epsilon(s)f ds$ does exist for all $f \in L$. An explanation of this phenomenon, based on the theory of so-called "integrated semigroups" (see [1]), may be found in [3].

On the other hand, let us note that if (a) holds, the condition $f \in L_0$ is necessary and sufficient for the convergence in (1.1) to be almost uniform in $[0, \infty)$ and, as it turns out, it may happen that (1.1) holds even for $f \notin L_0$, the convergence being only pointwise. An equivalent setting is that the Trotter-Kato theorem concerns strongly continuous semigroups, yet it may happen that strongly continuous semigroups approximate (in the sense of (1.1)) a semigroup that is strongly continuous only on a certain subspace of the original Banach space.

Examples of such a behaviour of semigroups were presented in [2]. To exhibit further ones is the aim of this article.

**2. Diffusion equation.** As noted by J.Kisyński there is a simple case, when the convergence of semigroups is implied by the convergence of their resolvents, even while the set $\{f : f = \lim_{\epsilon\to 0} R^\epsilon_\lambda g, g \in L\}$ is not dense in $L$. This is the case when the semigroups are "uniformly analytic". (For the definition of the analytic semigroup see for example [12; p.488]). Namely, the following theorem was proved in [2].

**Theorem 4.** *Let $L$ be a complex Banach space and $\{T_\epsilon(t), t \geq 0\}, 0 < \epsilon < 1$, be strongly continuous equibounded semigroups, ($\|T_\epsilon(t)\| \leq M$). Let us suppose that the resolvent set of the corresponding infinitesimal generators contains not only the half plane*

*$Re\lambda > 0$ but also a sector $|\arg\lambda| < \frac{\pi}{2} + \omega$, where $\omega > 0$ and for any $\delta > 0$,*

$$\|R_\lambda(A_\epsilon)\| \le \frac{M_\delta}{|\lambda|} \text{ for } |\lambda| \le \frac{\pi}{2} + \omega - \delta, \tag{2.1}$$

*with $\omega$ independent of $\epsilon$, and $M_\delta$ independent of $\lambda$. If, for all $\lambda$ such that $|\arg\lambda| < \frac{\pi}{2} + \omega$, and $f \in L$,*

$$\text{the limit } \lim_{\lambda\to\infty} R_\lambda(A_\epsilon)f \text{ exists,} \tag{2.2}$$

*then there exists a semigroup $\{T_0(t) : t \ge 0\}$ such that*

$$\lim_{\epsilon\to 0} T_\epsilon(t)f = T_0(t)f, \text{ for all } f \in L, t \ge 0. \tag{2.3}$$

*Furthermore, $T_0(t) \in \mathcal{L}(L_0, L_0)$ and $\{T_0(t)_{|L_0} : t \ge 0\}$ is a strongly continuous semigroup, where $L_0 = cl\{f \in L : f = \lim_{\epsilon\to 0} R_\lambda(A_\epsilon)g$, for some $g \in L\}$. (Note that $\lim_{t\to 0} T_0(t)f$ may not exist for $f \notin L_0$.)*

As an application of this theorem let us consider the following corollary. A similar example was presented in [2], but the calculations we present here are slightly more complicated. This example was also suggested by J.Kisyński.

**Corollary 1.** *Let $L = C_{[a,b]}$ be the space of all continuous functions $f : [a,b] \to C$ (a,b are given numbers). For any non-negative numbers $\mu, \nu$ let us define an operator $A_{\mu,\nu}$ by*

$$D(A_{\mu,\nu}) = \{f \in C^2_{[a,b]} : f(a) - \mu f'(a) = 0, f(b) + \nu f'(b) = 0\}$$

$$A_{\mu,\nu}f = \frac{1}{2}\frac{d^2 f}{dx^2}.$$

*The operators $A_{\mu,\nu}$ are infinitesimal generators of semigroups of linear operators $\{T_{\mu,\nu}(t), t \ge 0\}$ acting in $L$. The semigroup $\{T_{0,0}(t) : t \ge 0\}$ is strongly continuous only on the subspace $L_0 = \{f \in L : f(a) = f(b) = 0\}$, while the semigroups $\{T_{\mu,\nu}(t), t \ge 0\}$, $\mu, \nu > 0$, are strongly continuous on $L$. Furthermore, for $t \ge 0$, $f \in L$,*

$$\lim_{\mu,\nu\to 0} T_{\mu,\nu}(t)f = T_{0,0}(t)f.$$

**Proof.** The fact that the sets $D(A_{\mu,\nu})$, $\mu,\nu > 0$, are dense in $L$ can be proved in a similar way as in Proposition 2 in [2] (see [4], Proposition 2). We will prove that desired estimates concerning resolvent operators hold. To this end the resolvent operator will be expressed in the form

$$(2.4) \qquad R_\lambda(A_{\mu,\nu})g(x) = \frac{1}{\sqrt{2\lambda}}\int_{-\infty}^{\infty} e^{-\sqrt{2\lambda}|x-y|}g^*(y)dy$$

where the function $g^*$ is such that $\sup_{x\in R}|g^*(x)| = \|g\|_L$ ($g^*$ depends on $\mu,\nu$ but the subscript will be omitted in what follows). This formula will enable us to conclude that for $\lambda > 0$ and $\mu,\nu > 0$

$$\|R_\lambda(A_{\mu,\nu})g\| \le \frac{1}{\sqrt{2\lambda}}\int_{-\infty}^{\infty} e^{-\sqrt{2\lambda}|y|}dy \sup_{x\in R}|g^*(x)| = \frac{\|g\|}{\lambda},$$

which, by the Hille - Yosida theorem, proves that the operators $A_{\mu,\nu}$, $\mu,\nu > 0$, are generators of contraction semigroups. Furthermore, we have, for $|\arg\lambda| < \pi - \delta$,

$$\|R_\lambda(A_\epsilon)\| \le \frac{2}{|\sqrt{2\lambda}|}\frac{1}{\mathrm{Re}\sqrt{2\lambda}} \le \frac{1}{|\lambda|\cos(\frac{\pi}{2}-\frac{\delta}{2})} = \frac{1}{|\lambda|\sin\frac{\delta}{2}}$$

the last inequality following from $\frac{\mathrm{Re}\sqrt{2\lambda}}{|\sqrt{2\lambda}|} \ge \cos(\frac{\pi}{2}-\frac{\delta}{2})$. Thus, (2.1) will be proved with $M_\delta = 1/\sin\frac{\delta}{2}$, and, by Theorem 4, to complete the proof it will remain to justify (2.2).

Let us then prove (2.4).

To solve the resolvent equation for the operator $A_{\mu,\nu}$ let us note that the general solution to

$$\lambda f(x) - \frac{1}{2}f''(x) = g(x), \quad \text{for } x\in[a,b]$$

where $\arg\lambda \ne \pi$, may be written in the form

$$f(x) = f_{g,\lambda}(x) + C_1e^{-\sqrt{2\lambda}x} + C_2e^{\sqrt{2\lambda}x}$$

where $f_{g,\lambda}(x) = \frac{1}{\sqrt{2\lambda}}\int_a^b e^{-\sqrt{2\lambda}|x-y|}g(y)dy$. The boundary conditions $f(a) - \mu f'(a) = 0$, $f(b) + \nu f'(b) = 0$ lead to the system of equations which enables us to calculate constants $C_1, C_2$. We have namely,

$$C_1 e^{-\sqrt{2\lambda}a} + C_2 e^{\sqrt{2\lambda}a} + f_{g,\lambda}(a) = \mu\sqrt{2\lambda}\left(-C_1 e^{-\sqrt{2\lambda}a} + C_2 e^{\sqrt{2\lambda}a} + f_{g,\lambda}(a)\right),$$

$$C_1 e^{-\sqrt{2\lambda}b} + C_2 e^{\sqrt{2\lambda}b} + f_{g,\lambda}(b) = -\nu\sqrt{2\lambda}\left(-C_1 e^{-\sqrt{2\lambda}b} + C_2 e^{\sqrt{2\lambda}b} - f_{g,\lambda}(b)\right)$$

where we use $f'_{g,\lambda}(a) = \sqrt{2\lambda} f_{g,\lambda}(a)$, $f'_{g,\lambda}(b) = -\sqrt{2\lambda} f_{g,\lambda}(b)$. Thus

$$C_1 e^{-\sqrt{2\lambda}a}(1+\mu\sqrt{2\lambda}) + C_2 e^{\sqrt{2\lambda}a}(1-\mu\sqrt{2\lambda}) = (\mu\sqrt{2\lambda}-1) f_{g,\lambda}(a),$$

$$C_1 e^{-\sqrt{2\lambda}b}(1-\nu\sqrt{2\lambda}) + C_2 e^{\sqrt{2\lambda}b}(1+\nu\sqrt{2\lambda}) = (\nu\sqrt{2\lambda}-1) f_{g,\lambda}(b),$$

or, which is the same,

$$C_1 - H(\mu\sqrt{2\lambda}) e^{2\sqrt{2\lambda}a} C_2 = H(\mu\sqrt{2\lambda}) f_{g,\lambda}(a) e^{\sqrt{2\lambda}a},$$

$$-H(\nu\sqrt{2\lambda}) e^{-2\sqrt{2\lambda}b} C_1 + C_2 = H(\nu\sqrt{2\lambda}) f_{g,\lambda}(b) e^{-\sqrt{2\lambda}b}$$

where $H(z) = (z-1)/(z+1)$. Consequently,

$$C_1 = \frac{f_{g,\lambda}(a) H(\mu\sqrt{2\lambda}) e^{\sqrt{2\lambda}a} + f_{g,\lambda}(b) H(\mu\sqrt{2\lambda}) H(\nu\sqrt{2\lambda}) e^{\sqrt{2\lambda}(2a-b)}}{1 - H(\mu\sqrt{2\lambda}) H(\nu\sqrt{2\lambda}) e^{-2\sqrt{2\lambda}(b-a)}},$$

$$C_2 = \frac{f_{g,\lambda}(b) H(\nu\sqrt{2\lambda}) e^{-\sqrt{2\lambda}b} + f_{g,\lambda}(a) H(\mu\sqrt{2\lambda}) H(\nu\sqrt{2\lambda}) e^{\sqrt{2\lambda}(a-2b)}}{1 - H(\mu\sqrt{2\lambda}) H(\nu\sqrt{2\lambda}) e^{-2\sqrt{2\lambda}(b-a)}},$$

and

(2.5)

$$R_\lambda(A_{\mu,\nu}) g(x) = f_{g,\lambda}(x) + f_{g,\lambda}(a) \frac{H(\mu\sqrt{2\lambda}) e^{\sqrt{2\lambda}(a-x)} + H(\mu\sqrt{2\lambda}) H(\nu\sqrt{2\lambda}) e^{\sqrt{2\lambda}(a-2b+x)}}{1 - H(\mu\sqrt{2\lambda}) H(\nu\sqrt{2\lambda}) e^{-2\sqrt{2\lambda}(b-a)}} + f_{g,\lambda}(b) \frac{H(\nu\sqrt{2\lambda}) e^{-\sqrt{2\lambda}(b-x)} + H(\mu\sqrt{2\lambda}) H(\nu\sqrt{2\lambda}) e^{\sqrt{2\lambda}(2a-b-x)}}{1 - H(\mu\sqrt{2\lambda}) H(\nu\sqrt{2\lambda}) e^{-2\sqrt{2\lambda}(b-a)}}.$$

This formula may be rewritten in a form which is more convenient for our purposes. In order to do this let us set

$$\hat{g} = \begin{cases} g(x), & \text{for } x \in [a,b), \\ 0, & \text{for } x \notin [a,b), \end{cases} \qquad \hat{h} = \begin{cases} g(b+a-x), & \text{for } x \in [a,b), \\ 0, & \text{for } x \notin [a,b). \end{cases}$$

Observe that, by the definition of $f_{g,\lambda}$, for natural $k$ and for $x \in [a,b]$

$$\begin{aligned} f_{g,\lambda}(a)e^{-\sqrt{2\lambda}(x-a)}e^{-2k\sqrt{2\lambda}(b-a)} \\ &= \frac{1}{\sqrt{2\lambda}}\int_a^b e^{-\sqrt{2\lambda}\{|a-y|+(x-a)+2k(b-a)\}}g(y)dy \\ &= \frac{1}{\sqrt{2\lambda}}\int_a^b e^{-\sqrt{2\lambda}|y+x-2a+2k(b-a)|}g(y)dy \\ &= \frac{1}{\sqrt{2\lambda}}\int_{-\infty}^{\infty} e^{-\sqrt{2\lambda}|x-y+(2k+1)(b-a)|}\hat{h}(y)dy \\ &= \frac{1}{\sqrt{2\lambda}}\int_{-\infty}^{\infty} e^{-\sqrt{2\lambda}|x-y|}\hat{h}(y+(2k+1)(b-a))dy. \end{aligned}$$

As easily seen, $H$ maps the imaginary axis onto the unit circle and the right half-plane onto the open unit ball. Since

$$|H(\mu\sqrt{2\lambda})H(\nu\sqrt{2\lambda})e^{-2\sqrt{2\lambda}(b-a)}| < 1,$$

then

$$\begin{aligned} f_{g,\lambda}(a)&\frac{H(\mu\sqrt{2\lambda})e^{-\sqrt{2\lambda}(x-a)}}{1-H(\mu\sqrt{2\lambda})H(\nu\sqrt{2\lambda})e^{-2\sqrt{2\lambda}(b-a)}} \\ &= H(\mu\sqrt{2\lambda})\sum_{k=0}^{\infty}\Big[H(\mu\sqrt{2\lambda})H(\nu\sqrt{2\lambda})\Big]^k e^{-\sqrt{2\lambda}(x-a)}e^{-2k\sqrt{2\lambda}(b-a)}f_{g,\lambda}(a) \\ &= \frac{1}{\sqrt{2\lambda}}\int_{-\infty}^{\infty} e^{-\sqrt{2\lambda}|x-y|} \\ &\quad \times H(\mu\sqrt{2\lambda})\sum_{k=0}^{\infty}\Big[H(\mu\sqrt{2\lambda})H(\nu\sqrt{2\lambda})\Big]^k \hat{h}(y+(2k+1)(b-a))dy. \end{aligned}$$

Analogously one can prove that

$$\frac{f_{g,\lambda}(b)e^{-\sqrt{2\lambda}(b-a)}H(\mu\sqrt{2\lambda})}{1-H(\mu\sqrt{2\lambda})H(\nu\sqrt{2\lambda})e^{-2\sqrt{2\lambda}(b-a)}}$$
$$= \frac{1}{\sqrt{2\lambda}}\int_{-\infty}^{\infty} e^{-\sqrt{2\lambda}|x-y|}$$
$$\times H(\nu\sqrt{2\lambda})\sum_{k=0}^{\infty}\left[H(\mu\sqrt{2\lambda})H(\nu\sqrt{2\lambda})\right]^k \hat{h}(y-(2k+1)(b-a))dy,$$

and

$$\frac{f_{g,\lambda}(b)H(\mu\sqrt{2\lambda})H(\nu\sqrt{2\lambda})e^{\sqrt{2\lambda}(2a-b-x)}}{1-H(\mu\sqrt{2\lambda})H(\nu\sqrt{2\lambda})e^{-2\sqrt{2\lambda}(b-a)}}$$
$$+\frac{f_{g,\lambda}(a)H(\mu\sqrt{2\lambda})H(\nu\sqrt{2\lambda})e^{\sqrt{2\lambda}(a-2b+x)}}{1-H(\mu\sqrt{2\lambda})H(\nu\sqrt{2\lambda})e^{-2\sqrt{2\lambda}(b-a)}}$$
$$= -f_{g,\lambda}(x) + \frac{1}{\sqrt{2\lambda}}\int_{-\infty}^{\infty} e^{-\sqrt{2\lambda}|x-y|}$$
$$\times\left(\sum_{k=-\infty}^{\infty}\left[H(\mu\sqrt{2\lambda})H(\nu\sqrt{2\lambda})\right]^k \hat{g}(y+2k(b-a))\right)dy.$$

Thus (2.4) holds with

$$g^*(x) = \sum_{k=-\infty}^{\infty}\left[H(\mu\sqrt{2\lambda})H(\nu\sqrt{2\lambda})\right]^k \hat{g}(y+2k(b-a))$$
$$+ H(\nu\sqrt{2\lambda})\sum_{k=0}^{\infty}\left[H(\mu\sqrt{2\lambda})H(\nu\sqrt{2\lambda})\right]^k \hat{h}(y-(2k+1)(b-a))$$
$$+ H(\mu\sqrt{2\lambda})\sum_{k=0}^{\infty}\left[H(\mu\sqrt{2\lambda})H(\nu\sqrt{2\lambda})\right]^k \hat{h}(y+(2k+1)(b-a)).$$

(cf. [9; p.453, (5.16), (5.17)].)

This result is in accordance with the Lord Kelvin's method of images. Indeed, the function $g^*$ is a result of succesive approximations by repeated reflections of the function $g$ at the points $a$ and $b$. Note that while W.Feller considered (see [9; pp. 328-331]) odd (and even) extensions of the function $g$, in our case the left-hand side of the even extension has to be multilplied, while reflecting at the point $a$, by $H(\mu\sqrt{2\lambda})$, and the right-hand side of it is to be multiplied, while reflecting at $b$, by $H(\nu\sqrt{2\lambda})$ - compare [2; (2.26)]. (When $\epsilon = 0$ this is an odd extension, when $\epsilon = \infty$ the extension is even.)

By using this remark, or by noting that the functions appearing as coefficients in the definition of the function $g^*$ have pairwise disjoint supports, we see that $\sup_{x\in R}|g^*(x)| = \sup_{x\in[a,b]}|g(x)|$, since $|H(\mu\sqrt{2\lambda})| < 1$ and $|H(\nu\sqrt{2\lambda})| < 1$. Consequently, by the remarks made at the beginning of the proof, the operators $A_{\mu,\nu}$ $\mu,\nu > 0$ are proved to be infinitesimal generators of strongly continuous semigroups $\{T_{\mu,\nu}(t), t \geq 0\}$ acting in $L$, and, furthermore, the semigroups are "uniformly analytic".

Finally, since $\lim_{\mu\to 0} H(\mu\sqrt{2\lambda}) = \lim_{\nu\to 0} H(\nu\sqrt{2\lambda}) = H(0)$, by (2.5), $\lim_{\mu,\nu\to 0} R_\lambda(A_{\mu,\nu})g = R_\lambda(A_{0,0})g$, Theorem 4 applies, and the corollary is proved. ∎

**3. Kolmogorov backward equation for the Poisson - Kac process.** Yet another example we are going to present now is provided, once again, by the theory of stochastic processes. The Kolmogorov backward equation for the stochastic process $\left(\frac{1}{\sqrt{\epsilon}}\xi_{\frac{1}{2\epsilon}}(t), (-1)^{N_{\frac{1}{2\epsilon}}(t)}\right)_{t\geq 0}$ will be considered, where, for $a > 0$, $N_a(t)$ is a homogeneous Poisson process and $\xi_a(t) = \int_0^t (-1)^{N_a(s)} ds$. A proof will be given that the coresponding semigroups converge, as $\epsilon \to 0$even though $L_0 \neq L$. As shown e.g. in [14], the solutions to the telegraph equation may be obtained from the solutions of this Kolmogorov equation and the result presented below is parallel to that obtained in [2]. On the other hand, however, as a very example of convergence of semigroups it seems still to be worth considering here.

Let $C_{(-\infty,+\infty)}$ be the space of all real-valued uniformly continuous bounded functions, and let $C^n_{(-\infty,+\infty)}$ be the subset which consists of all $n$-times continuously differentiable functions $u$ with $u^{(n)} \in C_{(-\infty,+\infty)}$. For $\epsilon > 0$ define an operator $A_\epsilon$ acting in $L = C_{(-\infty,+\infty)} \times$

$C_{(-\infty,+\infty)}$ by

$$A_\epsilon = \frac{1}{\sqrt{\epsilon}}\begin{pmatrix} \frac{d}{dx} & 0 \\ 0 & -\frac{d}{dx}\end{pmatrix} + \frac{1}{2\epsilon}\begin{pmatrix} -1 & 1 \\ 1 & -1\end{pmatrix},$$
$$D(A_\epsilon) = C^1_{(-\infty,+\infty)} \times C^1_{(-\infty,+\infty)}.$$

The operator $A_\epsilon$ was proven to generate the semigroup of bounded operators $\{T_\epsilon(t), t \geq 0\}$ acting in $L$ and given by the formula (see [14; Lemma 1]):

(3.1)

$$T_\epsilon(t)\begin{pmatrix} u \\ v\end{pmatrix}(x) = \begin{pmatrix} \mathrm{E}1_{\Omega-\chi_{\frac{1}{2\epsilon}}(t)}u(x+\frac{1}{\sqrt{\epsilon}}\xi_{\frac{1}{2\epsilon}}(t)) + \mathrm{E}1_{\chi_{\frac{1}{2\epsilon}}(t)}v(x-\frac{1}{\sqrt{\epsilon}}\xi_{\frac{1}{2\epsilon}}(t)) \\ \mathrm{E}1_{\Omega-\chi_{\frac{1}{2\epsilon}}(t)}v(x-\frac{1}{\sqrt{\epsilon}}\xi_{\frac{1}{2\epsilon}}(t)) + \mathrm{E}1_{\chi_{\frac{1}{2\epsilon}}(t)}u(x+\frac{1}{\sqrt{\epsilon}}\xi_{\frac{1}{2\epsilon}}(t))\end{pmatrix}$$

(where $\chi_a(t) = \{\omega : N_a(t) = 1, 3, 5, ...\}$ ). Simple calculations show that the resolvent operator of $A_\epsilon$ is

$$R_\lambda(A_\epsilon)\begin{pmatrix} u \\ v\end{pmatrix} = \begin{pmatrix} \left(\lambda^2\epsilon + \lambda - \frac{d^2}{dx^2}\right)^{-1}\left(\frac{u+v}{2} + \epsilon\lambda v\right) + \sqrt{\epsilon}K_{\lambda^2\epsilon+\lambda}v \\ \left(\lambda^2\epsilon + \lambda - \frac{d^2}{dx^2}\right)^{-1}\left(\frac{u+v}{2} + \epsilon\lambda u\right) + \sqrt{\epsilon}K_{\lambda^2\epsilon+\lambda}u\end{pmatrix},$$

where $\lambda > 0$, $K_\alpha \in \mathcal{L}\left(C_{(-\infty,+\infty)}, C_{(-\infty,+\infty)}\right)$, $\alpha > 0$, is a bounded operator defined by

$$K_\alpha v(x) = \frac{1}{2}e^{\sqrt{\alpha}x}\int_x^\infty e^{-\sqrt{\alpha}y}v(y)dy - \frac{1}{2}e^{-\sqrt{\alpha}x}\int_{-\infty}^x e^{\sqrt{\alpha}y}v(y)dy,$$

and the operator $\frac{d^2}{dx^2}$ acts as follows $\frac{d^2}{dx^2} : C_{(-\infty,+\infty)} \supset C^2_{(-\infty,+\infty)} \to C_{(-\infty,+\infty)}$. Indeed, note that if $u, v \in C^1_{(-\infty,+\infty)}$ then

$$R_\lambda(A_\epsilon)\begin{pmatrix} u \\ v\end{pmatrix} = \begin{pmatrix} \left(\lambda^2\epsilon + \lambda - \frac{d^2}{dx^2}\right)^{-1}\left(\frac{u+v}{2} + \epsilon\lambda v + \sqrt{\epsilon}v'\right) \\ \left(\lambda^2\epsilon + \lambda - \frac{d^2}{dx^2}\right)^{-1}\left(\frac{u+v}{2} + \epsilon\lambda u + \sqrt{\epsilon}u'\right)\end{pmatrix}, \lambda > 0,$$

and, on the other hand, for $\alpha > 0$,

$$\left(\alpha - \frac{d^2}{dx^2}\right)^{-1} v'(x) = \frac{1}{2\sqrt{\alpha}}\int_{-\infty}^\infty e^{-\sqrt{\alpha}|y-x|}v'(y)dy = K_\alpha v(x).$$

Since $\|K_\alpha\|_{L(C_{(-\infty,+\infty)},C_{(-\infty,+\infty)})} \leq 1/\sqrt{\alpha}$ it is readily seen that

$$\lim_{\epsilon\to 0} R_\lambda(A_\epsilon)\begin{pmatrix} u \\ v \end{pmatrix} = R^0_\lambda \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \left(\lambda - \frac{d^2}{dx^2}\right)^{-1} \frac{u+v}{2} \\ \left(\lambda - \frac{d^2}{dx^2}\right)^{-1} \frac{u+v}{2} \end{pmatrix}.$$

Since the closure of the range of the operator $R^0_\lambda$ is equal to $L_0 = \left\{\begin{pmatrix} u \\ v \end{pmatrix} \in L:\ u = v\right\}$ condition (b) in Theorem 1 is certainly not satisfied.

On the subspace $L_0$ the operator $R^0_\lambda$ is equal to the resolvent of the operator $A_0 : L_0 \supset D(A_0) \to L_0, D(A_0) = \left\{\begin{pmatrix} u \\ u \end{pmatrix} : u \in C^2_{(-\infty,+\infty)}\right\}$,

$$A_0 = \begin{pmatrix} \frac{d^2}{dx^2} & 0 \\ 0 & \frac{d^2}{dx^2} \end{pmatrix}$$

which is the infinitesimal generator of the semigroup

$$T_0(t)\begin{pmatrix} u \\ u \end{pmatrix} = \begin{pmatrix} T(t)u \\ T(t)u \end{pmatrix}, \quad T(t)u(x) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} u(y)dy.$$

Thus, Theorem 2 applies and

$$\lim_{\epsilon\to 0} T_\epsilon(t)\begin{pmatrix} u \\ u \end{pmatrix} = T_0(t)\begin{pmatrix} u \\ u \end{pmatrix}.$$

For another simple proof of this fact see [8; p.471].

As proved below, convergence takes place for $u \neq v$ as well. Indeed, let us define the Borel measures $\mu^+_{t,\epsilon}, \mu^-_{t,\epsilon}$ as in [2; §3.3]

$$\mu^+_{t,\epsilon}(\mathcal{B}) = \text{Prob}\left(\frac{1}{\sqrt{\epsilon}}\xi_{\frac{1}{2\epsilon}}(t) \in \mathcal{B} | \chi_{\frac{1}{2\epsilon}}(t)\right),$$

$$\mu^-_{t,\epsilon}(\mathcal{B}) = \text{Prob}\left(\frac{1}{\sqrt{\epsilon}}\xi_{\frac{1}{2\epsilon}}(t) \in \mathcal{B} | \Omega - \chi_{\frac{1}{2\epsilon}}(t)\right).$$

Since $\text{Prob}(\chi_a(t)) = \text{Prob}(N_a(t) = 1,3,5,..) = e^{-at}\text{sh}(at)$ and $\text{Prob}(\Omega - \chi_a(t)) = \text{Prob}(N_a(t) = 0,2,4,..) = e^{-at}\text{ch}(at)$, then the formula (3.1) may be rewritten as

$$T_\epsilon(t)\begin{pmatrix} u \\ v \end{pmatrix}(x)$$
$$= \begin{pmatrix} e^{-\frac{t}{2\epsilon}}\text{ch}\frac{t}{2\epsilon}\int_R u(x+y)\mu^-_{t,\epsilon}(dy) + e^{-\frac{t}{2\epsilon}}\text{sh}\frac{t}{2\epsilon}\int_R v(x-y)\mu^+_{t,\epsilon}(dy) \\ e^{-\frac{t}{2\epsilon}}\text{ch}\frac{t}{2\epsilon}\int_R v(x-y)\mu^-_{t,\epsilon}(dy) + e^{-\frac{t}{2\epsilon}}\text{sh}\frac{t}{2\epsilon}\int_R u(x+y)\mu^+_{t,\epsilon}(dy) \end{pmatrix}$$

In [2; (3.4), (3.5)] it was proved that the characteristic functions $\varphi^+_{t,\epsilon}(u)$ and $\varphi^-_{t,\epsilon}(u)$ of the measures $\mu^+_{t,\epsilon}$, $\mu^-_{t,\epsilon}$ satisfy $\lim_{\epsilon\to 0}\varphi^+_{t,\epsilon}(u) = \lim_{\epsilon\to 0}\varphi^-_{t,\epsilon}(u) = e^{-u^2 t}$. In order to obtain this relations, the results of [19; p.9] may be also applied. Consequently, since

$$\lim_{\epsilon\to 0} e^{-\frac{t}{2\epsilon}}\text{ch}\frac{t}{2\epsilon} = \lim_{\epsilon\to 0} e^{-\frac{t}{2\epsilon}}\text{sh}\frac{t}{2\epsilon} = \frac{1}{2},$$

we have

$$\lim_{\epsilon\to 0} T_\epsilon(t)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{y^2}{4t}}[u(\cdot+y)+v(\cdot+y)]dy \\ \frac{1}{2}\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{y^2}{4t}}[u(\cdot+y)+v(\cdot+y)]dy \end{pmatrix} \tag{3.2}$$
$$= \begin{pmatrix} T(t)\frac{u+v}{2} \\ T(t)\frac{u+v}{2} \end{pmatrix}.$$

■

**Remark 1.** Note that the formula (3.2) provides a simple way of constructing semigroups that are strongly continuous only on a subspace of a Banach space. In fact if a strongly continuous semigroup $\{T(t) :\ t \geq 0\}$ acting in a Banach space $L$ is given, then the formula

$$\mathcal{T}_0(t)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} T(t)\frac{u+v}{2} \\ T(t)\frac{u+v}{2} \end{pmatrix}$$

defines the semigroup of linear operators in $L \times L$. The simple proof that the semigroup is strongly continuous only on the subspace $L_0 = \left\{\begin{pmatrix} u \\ v \end{pmatrix} :\ u = v\right\}$ is omitted.

**4. An example originating from the ergodic theory.** The importance of the following, simple example lies in the fact that, while the convergence of all the semigroups we have considered above seems to be closely connected to the continuity theorem, the case we will deal with now has nothing to do with that.

Let us give some preparatory definitions. Let $(X, \mu)$ be a measure space, and $L$ be the space of all real-valued integrable functions $f$ defined on $X$, with the norm $\|f\| = \int_X |f| d\mu$. Let us define the set $D$ of densities as $D = \{f \in L : \ f \geq 0, \|f\| = 1\}$. The linear operator $P$ is called a Markov operator iff $PD \subset D$. Note that any Markov operator is bounded and the inequality $\|Pf\| \leq \|f\|$ holds for all $f \in L$ (see [17; p.33]).

A semigroup $\{P(t), t \geq 0\}$ is called a stochastic semigroup iff all the operators $P(t), t \geq 0$ are Markov. A stochastic semigroup $\{P(t), t \geq 0\}$ is called asymptotically stable [16-18] iff there exists a unique density $f_*$ such that for all $f \in D$

$$\lim_{t \to \infty} P(t)f = f_*.$$

Now, let us suppose that $\{P(t), t \geq 0\}$ is asymptotically stable semigroup, and define the semigroups $\{P_\epsilon(t), t \geq 0\}$, $1 > \epsilon > 0$, by setting $P_\epsilon(t) = P(\frac{1}{\epsilon}t)$.

Of course, for $f \in L, t > 0$,

$$\lim_{\epsilon \to 0} P_\epsilon(t)f = \lim_{s \to \infty} P(s)f = \left(\int_X f d\mu\right) f_*. \tag{4.1}$$

Thus

$$\lim_{\epsilon \to 0} R_\lambda(A_\epsilon)f = \lim_{\epsilon \to 0} \int_0^\infty e^{-\lambda t} P_\epsilon(t) f dt = \left(\frac{1}{\lambda}\int_X f d\mu\right) f_*.$$

Since in the non-trivial cases, such as the Chandrasekhar-Münch equation [4], [16]-[18], the linear space $L_0$ generated by $f_*$ is not dense in $L$, the condition (a) in Theorem 1 is satisfied while (b) is not. On the other hand, (4.1) proves that the semigroups $\{P_\epsilon(t) : \ t \geq 0\}$, $1 > \epsilon > 0$, converge for all $f \in L$. The convergence is pointwise and the limiting semigroup is strongly continuous only on one-dimensional space $L_0$.

**5. The Yosida approximation of a sectorial operator.** An operator $A : \mathcal{D}(A) \to L$ acting in a complex Banach space $L$ is called sectorial ([5; p.307]) iff there exists $\omega > 0$ such that the sector $S_\omega = \{\lambda \in C : |\arg \lambda| < \frac{\pi}{2} + \omega\}$ is contained in the resolvent set of $A$ and there exists a constant $M > 0$ such that

$$\|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda|}, \quad \text{for all } \lambda \in S_\omega. \tag{5.1}$$

In particular if $\mathcal{D}(A)$ is dense in $L$, $A$ is the generator of an holomorphic semigroup (see [12; p. 488]).

Given an operator $A$ one can define a family of operators via the Dunford integral

$$S(t)f = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda - A)^{-1} f d\lambda, \qquad t > 0, \tag{5.2}$$

(see [5; p. 307] for details, comp. [2; p. 324]). The family $\{S(t), t \geq 0\}$ ($S(0) = I$) is a non-continuous semigroup. It is, however, strongly continuous for for $t > 0$, which will be of importance in what follows. Its further properties are listed in [5; pp. 307-309]. The aim of this paragraph is to show that the semigroups $\{e^{A_\mu t} : t \geq 0\}$, $\mu > 0$, where $A_\mu$ is a Yosida approximation of $A$, converge to $\{S(t), t \geq 0\}$. Since $\{S(t), t \geq 0\}$ is not strongly continuous except for $f \in L_0 = \mathrm{cl}\mathcal{D}(A)$, the convergence is, in general, only pointwise and represents another example of a phenomenon the paper is devoted to.

In what follows we will need the following lemma.

**Lemma.** *For any $r, \mu > 0$ and $\pi > \alpha \geq 0$,*

$$\left|\frac{\mu + r}{\mu + re^{i\alpha}}\right| \leq \sqrt{\frac{2}{1 + \cos\alpha}}. \tag{5.3}$$

**Proof.** Let us fix $\mu, r > 0$ and define $f(\alpha) = 2(\mu^2 + 2\mu r\cos\alpha + r^2) - (1 + \cos\alpha)(\mu + r)^2$, for $\alpha \in [0, \pi)$. We have $f(0) = 0$, $f'(\alpha) = (\mu - r)^2 \sin\alpha \geq 0$. Thus $f(\alpha) \geq 0$, for all $\alpha \in [0, \pi)$. Consequently, $2(1 + \cos\alpha)^{-1} \geq (\mu + r)^2(\mu^2 + 2\mu r\cos\alpha + r^2)^{-1}$, which is equivalent to (5.3) since $|\mu + re^{i\alpha}| = \sqrt{\mu^2 + 2\mu r\cos\alpha + r^2}$. ■

Let us assume that $A$ satisfies (5.1) with appropiate constants $M, \omega > 0$. One defines the Yosida approximation of $A$ as $A_\mu = \mu A R_\mu = \mu^2 R_\mu - \mu$, where $R_\mu = (\mu - A)^{-1}$.

**Proposition.** *The semigroups* $\{e^{A_\mu t} : \ t \geq 0\}, \mu > 0$ are *"uniformly analytic" in the sense of* Theorem 4.

**Proof.** Observe that by the Hilbert equation

$$(5.4) \qquad (I - \nu R_\mu)(I + \nu R_{\mu-\nu}) = (I + \nu R_{\mu-\nu})(I - \nu R_\mu) = I,$$

provided $\mu, \mu-\nu \in S_\omega$. Let us note that $S_\omega = \{\lambda : \ \cos \arg \lambda > \cos \omega\}$. Suppose that $\mu > 0$ and $\lambda = x + iy = re^{i\alpha} \in S_\omega$, and put $\beta = \arg \frac{\lambda\mu}{\lambda+\mu}$. Since $\frac{\lambda\mu}{\lambda+\mu} = \mu(x^2 + x\mu + y^2 + \mu y i)[(x+\mu)^2 + y^2]^{-1}$, we get $\cos\beta = (x^2 + x\mu + y^2)/\sqrt{(x^2 + x\mu + y^2)^2 + \mu^2 y^2}$. It is just a matter of standard calculations to prove that $(x^2 + x\mu + y^2)/\sqrt{(x^2 + x\mu + y^2)^2 + \mu^2 y^2} \geq x/\sqrt{x^2 + y^2}$, which means that $\cos\beta \geq \cos\alpha > \cos\omega$, and, consequently, $\frac{\lambda\mu}{\lambda+\mu} \in S_\omega$.

Thus, since $\frac{\lambda\mu}{\lambda+\mu} = \mu - \frac{\mu^2}{\lambda+\mu}$, we may apply (5.4) with $\nu = \frac{\mu^2}{\lambda+\mu}$ to obtain

$$\left(I - \frac{\mu^2}{\lambda+\mu} R_\mu\right)^{-1} = I + \frac{\mu^2}{\lambda+\mu} R_{\frac{\lambda\mu}{\lambda+\mu}}.$$

Consequently, for any $\mu > 0$, the resolvent set of $A_\mu$ contains $S_\omega$ and (see [7; p. 312]) for all $\mu > 0, \lambda \in S_\omega$

$$(5.5) \qquad \begin{aligned} (\lambda - A_\mu)^{-1} &= (\mu + \lambda - \mu^2 R_\mu)^{-1} = \frac{1}{\lambda+\mu}\left(I - \frac{\mu^2}{\lambda+\mu} R_\mu\right)^{-1} \\ &= \frac{1}{\lambda+\mu} + \left(\frac{\mu}{\lambda+\mu}\right)^2 R_{\frac{\lambda\mu}{\lambda+\mu}}. \end{aligned}$$

Furthermore, by (5.5), (5.1) and (5.3) we get

$$\begin{aligned} \|(\lambda - A_\mu)^{-1}\| &\leq \frac{1}{|\lambda+\mu|} + \frac{\mu^2}{|\lambda+\mu|^2}\|R_{\frac{\lambda\mu}{\lambda+\mu}}\| \\ &\leq \frac{1}{|\lambda+\mu|} + \frac{\mu}{|\lambda+\mu|}\frac{M}{|\lambda|} \leq M' \frac{|\lambda| + \mu}{|\mu+\lambda|}\frac{1}{|\lambda|} \\ &\leq \frac{\mu + |\lambda|}{|\mu + |\lambda| e^{i\alpha}|}\frac{M'}{|\lambda|} \leq \frac{M'\sqrt{\frac{2}{1+\cos\alpha}}}{|\lambda|}, \end{aligned}$$

(where $M' = \max\{M, 1\}$), provided $|\arg\lambda| \le \alpha < \frac{\pi}{2} + \omega$. Thus if $|\arg\lambda| \le \frac{\pi}{2} + \omega - \delta$ then

$$\|(\lambda - A_\mu)^{-1}\| \le M'\sqrt{2}|\lambda|^{-1}(1 - \sin(\omega - \delta))^{-1/2},$$

which is just (2.1) with $M_\delta = M'\sqrt{2}(1 - \sin(\omega - \delta))^{-1/2}$. ∎

Now, by Theorem 4, and Proposition, we may put

$$\hat{S}(t)f = \lim_{\mu\to\infty} e^{A_\mu t}f, \; t > 0, \; f \in L.$$

By the Lebesgue dominated convergence theorem and (5.4)

$$\int_0^\infty e^{-\lambda t}\hat{S}(t)f\,dt = \lim_{\nu\to\infty}\int_0^\infty e^{-\lambda t}e^{A_\mu t}f\,dt = \lim_{\nu\to\infty}(\lambda - A_\mu)^{-1}f$$

$$= \lim_{\nu\to\infty}\left\{\frac{1}{\lambda+\mu}f + \left(\frac{\mu}{\lambda+\mu}\right)^2 R_{\frac{\lambda\mu}{\lambda+\mu}}f\right\} = (\lambda - A)^{-1}f$$

Theorem 10.2 on p. 309 of [5] proves then that the Laplace transforms of $\hat{S}(t)$ and $S(t)$ defined by (5.2) coincide. Thus, $\hat{S}(t)f = S(t)f$ a.e. in $[0, \infty)$ for all $f \in L$. Any strongly measurable semigroup, however, is strongly continuous for $t > 0$, ([11] p.305, Th. 10.2.3.), and such is $\{\hat{S}(t), t \ge 0\}$. It implies that $\hat{S}(t)f = S(t)f, t > 0, f \in L$. As a consequence,

$$\lim_{\mu\to\infty} e^{A_\mu t}f = S(t)f, t > 0, f \in L,$$

which was the aim of this section.

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