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Some Remarks on the Isomorphism of Fuchsian Groups

Abstract. Let G, $\tilde{G}$ be Fuchsian groups of the first kind acting on the unit disk Δ and isomorphic under $\theta(g)$, $g \in G$. Under some further conditions an automorphism γ of $\partial\Delta$ can be associated with θ. A formula for reconstructing $\theta(g)$ by means of γ is established.

Introduction. Notations. This paper deals with an isomorphism θ between Fuchsian groups G and $\tilde{G}$ acting on the unit disk Δ, both being discontinuous and of the first kind. Discontinuity of G means that any $z \in \Delta$ has a neighbourhood which does not contain any pair of points equivalent under G. Moreover, G is said to be a Fuchsian group of the first kind if the fixed points of G are dense on $\mathbb{T} = \partial\Delta$.

In this case any $g \in G$ has either only one fixed point on $\mathbb{T}$ (*parabolic case*) or two different fixed points $\zeta_1, \zeta_2 \in \mathbb{T}$ *(hyperbolic case)*.

If $\mathfrak{M}$ stands for the group of all Möbius automorphisms of Δ then any $g \in \mathfrak{M}$ has the form $g(z) = e^{i\alpha}(z-a)(1-\overline{a}z)^{-1}$, where $|a| < 1$ and $0 \le \alpha < 2\pi$. Moreover, if $0 < |a| < 1$ then the inequality $\sin\frac{1}{2}\alpha < |a|$ distinguishes hyperbolic g, whereas the equality $\sin\frac{1}{2}\alpha = |a|$ is characteristic for parabolic g. For a hyperbolic $g \in \mathfrak{M}$ the circular arc joining in Δ the fixed points ζ_1, ζ_2 of g and orthogonal to $\mathbb{T}$ is said to be *the axis of g*.

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According to the below quoted Theorem A an isomorphism $\theta : G \to \widetilde{G}$ generates an automorphism γ of $\mathbb{T}$, so-called *boundary function*. Conversely, the isomorphism θ can be reconstructed in terms of γ, e.g. by formula (2), if γ fulfills suitable conditions.

In an earlier paper [1] the present author announced an analogous formula (6) in terms of the Poisson extension $P[\gamma] = h$. However, this formula holds only under an additional assumption on h to be quasiconformal. It is thus actually a particular case of formula (2). The characterization of γ whose Poisson extension is a quasiconformal self-mapping of Δ was given by Martio [4].

In this paper we establish a different representation of $\theta(g)$ under a much weaker assumption on γ (cf. formula (3)) which is supposed to be only a sense-preserving, homeomorphic self-mapping of $\mathbb{T}$.

Isomorphism and conjugation w. r. t. γ. Our starting point is the following basic

Theorem A [2], [5]. *Let θ be an isomorphism between the Fuchsian groups G and $\widetilde{G}$ of Möbius transformations acting on Δ, both fixed point free and of the first kind. Suppose $\theta(g)$ is parabolic if and only if g is. Then θ generates a mapping from the set X of fixed points of G onto the set $\widetilde{X}$ of fixed points of $\widetilde{G}$.*

This mapping can be extended to a homeomorphism γ of $\mathbb{T}$ if and only if the following axis condition is satisfied: $g_1, g_2 \in G$ have intersecting axes if and only if $\theta(g_1)$, $\theta(g_2)$ do. The homeomorphism $\gamma : \mathbb{T} \to \mathbb{T}$ is said to be the boundary function of the isomorphism θ. It satisfies the relation

$$\gamma \circ g = \theta(g) \circ \gamma \quad \text{on } \mathbb{T},\ g \in G. \tag{1}$$

A natural problem arises to express the isomorphism $\theta(g), g \in G$, in terms of the boundary automorphism γ of $\mathbb{T}$. This can be done for quasisymmetric γ which has a quasiconformal extension w to Δ. Then we have

$$\theta(g) = w \circ g \circ w^{-1} \tag{2}$$

cf.[2], [3, p. 134]. Note that the r. h. s. in (2) belongs to $\mathfrak{M}$ for any $g \in \mathfrak{M}$.

However, we are in a position to reconstruct the isomorphism $\theta(g)$ from its boundary function γ, without imposing any restrictions on γ. We have

Theorem. *Let $\theta(g)$ be an isomorphism between the Fuchsian groups G, $\widetilde{G}$ which satisfies the assumptions of* Theorem A *and let γ stand for the boundary function associated with the isomorphism θ. Then*

$$\theta(g) = P[\gamma^{-1} \circ g \circ \gamma] \quad \text{on } \Delta, \; g \in G, \tag{3}$$

where $P[\phi]$ denotes the Poisson extension to Δ of the homeomorphism $\phi : \mathbb{T} \to \mathbb{T}$.

Proof. According to Theorem A the isomorphism $\theta(g)$ and the associated boundary function γ are related by the identity $\gamma \circ \theta(g) = g \circ \gamma$ on $\mathbb{T}$ or equivalently, by the identity

$$\theta(g) = \gamma^{-1} \circ g \circ \gamma \quad \text{on } \mathbb{T}, \; g \in G. \tag{4}$$

This may be interpreted as a conjugation of groups G, $\widetilde{G}$ under γ. Now $\theta(g) \in \widetilde{G} \subset \mathfrak{M}$ and it is determined by its boundary values on $\mathbb{T}$ which are equal to the r. h. s. in (4). Hence (3) readily follows and the proof is complete.

The formula (3) may be applied only for $g \in G$ and a special boundary function γ intimately connected with the isomorphism $\theta : G \to \widetilde{G}$.

However, if γ is quasisymmetric on $\mathbb{T}$ and w is an arbitrary quasiconformal automorphism of Δ with boundary values γ then $w^{-1} \circ g \circ w \in \mathfrak{M}$ for any $g \in \mathfrak{M}$. To see this observe that quasiconformal mappings w and $g \circ w$ have the same complex dilatation on Δ and satisfy the same Beltrami equation. Therefore both mappings differ from each other by a conformal mapping which means that $w^{-1} \circ g \circ w$ is conformal. Then, as a conformal self-mapping of Δ, it is Möbius. In this way any quasisymmetric γ generates a group automorphism of $\mathfrak{M}$ given by formula (2). If γ fails to be quasisymmetric then (2) does not make sense, whereas (3) is still applicable.

References

[1] Krzyż, J. G., *Isomorphism of Fuchsian groups and harmonic maps*, Mitteilungen aus dem Mathem. Seminar Giessen, Heft **228**, 25–29.

[2] Lehto, O., *Group isomorphisms induced by quasiconformal mappings. Contributions to Analysis* (1974), A collection of papers dedicated to Lipman Bers. Academic Press, 241–244.

[3] ———, *Univalent functions and Teichmüller spaces*, Springer Verlag 1987.

[4] Martio, O., *On harmonic quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. AI. **425** (1968), 1–9.

[5] Tukia, P., *On discrete groups of the unit disk and their isomorphisms*, Ann. Sci. Fenn. Ser. AI. **504** (1972), 1–45.

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