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A Modulus for the Nearly Uniform Convexity

ABSTRACT. In this paper we define an "outside" modulus for the nearly uniform convexity and study its properties useful in fixed point theory for nonexpansive mappings. Moreover, we calculate this modulus in separable Hilbert spaces.

0. Introduction. In the geometric theory of Banach spaces the notion of modulus of convexity plays a very significant role. It allows us to classify Banach spaces from the point of view of their geometrical structure. In this context, the classical Clarkson modulus of convexity defined by Day [D] while considering the definition of uniform convexity due to Clarkson [C], is a useful tool in the fixed point theory. A lot of facts concerning this notion and its applications may be found in [GK], [ADL] and [O], for example.

Recently, K. Goebel, T. Sękowski, J. Banaś et al. [GS], [B], [DL] proposed several generalizations of the notion of modulus of convexity using some measures of noncompactness. With these moduli (so called moduli of noncompact convexity), they proved several interesting facts concerning the

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geometric theory of Banach spaces. Moreover, these moduli are suitable for the nearly uniformly convex spaces introduced in [H] in the same sense as the classical modulus of convexity of Clarkson is suitable for the uniformly convex spaces. In [Ku] a characterization of the nearly uniform convexity is proved considering sets which lie outside of the unit ball, instead of sets in the ball as in the original definition. The aim of this paper is to introduce an "outside"modulus of nearly uniform convexity and to prove its usefulness to the geometric theory of Banach spaces and to the theory of nonexpansive mappings. In the first section we shall show the relationship between this mapping and the moduli of noncompact convexity, deriving a fixed point result. In section 2 we compute the new modulus in separable Hilbert spaces.

1. Notations, definitions and first results. Let X be an infinite dimensional Banach space with closed unit ball B_X and unit sphere S_X. Let $B(x,r)$ denote the closed ball centered at x and of radius r and for any $A \subset X$, $\overline{A}$ and $\text{co}(A)$ will denote the closure and the convex hull of A, respectively. Let $\mathcal{B}$ be the family of bounded subsets of X. A map $\mu : \mathcal{B} \to [0,+\infty)$ is called a measure of noncompactness defined on X if $\mu(A) = 0$ if and only if A is a precompact set. The first measure of noncompactness (set-measure, denoted by $\alpha(A)$) was defined by Kuratowski in [K] as $\inf\{\varepsilon \geq 0 : A$ can be covered by finitely many sets with diameter $\leq \varepsilon\}$. Another measure of noncompactness (ball-measure, denoted by $\chi(A)$) was introduced by several authors (see [GGM] or [S]) as $\inf\{\varepsilon \geq 0 : A$ can be covered by finitely many balls with diameter $\leq \varepsilon\}$. In [WW, page 91] another measure of noncompactness is defined by $\beta(A) = \sup\{\varepsilon \geq 0 :$ there exists a sequence $\{x_n\}$ in A with $\text{sep}(\{x_n\}) > \varepsilon\}$, where $\text{sep}(\{x_n\}) = \inf\{\|x_n - x_m\|\ ,\ n \neq m\}$. It is easy to prove that $\chi(A) \leq \beta(A) \leq \alpha(A) \leq 2\chi(A)$ for every bounded subset A of X. Throughout this paper we denote by μ any of these measures of noncompactness. The main properties of these measures can be found in [AKPRS] or [ADL].

We will use in this paper the following:

(1) $\mu(A) = \mu(\overline{A})$,

(2) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$,

(3) $\mu(A) \leq \mu(B)$ if $A \subset B$,

(4) $\mu(tA) = |t|\mu(A)$,

(5) $\mu(co(A)) = \mu(A)$.

Associated with these measures of noncompactness, the following modulus of noncompact convexity for a Banach space X was considered in [GS] (with $\mu = \alpha$), [B] (with $\mu = \chi$) and [DL] (with $\mu = \beta$):

$\Delta_{X,\mu} : [0, \mu(B_X)] \to [0,1]$;

$\Delta_{X,\mu}(\varepsilon) = 1 - \sup\{\inf\{\|x\| : x \in A\} : A = \mathrm{co}(A) \subset B_X, \mu(A) > \varepsilon\}.$

Let us remember that $\alpha(B_X) = 2$ and $\chi(B_X) = 1$ on every Banach space. However, $\beta(B_X)$ is a real number in the interval $[1,2]$ which depends on the space X. For example, if $1 < p < +\infty$ then $\beta(B_{\ell^p}) = 2^{1/p}$ (see [ADL]).

This modulus measures the rotundity of the unit ball in similar way, as the classical Clarkson modulus of convexity given by $\delta_X : [0,2] \to [0,1]$:

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1,\ \|x-y\| \geq \varepsilon\right\}.$$

The coefficient of convexity of X is defined by

$$\delta_0(X) = \sup\{\varepsilon \geq 0 : \delta_X(\varepsilon) = 0\}$$

and the space X is said to be uniformly convex if and only if $\delta_0(X) = 0$. Analogously we can define the coefficient of noncompact convexity of X as $\Delta_{0,\mu}(X) = \sup\{\varepsilon \geq 0 : \Delta_{X,\mu}(\varepsilon) = 0\}$.

A Banach space X is said to be nearly uniformly convex if and only if $\Delta_{0,\mu}(X) = 0$. This property was originally introduced in [H] and was intensively studied in the last years (see [P], [KL1], [KL2], [Ku], [ADL] and references therein).

Let X be a Banach space. The drop D_x defined by an element $x \in X \backslash B_X$ is the set $\mathrm{co}(\{x\} \cup B_X)$, and we set $R_x = D_x \backslash B_X$. In [Ku] the following useful characterization of the nearly uniform convexity was given:

Proposition 1.1. *A Banach space X is nearly uniformly convex if and only if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in X$ with $1 < \|x\| < 1 + \delta$, then $\sup\{\mu(C) : C \subset R_x,\ C \text{ convex}\} < \varepsilon$.*

In order to simplify the notation, if A is a bounded subset of X we write $\tilde{\mu}(A) = \sup\{\mu(C) : C \subset A,\ C \text{ convex}\}$. Obviously $\tilde{\mu}$ is not in general a measure of noncompactness. For example, if $X = \ell^p$ and $A = \{e_n : n \in \mathbb{N}\}$ where $e_n = (0, 0, \ldots, 1^{n)}, 0, \ldots)$, then $\|e_n - e_m\| = 2^{1/p}$ for every $n \neq m$ and so $\alpha(A) = 2^{1/p}$. However, $\tilde{\alpha}(A) = 0$ because A does not contain any convex set with more than one point.

We are going to define a new modulus for the near uniform convexity using this characterization. We shall need the following lemma.

Lemma 1.2. *Let X be a Banach space and $x \in X \backslash B_X$. Then*

$$\frac{\mu(B_X)(\|x\| - 1)}{\|x\|} \leq \tilde{\mu}(R_x) \leq \mu(B_X).$$

Proof. Since $R_x \subset \operatorname{co}(\{x\} \cup B_X)$, we deduce from the properties of μ that $\tilde{\mu}(R_x) \leq \mu(B_X)$.

In order to prove the second inequality, we consider two arbitrary real numbers δ and d such that $\delta > 0$ and $0 < d < 1$. For every $y \in S_X$ we define the set $M(dy, \delta) = B(dy, 1-d+\delta) \backslash B_X$. Observe that $\delta\mu(B_X) \leq \tilde{\mu}(M(dy, \delta))$.

Indeed, consider the closed ball $B(y, \delta)$ and let $z \in B(y, \delta)$. Then

$$\|z - dy\| = \|z - (y - (1-d)y)\| = \|(1-d)y + z - y\| \leq 1 - d + \delta.$$

Therefore $B(y, \delta) \subset B(dy, 1-d+\delta)$. Let f be a linear functional in the dual space X^* such that $\|f\| = f(y) = 1$. Then

$$F = \{x \in B(y, \delta) : f(x) > 1\} \subset M(dy, \delta).$$

Since $\mu(F) = \delta\mu(B_X)$ and F is convex, we deduce that

$$\delta\mu(B_X) \leq \tilde{\mu}(M(dy, \delta)).$$

Let now $t > 1$ and $0 < \delta < (t-1)/t$. Choosing $d = t\delta/(t-1)$, we obtain $0 < \delta < d < 1$ and $t = d/(d-\delta)$. Using this and [KP, Proposition 1] we conclude that $M(dy, \delta) \subset R_{ty}$ and so $\delta\mu(B_X) \leq \tilde{\mu}(M(dy, \delta)) \leq \tilde{\mu}(R_{ty})$.

Since δ is arbitrary up to the condition $0 < \delta < (t-1)/t$, we obtain $\mu(B_X)(t-1)/t \leq \tilde{\mu}(R_{ty})$, which is the required inequality for every $x \in X \backslash B_X$. □

Corollary 1.3. *For every $\varepsilon \in [0, \mu(B_X))$ and $y \in S_X$ there exists $t > 1$ such that*

$$\tilde{\mu}(R_{ty}) \geq \frac{\mu(B_X)(t-1)}{t} > \varepsilon.$$

Let us define now the following function:

Definition 1.4. Let X be a Banach space. We define the following modulus associated to the measure of noncompactness μ: $D_{X,\mu} : [0, \mu(B_X)) \to [0, +\infty)$ given by

$$D_{X,\mu}(\varepsilon) = \inf\{\|x\| - 1 : x \in X \backslash B_X,\ \tilde{\mu}(R_x) \geq \varepsilon\}.$$

The coefficient of noncompact convexity of X corresponding to this modulus is the number $D_{0,\mu}(X) = \sup\{\varepsilon \geq 0 : D_{X,\mu}(\varepsilon) = 0\}$. From Proposition 1.1 we conclude that a Banach space X is nearly uniformly convex if and only if $D_{0,\mu}(X) = 0$.

Proposition 1.5. *Let X be a Banach space. The mapping $D_{X,\mu}$ has the following properties:*

(a) *$D_{X,\mu}$ is nondecreasing in $[0,\mu(B_X))$ and $D_{X,\mu}(0)=0$.*
(b) *For every $\varepsilon\in[0,\mu(B_X))$ we have $0\le D_{X,\mu}(\varepsilon)\le\varepsilon/(\mu(B_X)-\varepsilon)$.*
(c) *$D_{X,\mu}$ is continuous at zero.*
(d) *For every $\varepsilon\in(0,\mu(B_X)/2)$ we have $D_{X,\mu}(\varepsilon)<2\varepsilon/\mu(B_X)$.*

Proof.
(a) It is a trivial consequence of the definition.
(b) Let $\varepsilon\in[0,\mu(B_X))$ and $x\in X\backslash B_X$ such that $\|x\|>\mu(B_X)/(\mu(B_X)-\varepsilon)$. Then $\varepsilon<\mu(B_X)(\|x\|-1)/\|x\|\le\tilde{\mu}(R_x)$, and so

$$\inf\{\|x\|:\tilde{\mu}(R_x)\ge\varepsilon\}=1+D_{X,\mu}(\varepsilon)\le\frac{\mu(B_X)}{\mu(B_X)-\varepsilon}.$$

Hence $D_{X,\mu}(\varepsilon)\le\varepsilon/(\mu(B_X)-\varepsilon)$.

(c) It follows from (b).

(d) If $\varepsilon\in(0,\mu(B_X)/2)$ then $\mu(B_X)-\varepsilon>\mu(B_X)/2$ and the result follows from (b). □

The following theorem shows the relationship between the new modulus defined above and the noncompact modulus of convexity in the general class of all Banach spaces for the measure of noncompactness β.

Theorem 1.6. *Let X be a Banach space. Then $\Delta_{0,\beta}(X)\le 2D_{0,\beta}(X)\le 4\Delta_{0,\beta}(X)$.*

Proof. In order to obtain the first inequality, it suffices to prove that

$$D_{X,\beta}(\varepsilon)\le 2(D_{X,\beta}(\varepsilon)+2)\Delta_{X,\beta}(\varepsilon)\quad\text{for all }\varepsilon\in\left[0,\frac{\mu(B_X)}{2}\right).$$

Since the inequality is obvious if $D_{X,\beta}(\varepsilon)=0$, we can suppose that $D_{X,\beta}(\varepsilon)=s>0$. Let r be a real number such that $0<r<s$ and A a convex subset of B_X with $\beta(A)>2\varepsilon$. Let us consider a sequence $\{x_n\}$ in A with $\operatorname{sep}(\{x_n\})\ge 2\varepsilon$. We are proving by contradiction that

$$\operatorname{co}(\{x_n\})\cap\left(1-\frac{r}{s+2}\right)B_X\neq\emptyset.$$

To reach a contradiction we take $y=(1+r)x_1$, $y_n=(y+x_n)/2$, $n\in\mathbb{N}$. Observe that $1<\|y\|<1+s$ and $\{y_n\}\subset R_y$. Indeed,

$$\|y\|=(1+r)\|x_1\|<1+s,$$

$$\|y\| > (1+r)\left(1-\frac{r}{s+2}\right) = 1+\frac{r(1+s-r)}{s+2} > 1,$$

$$\|y_n\| = \left\|\left(1+\frac{r}{2}\right)\left(\frac{1+r}{2+r}x_1+\frac{1}{2+r}x_n\right)\right\|$$
$$> \left(1+\frac{r}{2}\right)\left(1-\frac{r}{s+2}\right) = 1+\frac{r(s-r)}{2(s+2)} > 1.$$

Therefore $\{y_n\} \subset R_y$. Actually $\operatorname{co}(\{y_n\}) \subset R_y$. Indeed, for any choice y_{n_i}, $i=1,2,\ldots,k$ and positive coeffcients γ_i, $i=1,2,\ldots,k$, $\sum_{i=1}^k \gamma_i = 1$, we have

$$\sum_{i=1}^k \gamma_i y_{n_i} = \left(1+\frac{r}{2}\right)\left(\frac{1+r}{2+r}x_1+\frac{1}{2+r}\sum_{i=1}^k \gamma_i x_{n_i}\right).$$

Thus $\left\|\sum_{i=1}^k \gamma_i y_{n_i}\right\| > 1$ and so $\operatorname{co}(\{y_n\}) \subset R_y$. Since $\bar{\beta}(R_y) < \varepsilon$, we deduce that $\operatorname{sep}(\operatorname{co}(\{y_n\})) < \varepsilon$ and so $\operatorname{sep}(\{y_n\}) < \varepsilon$.

It follows that $\operatorname{sep}(\{x_n\}) < 2\varepsilon$ and this contradicts our assumption. Therefore, there exists $z \in \operatorname{co}(\{x_n\})$ such that

$$\|z\| \le 1-\frac{r}{s+2}.$$

Hence, we have proved that if A is a convex set contained in B_X with $\beta(A) > 2\varepsilon$, we have $\inf\{\|x\| : x \in A\} \le 1 - r/(s+2)$ and so

$$\sup\{\inf\{\|x\| : x \in A\} : A \subset B_X,\, A \text{ convex},\, \beta(A) > 2\varepsilon\} \le 1-\frac{r}{s+2}.$$

It follows that $r \le (s+2)\Delta_{X,\beta}(2\varepsilon)$ and since r was chosen arbitrarily smaller than $s = D_{X,\beta}(\varepsilon)$, we can conclude the required inequality.

In order to obtain the second inequality we are proving that

$$\Delta_{X,\beta}(\varepsilon) \le 2 \lim_{\varepsilon' \to \varepsilon^+} D_{X,\beta}(2\varepsilon') \text{ for all} \varepsilon \in [0, \beta(B_X)/2].$$

Suppose $\Delta_{X,\beta}(\varepsilon) > 0$. Let $x \in X \setminus B_X$ such that $1 < \|x\| < 1+\Delta_{X,\beta}(\varepsilon)/2$. Let $\{x_n\} \subset R_x$ such that $\operatorname{co}(\{x_n\}) \subset R_x$. Then if $\gamma_i \ge 0$, $i=1,2,\ldots,k$; $\sum_{i=1}^k \gamma_i = 1$, we have $\sum_{i=1}^k \gamma_i x_{n_i} \in R_x$ and so

$$1 < \left\|\sum_{i=1}^k \gamma_i x_{n_i}\right\| < 1+\frac{\Delta_{X,\beta}(\varepsilon)}{2}.$$

For every $n \in \mathbb{N}$ we define $y_n = \left(1 - \frac{\Delta_{X,\beta}(\varepsilon)}{2}\right) x_n$. Then

$$\|y_n\| \leq \left(1 - \frac{\Delta_{X,\beta}(\varepsilon)}{2}\right)\left(1 + \frac{\Delta_{X,\beta}(\varepsilon)}{2}\right) < 1$$

and so $\mathrm{co}(\{y_n\}) \subset B_X$. Moreover,

$$\left\|\sum_{i=1}^{k} \gamma_i y_{n_i}\right\| = \left(1 - \frac{\Delta_{X,\beta}(\varepsilon)}{2}\right)\left\|\sum_{i=1}^{k} \gamma_i x_{n_i}\right\| > 1 - \frac{\Delta_{X,\beta}(\varepsilon)}{2}$$

$$\Rightarrow \inf\{\|z\| : z \in \mathrm{co}(\{y_n\})\} \geq 1 - \frac{\Delta_{X,\beta}(\varepsilon)}{2} > 1 - \Delta_{X,\beta}(\varepsilon)$$

and consequently,

$$\beta(\mathrm{co}(\{x_n\})) = \frac{\beta(\mathrm{co}(\{y_n\}))}{1 - \frac{\Delta_{X,\beta}(\varepsilon)}{2}} \leq \frac{2\varepsilon}{2 - \Delta_{X,\beta}(\varepsilon)} \leq 2\varepsilon.$$

Therefore, we have proved that if $\{x_n\} \subset R_x$ and $\mathrm{co}(\{x_n\}) \subset R_x$ with $1 < \|x\| < 1 + \Delta_{X,\beta}(\varepsilon)/2$, then $\beta(\mathrm{co}(\{x_n\})) \leq 2\varepsilon$ and so $\tilde{\beta}(R_x) \leq 2\varepsilon$. This implies that

$$D_{X,\beta}(2\varepsilon') \geq \frac{\Delta_{X,\beta}(\varepsilon)}{2}$$

and the required inequality is obtained. □

From [ADL, Chapters V and VI] and Theorem 1.6 we obtain the following result which is useful in fixed point theory for nonexpansive mappings [Ki]:

Corollary 1.7. *If $D_{0,\beta}(X) < 1/2$, then the space X is reflexive and has normal structure.*

2. Computation of the modulus in Hilbert spaces.

Let us remember that if H is a Hilbert space, for every $x \in H$, $x \neq 0$, there exists a unique $f \in H^*$ with $\|f\| = 1$ such that $f(x) = \|x\|$. Hence, if $x \in S_H$ then there is a unique $f \in H^*$ such that $\|f\| = 1 = f(x)$. We shall denote this functional by G_x.

The purpose of the following lemmas is to calculate the measure $\tilde{\mu}$ of R_x in this class of spaces.

Lemma 2.1. *Let H be a separable and infinite dimensional Hilbert space, $x \in H$ with $\|x\| = 1$ and $\delta > 1$. Then*

$$\tilde{\mu}(R_{\delta x}) = \mu(\{z \in R_{\delta x} : G_x(z) \geq 1\}).$$

Proof. Let C be a convex set contained in $R_{\delta x}$. Using the Hahn-Banach theorem we obtain a functional $f \in H^*$ such that $\|f\| = 1$ and $f(z) \geq 1$ for every $z \in C$. Moreover, since H is reflexive and strictly convex, there exists a unique $y \in S_H$ such that $f(y) = 1$ and so, we write $f = G_y$.

Let us see that $G_y(\delta x) \geq 1$. Indeed, if $G_y(\delta x) < 1$, we take $z \in C$, $z = \lambda u + (1-\lambda)\delta x$, with $u \in B_H$ and $0 \leq \lambda < 1$ and we have

$$G_y(z) = \lambda G_y(u) + (1-\lambda) G_y(\delta x) < 1$$

which is a contradiction.

Let us consider now the following sets:

$$A = \{z \in R_{\delta x} : G_x(z) \geq 1\}, \quad B = \{z \in R_{\delta x} : G_y(z) \geq 1\},$$

$$A_1 = \{z \in R_{\delta x} : G_x(z) \geq 1, \; G_y(z) \leq 1\},$$
$$B_1 = \{z \in R_{\delta x} : G_x(z) \leq 1, \; G_y(z) \geq 1\},$$
$$D = \{z \in R_{\delta x} : G_x(z) \geq 1, \; G_y(z) \geq 1\}.$$

We have $A = A_1 \cup D$ and $B = B_1 \cup D$. In order to prove the proposition it suffices to obtain that $\mu(B_1) \leq \mu(A_1)$.

If $x = y$ then $A_1 = B_1$ and the result is obvious. So we can suppose that $x \neq y$. Let us consider the unique vector $u = \lambda x + \mu y$ such that $G_x(u) = 1$ and $G_y(u) = 1$. Let σ be the symmetry of centre u. We shall see that $\sigma(B_1) \subset A_1$.

Let $z \in B_1$. Then $\sigma(z) = z' = 2u - z$, $G_x(z') = 2 - G_x(z) \geq 1$ and $G_y(z') = 2 - G_y(z) \leq 1$. To obtain that $z' \in A_1$, we only have to show that $z' \in R_{\delta x}$.

If $z \in B_1$ with $G_y(z) > 1$ and $G_x(z) < 1$, then it is not difficult to check that the line passing through δx and z intersects the sets $H_x = \{v \in H : G_x(v) = 1\}$ and $H_y = \{v \in H : G_y(v) = 1\}$ at two points, say z_x and z_y, in such a way that z is on the segment line $[z_x, z_y]$. Since $A \cup \{x\}$ is a convex set, it is enough to prove the result for the points $z \in B_1 \cap H_x$, or $z \in B_1 \cap H_y$.

Let z be a point of $B_1 \cap H_x$. We can suppose without loss of generality that x, y and z are linearly independent vectors. Let $\{e_1, e_2, e_3, ...\}$ be a complete orthonormal system of H.

We consider the sequence $\{x, y, z, e_1, e_2, ...\}$. Applying the Gram-Schmidt orthonormalization process we obtain a Schauder basis $\{u_1, u_2, u_3, ...\}$ of H such that

$$x = (1, 0, 0, 0, ...),\quad y = (y_1, y_2, 0, 0, ...),\quad z = (z_1, z_2, z_3, 0, ...),$$

$$G_x = (1, 0, 0, 0, \ldots),\quad G_y = (y_1, y_2, 0, 0, \ldots).$$

Moreover, from the conditions for y and z, we have the following equalities and inequalities:

$$y_1^2 + y_2^2 = 1,$$
$$\frac{1}{\delta} \le y_1 < 1,$$
$$G_x(z) = 1 \Rightarrow z_1 = 1,$$
$$G_y(z) \ge 1 \Rightarrow y_1 + y_2 z_2 \ge 1,$$
$$u = \Big(1, \frac{1 - y_1}{y_2}, 0, 0, 0, \ldots\Big).$$

Furthermore, the symmetric point of $z = (1, z_2, z_3, 0, 0, \ldots)$ with respect to u is

$$z' = (z_1', z_2', z_3', 0, 0, ...) = \Big(1, \frac{2(1 - y_1)}{y_2} - z_2, -z_3, 0, 0, 0, \ldots\Big).$$

Since $y_1 + y_2 z_2 \ge 1$, it is not difficult to check that

$$\delta^2 \big(z_2'^2 + z_3'^2\big)^2 \le (\delta - z_1')^2 + z_2'^2 + z_3'^2 < \delta(\delta - z_1')$$

and this condition implies that the line $\{\lambda z' + (1 - \lambda)\delta x : \lambda \in \mathbb{R}\}$ intersects the unit ball in one or two points for $\lambda > 1$, that is, $z' \in R_{\delta x}$.

Similarly, if $z \in B_1 \cap H_y$ we obtain the same result, and so $\sigma(B_1) \subset A_1$. This implies that $\mu(B_1) \le \mu(A_1)$ and the proof is complete. □

Lemma 2.2. *Let H be a separable and infinite dimensional Hilbert space, $\{e_1, e_2, e_3, ...\}$ a complete orthonormal system of H and $\delta > 1$. Then:*

(a) *If $x \in H$ with $\|x\| = \delta > 1$, we have $\tilde{\mu}(R_x) = \tilde{\mu}(R_{\delta e_1})$.*

(b) *Let $S = \Big\{x \in H : x = \sum_{i=1}^{\infty} x_i e_i = (x_1, x_2, x_3, ...), \|x\| \le 1, x_1 = \frac{1}{\delta}\Big\}$. Then $\overline{co}(R_{\delta e_1}) = \overline{co}(S \cup \{\delta e_1\})$.*

(c) *Let $A = \{x \in R_{\delta e_1} : G_{e_1}(x) \ge 1\}$ and h the homothety with centre δe_1 and ratio $\frac{\delta + 1}{\delta}$. Then $h(A \cup \{e_1\}) = co(S \cup \{\delta e_1\})$.*

Proof. (a) Let us consider the sequence $\{x, e_1, e_2, e_3, ...\}$. Applying the Gram-Schmidt orthonormalization process we can obtain a Schauder basis $\{u_1, u_2, u_3, ...\}$ of H such that $x = \delta u_1$. The mapping $f : H \to H$ defined by $f(u_n) = e_n$ for every $n \in \mathbb{N}$ is an isometry such that $f(x) = f(\delta u_1) = \delta e_1$. Hence, we have $f(\operatorname{co}(\{x\} \cup B_H) = \operatorname{co}(f(x) \cup f(B_H)) = \operatorname{co}(\{\delta e_1\} \cup B_H)$. Therefore $f(R_x) = R_{\delta e_1}$ and we have just proved (a).

(b) Let $C = \{x \in S : \|x\| = 1\}$. We shall prove the following facts:
(i) $S = \operatorname{co}(C)$.
(ii) $C \subset \overline{\operatorname{co}}(R_{\delta e_1})$.
(iii) $R_{\delta e_1} \subset \operatorname{co}(\{\delta e_1\} \cup S)$.
Indeed,

(i) It suffices to note that $S - e_1/\delta$ is a closed ball, centered at 0 with radius $\left(1 - 1/\delta^2\right)^{1/2}$ and $C - e_1/\delta$ is the corresponding sphere.

(ii) If $x \in C$ then $G_x(\delta e_1) = 1$ and so $x \in \overline{\operatorname{co}}(R_{\delta e_1})$ (see [AF, Lemma 2.1]).

(iii) Let $z \in R_{\delta e_1}$. Then, we can write $z = \delta e_1 + \lambda(y - \delta e_1)$ with $y = (y_1, y_2, y_3, ...) = y_1 e_1 + y'$, $\|y\| = 1$, $y_1 \geq 1/\delta$, $G_y(\delta e_1) \geq 1$ and $0 \leq \lambda < 1$ (see [ADF, Lemma 2.1] and [AF, Lemma 2.1 and Example]). It is easy to check that the line $\{\delta e_1 + t(y - \delta e_1) : t \in \mathbb{R}\}$ intersects S when $t = (\delta - 1/\delta)/(\delta - y_1) \geq 1$, and so $z \in \operatorname{co}(\{\delta e_1\} \cup S)$. Hence $R_{\delta e_1} \subset \operatorname{co}(\{\delta e_1\} \cup S)$.

From (i) and (ii) we deduce that $S \subset \overline{\operatorname{co}}(R_{\delta e_1})$ and so, $\operatorname{co}(\{\delta e_1\} \cup S) \subset \overline{\operatorname{co}}(R_{\delta e_1})$. The converse follows from (iii).

(c) The equation of the homothety is the following:

$$h(x) = x' = \frac{\delta + 1}{\delta} x - e_1.$$

Let now $x \in A$. Then $x \in R_{\delta e_1}$ and $x_1 \geq 1$. Set $x = tz + (1 - t)\delta e_1$ with $z \in S$, $z = (1/\delta, z_2, z_3, ...)$ and $0 < t \leq 1$. Then

$$h(x) = \frac{t\delta + t}{\delta} z + \frac{\delta - \delta t - t}{\delta} \delta e_1 = t_0 z + t_1 \delta e_1,$$

where $t_0 \geq 0$, $t_1 \geq 0$ and $t_0 + t_1 = 1$. So $h(x) \in \operatorname{co}(S \cup \{\delta e_1\})$.

Conversely, let $x' \in \operatorname{co}(S \cup \{\delta e_1\})$, $x' = sz + (1 - s)\delta e_1$ with $z \in S$ and $0 \leq s \leq 1$. Then x' is the image of $x = \frac{s\delta}{\delta+1} z + \frac{\delta}{\delta+1}(1 - s + \frac{1}{\delta})\delta e_1 = s_0 z + s_1 \delta e_1$ with $0 \leq s_0 < 1$, $0 < s_1 \leq 1$ and $s_0 + s_1 = 1$. Moreover,

$$\frac{s\delta}{\delta + 1} \frac{1}{\delta} + \frac{\delta}{\delta + 1}(1 - s + \frac{1}{\delta})\delta = (\delta - 1)(1 - s) + 1 \geq 1$$

and so $x \in A$. □

Corollary 2.3. *Let H be a separable and infinite dimensional Hilbert space and $x \in H$ with $\|x\| > 1$. Then*

$$\tilde{\alpha}(R_x) = 2\left(\frac{\|x\|-1}{\|x\|+1}\right)^{1/2},$$

$$\tilde{\chi}(R_x) = \left(\frac{\|x\|-1}{\|x\|+1}\right)^{1/2},$$

$$\tilde{\beta}(R_x) = \left(\frac{2(\|x\|-1)}{\|x\|+1}\right)^{1/2}.$$

Proof. The result follows from [ADF, Theorem 2.4 and Corollary 2.5] and the equality $\tilde{\mu}(R_x) = \tilde{\mu}(R_{\delta e_1}) = \frac{\delta}{\delta+1}\mu(R_{\delta e_1})$ with $\delta = \|x\|$ and μ any of the measures of noncompactness α, χ or β. □

Theorem 2.4. *Let H be a separable and infinite dimensional Hilbert space. Then*

$$D_{H,\alpha}(\varepsilon) = \frac{2\varepsilon^2}{4-\varepsilon^2}, \quad \varepsilon \in [0,2),$$

$$D_{H,\chi}(\varepsilon) = \frac{2\varepsilon^2}{1-\varepsilon^2}, \quad \varepsilon \in [0,1),$$

$$D_{H,\beta}(\varepsilon) = \frac{2\varepsilon^2}{2-\varepsilon^2}, \quad \varepsilon \in [0,\sqrt{2}).$$

Proof. We present the proof only for the measure α. The other cases are similar. Since the mapping $\delta \to 2[(\delta-1)/(\delta+1)]^{1/2}$ is strictly increasing, the infimum in the definition of $D_{H,\alpha}(\varepsilon)$ is attained for $2[(\delta-1)/(\delta+1)])^{1/2} = \varepsilon$, that is for $\delta = (4+\varepsilon^2)/(4-\varepsilon^2)$. Hence $D_{H,\alpha}(\varepsilon) = (4+\varepsilon^2)/(4-\varepsilon^2) - 1 = 2\varepsilon^2/(4-\varepsilon^2)$. □

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