ANNALES
UNIVERSITATIS MARIAE CURIE - SKŁODOWSKA
LUBLIN - POLONIA

VOL. LIII, 19 SECTIO A 1999

HITOSHI SAITOH

Univalence and starlikeness of solutions of $W'' + aW' + bW = 0$

ABSTRACT. We consider the differential equation

$$w''(z) + a(z)w'(z) + b(z)w(z) = 0,$$

where $a(z)$ and $b(z)$ are analytic in the unit disc Δ. We show that this differential equation has a solution $w(z)$ univalent and starlike in Δ under some conditions imposed on $a(z)$ and $b(z)$. It is related to results of S. S. Miller and M. S. Robertson.

1. Introduction. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be an analytic function defined in the unit disc $\Delta = \{z : |z| < 1\}$. We denote the class of such functions by A. If in addition $f(z)$ is univalent, then we say $f(z) \in S$. If $f'(z) \neq 0$ in Δ, then we define

$$S(f,z) = \left(\frac{f''}{f'}\right)'(z) - \frac{1}{2}\left(\frac{f''}{f'}(z)\right)^2$$

to be the *Schwarzian derivative* of $f(z)$.

Our starting point is the following result of S. S. Miller.

Theorem A (Miller [4]). *Let $p(z)$ be analytic in the unit disc Δ with $|zp(z)| < 1$. Let $v(z)$, $z \in \Delta$, be the unique solution of*

$$v''(z) + p(z)v(z) = 0 \tag{1.1}$$

with $v(0) = 0$ and $v'(0) = 1$. Then

$$\left| \frac{zv'(z)}{v(z)} - 1 \right| < 1, \tag{1.2}$$

and $v(z)$ is a starlike conformal map of the unit disc.

Theorem A is related to the next results of M. S. Robertson and Z. Nehari.

Theorem B (Robertson [8]). *Let $zp(z)$ be analytic in Δ and*

$$\operatorname{Re}\left\{z^2 p(z)\right\} \le \frac{\pi^2}{4}|z|^2 \qquad (z \in \Delta). \tag{1.3}$$

Then the unique solution $W = W(z)$, $W(0) = 0$, $W'(0) = 1$ of

$$W''(z) + p(z)W(z) = 0 \tag{1.4}$$

is univalent and starlike in Δ. The constant $\pi^2/4$ is best possible.

Theorem C (Nehari [6]). *If $f(z) \in A$ satisfies*

$$|S(f,z)| \le \frac{\pi^2}{2} \qquad (z \in \Delta), \tag{1.5}$$

then $f(z)$ is univalent. The result is sharp.

Remark 1. The constant $\pi^2/2$ is best possible as shown by the example $[e^{i\pi z} - 1]/i\pi$. We note that setting $p(z) = \frac{1}{2}S(f,z)$ and using (1.5) we obtain (1.3). Therefore, Nehari's theorem has a stronger hypothesis. Robertson proved that the unique solution of the equation (1.4) is starlike whereas Nehari proved the quotient of the linearly independent solution of (1.4) is univalent.

We have also

Theorem D (Gabriel [2]). *Suppose* $f(z) \in A$ *and*

$$|S(f,z)| \le 2c_0 \approx 2.73 \qquad (z \in \Delta), \tag{1.6}$$

where c_0 *is the smallest positive root of the equation* $2\sqrt{x} - \tan\sqrt{x} = 0$. *Then* $f(z)$ *maps* Δ *onto a starlike domain.*

Recall that $f(z) \in S$ is starlike with respect to the origin if and only if $\operatorname{Re}\{zf'(z)/f(z)\} > 0$ for all $z \in \Delta$. We denote the class of starlike functions by S^*.

2. A class of bounded functions. Let B_J denote the class of bounded functions $w(z) = w_1 z + w_2 z^2 + \cdots$ analytic in the unit disc Δ for which $|w(z)| < J$. If $g(z) \in B_J$, then by using the Schwarz lemma we can show that the function $w(z)$ defined by $w(z) = z^{-\frac{1}{2}} \int_0^z g(t) t^{-\frac{1}{2}} dt$ is also in B_J. Writing this result in terms of derivatives we have

$$\left|\frac{1}{2}w(z) + zw'(z)\right| < J \quad (z \in \Delta) \quad \Rightarrow \quad |w(z)| < J \quad (z \in \Delta). \tag{2.1}$$

If we set $h(u,v) = \frac{1}{2}u + v$, we can write (2.1) as an implication

$$|h(w(z), zw'(z))| < J \quad \Rightarrow \quad |w(z)| < J. \tag{2.2}$$

In this section we show that (2.2) holds for functions $h(u,v)$ satisfying the following definition.

Definition 1. Let H_J be the set of complex functions $h(u,v)$ satisfying:

(i) $h(u,v)$ is continuous in a domain $D \subset \mathbb{C} \times \mathbb{C}$,

(ii) $(0,0) \in D$ and $|h(0,0)| < J$,

(iii) $|h(Je^{i\theta}, Ke^{i\theta})| \ge J$ when $(Je^{i\theta}, Ke^{i\theta}) \in D$, θ is real and $K \ge J$.

Example 1. It is easy to check that the following function $h(u,v)$ is in H_J: $h(u,v) = \alpha u + v$ where α is complex with $\operatorname{Re}\alpha \ge 0$, and $D = \mathbb{C} \times \mathbb{C}$.

Definition 2. Let $h \in H_J$ with corresponding domain D. We denote by $B_J(h)$ the class those functions $w(z) = w_1 z + w_2 z^2 + \cdots$ which are analytic in Δ and satisfy

(i) $(w(z), zw'(z)) \in D$,

(ii) $|h(w(z), zw'(z))| < J \qquad (z \in \Delta)$.

The set $B_J(h)$ is not empty, since for any $h \in H_J$ we have $w(z) = w_1 z \in B_J(h)$ for $|w_1|$ sufficiently small depending on h.

We need the following lemma to prove our results.

Lemma 1 (Miller and Mocanu [5]). *Let* $w(z) = w_1 z + w_2 z^2 + \cdots$ *be analytic in* Δ *with* $w(z) \not\equiv 0$. *If* $z_0 = r_0 e^{i\theta_0}$, $0 < r_0 < 1$, *and* $|w(z_0)| = \max_{|z|\le r_0} |w(z)|$, *then*

(i)
$$\frac{z_0 w'(z_0)}{w(z_0)} = m$$

and

(ii)
$$\mathrm{Re}\left[\frac{z_0 w''(z_0)}{w'(z_0)}\right] + 1 \ge m,$$

where $m \ge 1$.

Theorem 1. *For any* $h \in H_J$, *we have* $B_J(h) \subset B_J$.

Proof. Let $w(z) \in B_J(h)$. Suppose that $\exists z_0 = r_0 e^{i\varphi_0} \in \Delta$ $(0 < r_0 < 1)$ such that

$$\max_{|z|\le r_0} |w(z)| = |w(z_0)| = J.$$

Then $w(z_0) = Je^{i\theta}$ and by Lemma 1

$$z_0 w'(z_0)/w(z_0) = m \ge 1,$$

we have $z_0 w'(z_0) = Ke^{i\theta}$, $(K = mJ \ge J)$ and thus

$$h(w(z_0), z_0 w'(z_0)) = h(Je^{i\theta}, Ke^{i\theta}).$$

Since $h \in H_J$, this implies

$$|h\left(w(z_0), z_0 w'(z_0)\right)| \ge J$$

which contradicts $w(z) \in B_J(h)$. Hence $|w(z)| < J$ $(z \in \Delta)$, and thus $w(z) \in B_J$. $\square$

Remark 2. In other words, Theorem 1 shows that if $h \in H_J$, with corresponding domain D and if $w(z) = w_1 z + w_2 z^2 + \cdots$ is analytic in Δ and $(w(z), zw'(z)) \in D$, then

$$|h(w(z), zw'(z))| < J \quad \Rightarrow \quad |w(z)| < J.$$

Furthermore, Theorem 1 can be used to show that certain first order differential equations have bounded solutions. The proof of the following theorem follows immediately from Theorem 1.

Theorem 2. *Let $h \in H_J$ and $b(z)$ be an analytic function in Δ with $|b(z)| < J$. If the differential equation $h(w(z), zw'(z)) = b(z)$, $(w(0) = 0)$ has a solution $w(z)$ analytic in Δ, then $|w(z)| < J$.*

3. Main results.

Our main result is the following theorem.

Theorem 3. *Let $a(z)$ and $b(z)$ be analytic in Δ with*

$$\left| z\left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}a^2(z)\right)\right| < \frac{1}{2}$$

and $|a(z)| < 1$. Let $w(z)$ $(z \in \Delta)$ be the solution of the following second order linear differential equation

$$w''(z) + a(z)w'(z) + b(z)w(z) = 0 \tag{3.1}$$

with $w(0) = 0$, $w'(0) = 1$. Then $w(z)$ is starlike in Δ.

Proof. The transformation

$$w(z) = \exp\left(-\frac{1}{2}\int_0^z a(\xi)d\xi\right) v(z) \tag{3.2}$$

leads to the normal form

$$v''(z) + \left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}a^2(z)\right) v(z) = 0 \tag{3.3}$$

and $v(0) = 0$, $v'(0) = 1$. If we put

$$u(z) = \frac{zv'(z)}{v(z)} - 1 \qquad (z \in \Delta), \tag{3.4}$$

then $u(z)$ is analytic in Δ, $u(0) = 0$ and (3.3) becomes

$$u^2(z) + u(z) + zu'(z) = -z^2\left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}a^2(z)\right), \tag{3.5}$$

or equivalently

$$h(u(z), zu'(z)) = -z^2\left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}a^2(z)\right), \tag{3.6}$$

where $h(u,v) = u^2 + u + v$. It is easy to check $h(u,v) \in H_{\frac{1}{2}}$. i.e.,

(i) $h(u,v)$ is continuous in $D=\mathbb{C}\times\mathbb{C}$,
(ii) $(0,0)\in D, |h(0,0)|=0<\frac{1}{2}$
(iii) $\left|h\left(\frac{1}{2}e^{i\theta}, Ke^{i\theta}\right)\right|\geq\frac{1}{2}\qquad\left(K\geq\frac{1}{2}\right)$.

From assumption, we have

$$\left|-z^2\left(b(z)-\frac{1}{2}a'(z)-\frac{1}{4}a^2(z)\right)\right|<\frac{1}{2}\qquad(z\in\Delta).$$

By using Theorem 2, we have $|u(z)|<1/2, (z\in\Delta)$. Therefore, we obtain

$$\left|\frac{zv'(z)}{v(z)}-1\right|<\frac{1}{2}\qquad(z\in\Delta).$$

This implies

$$\frac{1}{2}<\mathrm{Re}\left\{\frac{zv'(z)}{v(z)}\right\}<\frac{3}{2}\qquad(z\in\Delta). \tag{3.7}$$

From (3.2), we have

$$\exp\left(\frac{1}{2}\int_0^z a(\xi)d\xi\right)w(z)=v(z). \tag{3.8}$$

Logarithmically differentiating of (3.8) leads to

$$\frac{zw'(z)}{w(z)}=\frac{zv'(z)}{v(z)}-\frac{z}{2}a(z). \tag{3.9}$$

Combining (3.9) and $|a(z)|<1$, we obtain

$$\mathrm{Re}\left\{\frac{zw'(z)}{w(z)}\right\}\geq\mathrm{Re}\left\{\frac{zv'(z)}{v(z)}\right\}-\frac{1}{2}|za(z)|>0\qquad(z\in\Delta),$$

and thus $w(z)$ is starlike in Δ. □

Example 2. Let $a(z)=-z$, $b(z)=z^2/4$ in Theorem 3, then the solution of

$$w''(z)-zw'(z)+\frac{z^2}{4}w(z)=0 \tag{3.10}$$

is $w(z)=\sqrt{2}\exp(z^2/4)\sin(z/\sqrt{2})\in S^*$.

Let $a(z)=-z$, $b(z)=\lambda$ $(\lambda\in\mathbb{C})$ in Theorem 3, then differential equation (3.1) is

$$w''(z)-zw'(z)+\lambda w(z)=0. \tag{3.11}$$

The differential equation (3.11) is called Hermite's differential equation.

By the transformation $w(z)=v(z)\exp(z^2/4)$, (3.11) lead to

$$v''(z)+\left(\lambda+\frac{1}{2}-\frac{z^2}{4}\right)v(z)=0. \tag{3.12}$$

This differential equation is a well-known, *Weber's equation* (see [9]).

Theorem 4. *We consider Weber's differential equation* (3.12). *If*

$$\left|\lambda + \frac{1}{2} - \frac{z^2}{4}\right| < 1,$$

then the solution $v(z)$ *is starlike in* Δ.

Proof. We put

$$u(z) = \frac{zv'(z)}{v(z)} - 1. \tag{3.13}$$

Then $u(z)$ is analytic in Δ, $u(0) = 0$ and

$$u^2(z) + u(z) + zu'(z) = -z^2\left(\lambda + \frac{1}{2} - \frac{z^2}{4}\right) \tag{3.14}$$

or equivalently

$$h(u(z), zu'(z)) = -z^2\left(\lambda + \frac{1}{2} - \frac{z^2}{4}\right), \tag{3.15}$$

where $h(u, v) = u^2 + u + v$. It is easy to check $h(u, v) \in H_1$, i.e.,

(i) $h(u, v)$ is continuous in $D = \mathbb{C} \times \mathbb{C}$,

(ii) $(0, 0) \in D$, $|h(0, 0)| = 0 < 1$

(iii) $|h(e^{i\theta}, Ke^{i\theta})| \geq 1 \qquad (K \geq 1)$.

From assumption we have

$$\left|-z^2\left(\lambda + \frac{1}{2} - \frac{z^2}{4}\right)\right| < 1 \qquad (z \in \Delta).$$

By using Theorem 2, we obtain $|u(z)| < 1$, $(z \in \Delta)$. Therefore, this shows that

$$\left|\frac{zv'(z)}{v(z)} - 1\right| < 1,$$

which implies $\operatorname{Re}\{zv'(z)/v(z)\} > 0$ $(z \in \Delta)$, so $v(z)$ is starlike in Δ. □

Remark 3. The solutions of Weber's differential equation

$$v''(z) + \left(\lambda + \frac{1}{2} - \frac{z^2}{4}\right) v(z) = 0$$

are

(3.16)

$$D_\lambda(z) = 2^{\frac{\lambda}{2}} \sqrt{\pi} e^{-\frac{z^2}{4}} \left[\frac{1}{\Gamma\left(\frac{1-\lambda}{2}\right)} F\left(-\frac{\lambda}{2}, \frac{1}{2}; \frac{z^2}{2}\right) - \frac{\sqrt{2}z}{\Gamma\left(-\frac{\lambda}{2}\right)} F\left(\frac{1-\lambda}{2}, \frac{3}{2}; \frac{z^2}{2}\right)\right]$$

(*Weber's function*), where F is the confluent hypergeometric function. The following $D_{\frac{1}{4}}(z)$, $D_{-\frac{1}{4}}(z)$ are the solutions of (3.12) in Theorem 4.

$$D_{\frac{1}{4}}(z) = 2^{\frac{1}{8}}\sqrt{\pi}e^{-\frac{z^2}{4}}\left[\frac{1}{\Gamma\left(\frac{3}{8}\right)}F\left(-\frac{1}{8},\frac{1}{2};\frac{z^2}{2}\right) - \frac{\sqrt{2}z}{\Gamma\left(-\frac{1}{8}\right)}F\left(\frac{3}{8},\frac{3}{2};\frac{z^2}{2}\right)\right]$$

$$D_{-\frac{1}{4}}(z) = 2^{-\frac{1}{8}}\sqrt{\pi}e^{-\frac{z^2}{4}}\left[\frac{1}{\Gamma\left(\frac{5}{8}\right)}F\left(\frac{1}{8},\frac{1}{2};\frac{z^2}{2}\right) - \frac{\sqrt{2}z}{\Gamma\left(\frac{1}{8}\right)}F\left(\frac{5}{8},\frac{3}{2};\frac{z^2}{2}\right)\right]$$

References

1. Duren, P. L., *Univalent Functions*, Springer-Verlag, New York, 1983.
2. Gabriel, R. F., *The Schwarzian derivative and convex functions*, Proc. Amer. Math. Soc. **6** (1955), 58–66.
3. Hille, E., *Ordinary Differential Equations in the Complex Plane*, Wiley, New York, 1976.
4. Miller, S. S., *A class of differential inequalities implying boundedness*, Illinois J. Math. **20** (1976), 647–649.
5. Miller, S. S. and P. T. Mocanu, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl. **65** (1978), 289–305.
6. Nehari, Z., *The Schwarzian derivative and schlicht functions*, Bull. Amer. Math. Soc. **55** (1949), 545–551.
7. Pommerenke, Ch., *Univalent Functions*, Vanderhoeck and Ruprecht, Göttingen, 1975.
8. Robertson, M. S., *Schlicht solution of $W'' + pW = 0$*, Trans. Amer. Math. Soc. **76** (1954), 254–274.
9. Whittaker, E. T. and G. N. Watson, *A Course of Modern Analysis*, Cambridge Univ. Press, 1902.

Department of Mathematics
Gunma National College of Technology
Maebashi, Gunma 371-8530, Japan

received December 19, 1998