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Uniformly starlike and convex functions and other related classes of univalent functions

ABSTRACT. In this paper we investigate the classes of functions called uniformly convex and uniformly starlike, and some related classes of univalent functions. We also introduce a class of functions $ST(\zeta)$ which is given by the property that the image of any circular arc centered at ζ and contained in the unit disk U is starlike with respect to $f(\zeta)$. These functions are normalized so that $f(\zeta) = f'(\zeta) - 1 = 0$. Corresponding to $ST(\zeta)$ we define $CV(\zeta)$ such that $f \in CV(\zeta)$ if and only if $(z-\zeta)f'(z) \in ST(\zeta)$.

1. Introduction. Denote by H the class of functions f analytic in the unit disk U and normalized by $f(0) = f'(0) - 1 = 0$. Let S denote the class of functions in H that are univalent in U. Further on, let UST and UCV denote the classes of uniformly starlike and uniformly convex functions characterized by

$$(1.1) \qquad UST = \left\{ f \in S : \operatorname{Re} \frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)} > 0\,,\ (z,\zeta) \in U \times U \right\}$$

and

$$(1.2) \qquad UCV = \left\{ f \in S : \operatorname{Re} \left[1 + (z-\zeta)\frac{f''(z)}{f'(z)} \right] > 0\,,\ (z,\zeta) \in U \times U \right\}.$$

A geometric characterization of these classes is that the class UST (respectively UCV) is the collection of functions f which map each circular arc with center at the point $\zeta \in U$ onto an arc which is starlike with respect to $f(\zeta)$ (respectively convex) (see [2], [3]). We shall also consider the class

$$S_{\mathrm{par}} = \left\{ f \in S : \left| \frac{zf'(z)}{f(z)} - 1 \right| \le \operatorname{Re} \frac{zf'(z)}{f(z)} \,,\ z \in U \right\}, \tag{1.3}$$

introduced in [5], and the well known functions starlike of order α, defined by

$$S_\alpha^* = \left\{ f \in S : \operatorname{Re} \frac{zf'(z)}{f(z)} \ge \alpha \,,\ z \in U \right\}.$$

2. Uniformly convex functions and the class S_{par}. The original analytic characterization of UCV by Goodman [2] was the one given in (1.2), in terms of two variables. Later a one-variable characterization of UCV was found [4], [5], namely

$$g \in UCV \iff \operatorname{Re}\left[1 + \frac{zg''(z)}{g'(z)}\right] \ge \left| \frac{zg''(z)}{g'(z)} \right| \,,\ z \in U \,. \tag{2.1}$$

From this the class S_{par} in a natural way emerged as the class of functions with the property that $g \in UCV \iff zg' \in S_{\mathrm{par}}$. Many properties are easier to obtain from a one-variable characterization than from a two-variable characterization, and therefore (2.1) has been very helpful in the investigation of the classes UCV and S_{par}. However, in some cases it may be helpful to have a description in terms of two or more variables. Our first result is a three-variable characterization of UCV that resembles the two-variable characterization of the classical convex functions given by Sheil-Small [7] and Suffridge [8].

Theorem 2.1. *Let f be analytic in U. Then $f \in UCV$ if and only if*

$$\operatorname{Re} F(z,\zeta,\eta) \ge 0 \,,\ z,\zeta,\eta \in U \,, \tag{2.2}$$

where

$$F(z,\zeta,\eta) = \begin{cases} \frac{2(z-\zeta)f'(z)}{f(z)-f(\eta)} - \frac{z+\eta-2\zeta}{z-\eta} & \text{for } z \ne \eta \\ 1 + \frac{(z-\zeta)f''(z)}{f'(z)} & \text{for } z = \eta. \end{cases}$$

Proof. We first observe that

$$\lim_{\eta \to z} \left(\frac{2(z-\zeta)f'(z)}{f(z)-f(\eta)} - \frac{z+\eta-2\zeta}{z-\eta} \right) = 1 + \frac{(z-\zeta)f''(z)}{f'(z)}$$

so that $F(z,\zeta,\eta)$ is continuous and hence analytic in z,ζ and η. It is clear that (2.2) implies $f \in UCV$.

Now suppose $f \in UCV$. We will show that (2.2) holds. If $z = \eta$ then (2.2) clearly holds. Consider then the case $z \neq \eta$, but $|z-\zeta| = |\eta - \zeta| = r$. Since $f \in UCV$, the part of the arc $z(t) = \zeta + re^{it}$ which lies inside U will be mapped onto a convex arc containing $f(\eta)$. A convex arc is starlike with respect to all points in its interior or on its boundary, so therefore

$$\operatorname{Re} \frac{(z-\zeta)f'(z)}{f(z)-f(\eta)} \geq 0 .$$

Moreover,

$$\operatorname{Re} \frac{z+\eta-2\zeta}{z-\eta} = \operatorname{Re} \frac{(z-\zeta)+(\eta-\zeta)}{(z-\zeta)-(\eta-\zeta)} = 0 ,$$

for $|z-\zeta| = r$, $z \neq \eta$.

From this we conclude that $\operatorname{Re} F(z,\zeta,\eta) \geq 0$ when $|z-\zeta| = |\eta-\zeta| = r$. Since the function $\operatorname{Re} F(z,\zeta,\eta)$ is a harmonic function in z for fixed ζ and η, an application of the minimum principle gives (2.2) in the case $|z-\zeta| < |\eta - \zeta|$. Similarly (2.2) holds when $|z-\zeta| > |\eta-\zeta|$, and the proof is complete.■

The next result is an alternative two-variable characterization of UCV, obtained from Theorem 2.1 in the same way as (2.1) was obtained from (1.2).

Corollary 2.2. *The $f \in UCV$ if and only if*

(2.3)
$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)-f(\eta)} + \frac{\eta}{\eta - z}\right] > \frac{1}{2} + \left|\frac{zf'(z)}{f(z)-f(\eta)} + \frac{z}{\eta-z}\right| , \ (z,\eta) \in U \times U .$$

Proof. Assume $z \neq \eta$, and write $F(z,\zeta,\eta)$ as

$$\begin{aligned} F(z,\zeta,\eta) &= \frac{2zf'(z)}{f(z)-f(\eta)} - \frac{z+\eta}{z-\eta} - \left[\frac{2\zeta f'(z)}{f(z)-f(\eta)} - \frac{2\zeta}{z-\eta}\right] \\ &= 2\left[\frac{zf'(z)}{f(z)-f(\eta)} + \frac{\eta}{\eta-z}\right] - \left[1 + \frac{2\zeta f'(z)}{f(z)-f(\eta)} + \frac{2\zeta}{\eta-z}\right]. \end{aligned}$$

Then we get $\operatorname{Re} F(z,\zeta,\eta) \geq 0$ if and only if

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)-f(\eta)} + \frac{\eta}{\eta-z}\right] \geq \frac{1}{2} + \operatorname{Re}\left[\frac{\zeta f'(z)}{f(z)-f(\eta)} + \frac{\zeta}{\eta - z}\right].$$

Choose $\zeta = e^{i\alpha} z$ such that

$$\operatorname{Re}\left[\frac{\zeta f'(z)}{f(z)-f(\eta)} + \frac{\zeta}{\eta-z}\right] = \left|\frac{zf'(z)}{f(z)-f(\eta)} + \frac{z}{\eta-z}\right|$$

and (2.3) follows. Assume that (2.3) holds. Clearly (2.3) implies (2.2) if $|z| > |\zeta|$. Again, applying the minimum principle for harmonic functions we see that this implies $\operatorname{Re} F(z,\zeta,\eta) \geq 0$ for all $z,\zeta,\eta) \in U$, hence $f \in UCV$. Taking the limit as $\eta \to z$ in (2.3), we see that this inequality turns into

$$\operatorname{Re}\left[1+\frac{1}{2}\frac{zf''(z)}{f'(z)}\right] > \frac{1}{2}+\frac{1}{2}\left|\frac{zf''(z)}{f'(z)}\right|,$$

which is equivalent to (2.1). Hence, the result holds also in the case $\eta = z$. ■

These alternative characterizations of UCV can be used to derive some new properties of UCV which we state as

Corollary 2.3. *Assume $f \in UCV$. Then*

$$\operatorname{Re}\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)} > \frac{1}{2},\ (z,\zeta) \in U \times U, \tag{2.4}$$

$$\operatorname{Re}\left[\frac{(z-\zeta)f'(z)}{f(z)} + \frac{\zeta}{z}\right] > \frac{1}{2},\ (z,\zeta) \in U \times U \tag{2.5}$$

and

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \operatorname{Re}\frac{zf'(z)}{f(z)} - \frac{1}{2},\ z \in U. \tag{2.6}$$

Proof. The inequality (2.4) follows from (2.2) by taking $\eta = \zeta$ and (2.5) follows from (2.2) by taking $\eta = 0$. Finally (2.6) follows from (2.3) by taking $\eta = 0$. In fact, (2.6) can also be obtained directly from (2.5) in the same way as (2.1) is obtained from (1.2). ■

It is obvious that any uniformly convex function is uniformly starlike. If one were to introduce a concept "order of uniform starlikeness", then the inequality (2.4) can be read as 'any uniformly convex function is uniformly starlike of order 1/2', a result corresponding to the classical result that any convex function is starlike of order 1/2 (in the usual sense). In [6] the classes $S_{\text{par}}(\alpha)$ were introduced as functions with the property

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \operatorname{Re}\frac{zf'(z)}{f(z)} - \alpha,\ z \in U.$$

It is easily seen that $f \in S_{\text{par}}(\alpha)$ implies that f is starlike of order $(1+\alpha)/2$. The statement in (2.6) is that if $f \in UCV$ then $f \in S_{\text{par}}(1/2)$, and from the above we then get the following corollary from (2.6).

Corollary 2.4. *If $f \in UCV$ then $f \in S^*_{3/4}$.*

Remark. From (2.1) we see that for $f \in UCV$ we have

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} \geq \frac{1}{2},$$

hence a function in UCV is convex of order $1/2$. Given the order of convexity β, we can compute the order of starlikeness $\alpha = \alpha(\beta)$ using a result by Wilken and Feng [10]. For $\beta = 1/2$ this result gives $\alpha(1/2) = 1/(2\log 2) = 0.72\ldots$. We see from Corollary 2.4 that replacing convexity of order $1/2$ by uniform convexity increases the order of starlikeness even further.

Corollary 2.5. *Let $f \in UCV$ and $\zeta \in U$. Then for each fixed $\eta \in U$ the function*

$$g(z) = (z-\zeta)\left(\frac{f(z)-f(\eta)}{z-\eta}\right)^2 + \zeta\left(\frac{f(\eta)}{\eta}\right)^2 \tag{2.7}$$

maps $|z-\zeta| = r$, $r < 1-|\zeta|$, onto curves that are starlike with respect to $g(\zeta) = \zeta(f(\eta)/\eta)^2$.

Proof. Given $\zeta, \eta \in U$, $f \in UCV$ and let the function g be defined by (2.7). Then g is analytic in U and $g(0) = 0$. Moreover

$$\frac{(z-\zeta)g'(z)}{g(z)-g(\zeta)} = \frac{2(z-\zeta)f'(z)}{f(z)-f(\eta)} - \frac{z+\eta-2\zeta}{z-\eta}.$$

Hence, using Theorem 2.1 we obtain the desired result. ∎

Because of the close connection between UCV and S_{par} it is clear that many results about UCV can be directly translated to results about S_{par}. We will not include such results. The class S_{par} has always been investigated while basing on the characterization (1.3). However, using (1.2) and the fact that $g \in UCV \iff f = zg' \in S_{\text{par}}$ it is clear that we can obtain an alternative characterization of S_{par} in terms of two variables and this characterization will be as follows.

Theorem 2.6. *Let*

$$F(z,\zeta) = \frac{(z-\zeta)f'(z)}{f(z)} + \frac{\zeta}{z}. \tag{2.8}$$

Then $f \in S_{\text{par}}$ if and only if $\operatorname{Re} F(z,\zeta) \geq 0$ for all $(z,\zeta) \in U \times U$.

We include here one application of this alternative characterization of S_{par}.

Theorem 2.7. *If $f(z) = z + a_2 z^2 + \cdots \in S_{\text{par}}$ then*

$$|a_2 + (a_2^2 - 2a_3)\zeta| \leq 2\,\mathrm{Re}(1 - a_2\zeta)\,, \quad |\zeta| < 1. \tag{2.9}$$

Proof. Let $F(z,\zeta)$ be as in (2.8) and $f \in S_{\text{par}}$. Then $\mathrm{Re}\, F(z,\zeta) > 0$. Writing f and f' in (2.8) as a power series, we see, after cancelling the coefficient of z^2, that $F(0,\zeta) = 1 - a_2\zeta \neq 0$ since $|a_2| < 1$. Similarly, we see by computing $(\partial/\partial z)F(z,\zeta)$ that

$$\frac{\partial}{\partial z}F(0,\zeta) = a_2 + (a_2^2 - 2a_3\zeta).$$

It is well known that if $p(z)$ is analytic and has positive real part in U then $|p'(0)| \leq 2\,\mathrm{Re}\, p(0)$. In our case this will turn into

$$\left|\frac{\partial}{\partial z}F(0,\zeta)\right| \leq 2\,\mathrm{Re}\, F(0,\zeta),$$

and substituting the above expressions for the left and right hand side of this inequality the result follows. ∎

Corollary 2.8. *Let $f(z) = z + a_2 z^2 + a_4 z^4 + \ldots$ (i.e. $a_3 = 0$). Then*

$$|a_2| \leq \frac{\sqrt{17} - 3}{2} = 0.56\ldots.$$

Proof. Set $a_3 = 0$ and choose $\zeta = \overline{a}_2/|a_2|$. Then we get from (2.9)

$$|a_2| \cdot |1 + |a_2|| \leq 2(1 - |a_2|).$$

Solving for $|a_2|$ the result follows. ∎

Remark. The bound for $|a_2|$ in S_{par} without the restriction $a_3 = 0$ is $|a_2| \leq 8/\pi^2 = 0.81\ldots$ cf. [5].

3. Uniformly starlike functions. The class of uniformly starlike functions is not as well described as the class of uniformly convex functions. One reason for this is that the characterization of UST includes the value $f(\zeta)$, and a one-variable characterization of UST does not seem to be available. We present here some new results on UST that can be derived from the definition (1.1).

Theorem 3.1. *For $f(z) = z + a_2z^2 + \cdots \in UST$ we have*

$$\left|\frac{f(z)}{z} - 1 - 2a_2f(z)\right| \le 2|z| \operatorname{Re}\frac{f(z)}{z}, \quad z \in U.$$

Proof. Let $F(z,\zeta) = [f(z) - f(\zeta)]/[(z-\zeta)f'(z)]$. Then $f \in UST \iff \operatorname{Re} F(z,\zeta) > 0$, $z, \zeta \in U$. Using again

$$\left|\frac{\partial}{\partial z}F(0,\zeta)\right| \le 2 \operatorname{Re} F(0,\zeta)$$

in a similar way as in the proof of Theorem 2.7, the result follows. ∎

Theorem 3.1 immediately gives the following corollary.

Corollary 3.2. *For $f \in UST$ we have $\operatorname{Re} f(z)/z > 0$. If $f(z) = z + a_3z^3 + \cdots$ (i.e. $a_2 = 0$) we get the stronger result*

$$\left|\frac{f(z)}{z} - 1\right| \le 2 \operatorname{Re}\frac{f(z)}{z}, \ z \in U. \tag{3.1}$$

Remark. From (3.1) we see that if $f \in UST$ and $a_2 = 0$ we have $\operatorname{Re} f(z)/z > 1/3$. In general it is only known [3] that

$$\left|\frac{f(z)}{z} - 1\right| \le 2\left|\frac{f(z)}{z}\right|,$$

which implies $|f(z)/z| > 1/3$. We do not know whether it is true in general that $\operatorname{Re} f(z)/z > 1/3$ for every function in UST.

4. The classes $ST(\zeta)$ and $CV(\zeta)$. In this section we consider classes with a normalization different from the usual one. Given ζ, let $S(\zeta)$ denote the class of all functions f analytic and univalent in the unit disk with the normalization $f(\zeta) = f'(\zeta) - 1 = 0$. In this class we shall define subclasses of starlike and convex functions, denoted by $ST(\zeta)$ and $CV(\zeta)$. Here $ST(\zeta)$ is defined by the geometric property that the image of any circular arc centered at ζ is starlike with respect to $f(\zeta)$, which due to normalization is the origin. The corresponding class of convex functions, $CV(\zeta)$, is defined by the property that the image of any circular arc centered at ζ is convex. Hence, the definitions are somewhat similar to the ones for uniformly starlike and convex functions, except that in this case the point ζ is fixed. The analytic characterizations will then be as follows.

$$ST(\zeta)=\left\{f\in S(\zeta):\operatorname{Re}\frac{(z-\zeta)f'(z)}{f(z)}>0\,,\ z\in U\right\},$$

$$CV(\zeta)=\left\{g\in S(\zeta):1+\operatorname{Re}\frac{(z-\zeta)g''(z)}{g'(z)}>0\,,\ z\in U\right\}.$$

It is obvious that we have a natural 'Alexander relation' between these classes, i.e.

Proposition 4.1. *A function g is in $CV(\zeta)$ if and only if*

$$f(z)=(z-\zeta)g'(z)\in ST(\zeta).$$

As with the usual starlike and convex functions the sets $ST(\zeta)$ and $CV(\zeta)$ will be associated with a set of functions with positive real part. We denote this set by $P(\zeta\,)$, and this will be the class of all functions

$$p(z)=1+\sum_{n=1}^{\infty}B_n(z-\zeta)^n \tag{4.1}$$

that are regular in U, and satisfy $p(\zeta)=1$ and $\operatorname{Re}p(z)>0$ for $z\in U$.

Theorem 4.2. *Let $f\in CV(\zeta)$. Then for $z,\alpha,w\in U$ we have*

$$\operatorname{Re}\left\{\frac{z-\zeta}{z-\alpha}\,\frac{\alpha-w}{z-w}\,\frac{f(z)-f(w)}{f(\alpha)-f(w)}-\frac{\alpha-\zeta}{z-\alpha}\right\}>\frac{1}{2}. \tag{4.2}$$

Proof. The function

$$F(z,\zeta,\alpha,w)=\frac{2(z-\zeta)}{z-\alpha}\,\frac{\alpha-w}{z-w}\,\frac{f(z)-f(w)}{f(\alpha)-f(w)}-\frac{z+\alpha-2\zeta}{z-\alpha}$$

is analytic for $z,\alpha,w\in U$. For distinct points $z=\zeta+re^{i\theta_1}$, $\alpha=\zeta+re^{i\theta_2}$, $w=\zeta+re^{i\theta_3}$, all in U, we have

$$\begin{aligned}\frac{2(z-\zeta)}{z-\alpha}\,\frac{\alpha-w}{z-w}&=\frac{2e^{i\theta_1}}{e^{i\theta_1}-e^{i\theta_2}}\,\frac{e^{i\theta_2}-e^{i\theta_3}}{e^{i\theta_1}-e^{i\theta_3}}\\&=-i\,\frac{\sin((\theta_2-\theta_3)/2)}{\sin((\theta_1-\theta_2)/2))\sin((\theta_1-\theta_3)/2))}.\end{aligned}$$

This gives

$$\begin{aligned}
&\operatorname{Re} F(\zeta + re^{i\theta_1}, \zeta, \zeta + re^{i\theta_2}, \zeta + re^{i\theta_3}) \\
(4.3)\qquad &= \frac{\sin((\theta_2-\theta_3)/2)}{\sin((\theta_1-\theta_2)/2))\sin((\theta_1-\theta_3)/2))}\,\Im \frac{f(\zeta+re^{i\theta_1}) - f(\zeta+re^{i\theta_3})}{f(\zeta+re^{i\theta_2}) - f(\zeta+re^{i\theta_3})}.
\end{aligned}$$

The three points z, α and w all lie on the arc $\zeta + re^{i\theta}$, and since $f \in UCV(\zeta)$, the image of this arc is convex. Using this and discussing all possible relative locations of z, α and w, we find that the expression in (4.3) is always positive. Hence we have $\operatorname{Re} F(z,\zeta,\alpha,w) > 0$, $|z-\zeta| = |\alpha-\zeta| = r$. Using the maximum principle for harmonic functions we conclude that for fixed α and w we have $\operatorname{Re} F(z,\zeta,\alpha,w) > 0$, $z \in U$. A similar application of the maximum principle, for fixed α and z, and for fixed z and w gives the statement in the theorem. ■

Corollary 4.3. *If $f \in CV(\zeta)$ then*

$$\operatorname{Re} \frac{z-\zeta}{f(z)} \frac{f(z)-f(w)}{z-w} > \frac{1}{2}, \ (z,w) \in U \times U, \tag{4.4}$$

$$\operatorname{Re} \frac{(z-\zeta)f'(z)}{f(z)} > \frac{1}{2}, \ z \in U \tag{4.5}$$

and

$$\operatorname{Re} \frac{f(z)}{z-\zeta} > \frac{1}{2}, \ z \in U. \tag{4.6}$$

Proof. The inequalities (4.4) - (4.6) are special cases of (4.2). We obtain (4.4) by taking $\alpha = \zeta$ in (4.2) and changing the role of z and w. We get (4.5) from (4.4) by letting $w \to z$, and (4.6) follows from (4.4) if we let $z \to \zeta$, and afterwards replace w by z.

■

It is natural to ask about the coefficients of functions from the classes $CV(\zeta)$ and $ST(\zeta)$. If $p \in P(\zeta)$ and has the series expansions (4.1) then for all $n \geq 1$ a result by Wald [9] (see also [1, p. 158]) gives the sharp bounds for the coefficients B_n of the function p. This result is

$$|B_n| \leq \frac{2}{(1+d)(1-d)^n}, \quad d = |\zeta|. \tag{4.7}$$

Theorem 4.4. *Let* $f \in ST(\zeta)$ *and* $f(z) = (z-\zeta) + a_2(z-\zeta)^2 + \cdots$. *Then*

$$|a_2| \le \frac{2}{1-d^2}, \quad |a_3| \le \frac{3+d}{(1-d^2)^2}, \quad |a_4| \le \frac{2}{3}\,\frac{(2+d)(3+d)}{(1-d^2)^3},$$

$$|a_5| \le \frac{1}{6}\,\frac{(2+d)(3+d)(3d+5)}{(1-d^2)^4}.$$

Proof. Let $f \in ST(\zeta)$. Then there exists a function $p \in P(\zeta)$ such that

$$\frac{(z-\zeta)f'(z)}{f(z)} = p(z).$$

Equating coefficients of $(z-\zeta)^n$ in the power series of both sides we obtain

$$(n-1)a_n = \sum_{k=1}^{n-1} a_k B_{n-k}, \quad n = 2,3,4,\ldots, \; a_1 = 1.$$

Hence we get

$$a_2 = B_1, \quad a_3 = \frac{1}{2}(B_2 + B_1^2), \quad a_4 = \frac{1}{3}\left(B_3 + \frac{3}{2}B_1B_2 + \frac{1}{2}B_1^3\right),$$

$$a_5 = \frac{1}{4}\left(B_4 + \frac{4}{3}B_1B_3 + B_1^2B_2 + \frac{1}{2}B_2^2 + \frac{1}{6}B_1^4\right).$$

Applying the above and the estimates (4.7) we get the result. ■

Remark. It is clear that Theorem 4.1 also provides bounds for the coefficients of functions in $CV(\zeta)$, due to the relation between $CV(\zeta)$ and $ST(\zeta)$ given in Proposition 4.1.

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