ANNALES
UNIVERSITATIS MARIAE CURIE – SKŁODOWSKA
LUBLIN – POLONIA

VOL. LVII, 9 SECTIO A 2003

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Convoluting to a challenge function

Abstract. Let $\mathcal{A}$ denote the class of functions f that are analytic in $\Delta = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. The subclasses of $\mathcal{A}$ consisting of functions that are univalent in Δ, starlike with respect to the origin, and convex will be denoted by S, S^* and K, respectively. In this paper, we investigate conditions under which $f \in S^*$ has a starlike inverse; i.e., a $g \in S^*$ for which the convolution $f * g = \dfrac{z}{1-z}$. We also determine conditions under which a fixed $h \in K$ can be expressed as $h = f * g$ where f and g are in S^* (or S).

1. Introduction. Let $\mathcal{A}$ denote the class of functions f that are analytic in $\Delta = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. The subclasses of $\mathcal{A}$ consisting of functions that are univalent in Δ, starlike with respect to the origin, and convex will be denoted by S, S^* and K, respectively. For $h(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ and $\delta > 0$, a δ-neighborhood of h is defined by

$$N_\delta(h) = \left\{ z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A} : \sum_{n=2}^{\infty} n |a_n - b_n| \le \delta \right\}.$$

In [12], St. Ruscheweyh introduced the notion of δ-neighborhoods and proved the following two results.

2000 *Mathematics Subject Classification.* Primary 30C45; Secondary 30C50.
Key words and phrases. Convex, Starlike, Neighborhoods.

Theorem A. *If $f \in \mathcal{A}$ is such that $\dfrac{f(z) + \varepsilon z}{1 + \varepsilon}$ is starlike for all $\varepsilon \in \mathbb{C}$ with $|\varepsilon| < \delta$, then $N_\delta(f) \subset S^*$.*

Theorem B. *If $f \in K$, then $N_{1/4}(f) \subset S^*$.*

Theorem B shows that the well-known result $N_1(z) \subset S^*$ [6] can be extended to claim the existence of neighborhoods of arbitrary convex functions that consist of starlike functions. For extensions and generalizations of the work that was initiated by St. Ruscheweyh, see [2], [4], [5], and [13].

For $f(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ and $g(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{A}$, the convolution or Hadamard product of f and g is $(f * g)(z) = z + \sum_{n=2}^{\infty} b_n c_n z^n$. One motivation for looking at convolutions over various subclasses of $\mathcal{A}$ was the Pólya-Schoenberg conjecture [8] that the convolution of two convex functions is convex. In addition to proving the conjecture, St. Ruscheweyh and T. Sheil-Small [9] showed that convolution with convex functions also preserves the classes of starlike and close-to-convex functions. Such preservation results enable us to determine geometric properties associated with various operators that can be realized as convolutions with specific convex functions. For example, $g_\gamma(z) = \sum_{n=1}^{\infty} \dfrac{1+\gamma}{n+\gamma} z^n \in K$ whenever $\operatorname{Re}\gamma \geq 0$ [10]. Hence, $I(f) = f * g_\gamma$ yields that, for $\operatorname{Re}\gamma \geq 0$, the operator $I(f) = \dfrac{1+\gamma}{z^\gamma}\displaystyle\int_0^z t^{\gamma-1} f(t)\, dt \in S^*$ whenever $f \in S^*$. Other operator applications can be found in [1] and [15].

In another direction, one can specify a function g and define a class $\mathcal{F}$ consisting of all $f \in \mathcal{F}$ for which $f * g$ satisfies a particular property. The best known example of this is with the class of prestarlike functions of order α, denoted by $\mathcal{R}_\alpha$, that was introduced by St. Ruscheweyh [11]: A function $f \in \mathcal{A}$ is prestarlike of order α for $0 \leq \alpha < 1$ if $f * \left(z/(1-z)^{2(1-\alpha)}\right)$ is starlike of order α. It is known [14] that $\mathcal{R}_\alpha \subset S$ if and only if $\alpha \leq \dfrac{1}{2}$.

Note that $H(z) = \dfrac{z}{1-z}$ is the identity function under convolution. In this paper, we investigate conditions under which $f \in S^*$ has a starlike inverse; i.e., a $g \in S^*$ for which $f * g = H$. We also determine conditions under which a fixed $h \in K$ can be expressed nontrivially as $h = f * g$ where f and g are in S^* (or S).

2. Some preliminaries. In this section, we identify some subclasses $\mathcal{F}$ and $\mathcal{G}$ of S^* for which corresponding to each $f \in \mathcal{F}$ there exists a $g \in \mathcal{G}$ such that $f * g = \dfrac{z}{1-z}$.

Lemma 1. *Let $F_B(z) = \dfrac{z + Bz^2}{1-z}$ for $B \in \mathbb{C}$. Then $F_B \in S^*$ if and only if*

$|B| \leq \dfrac{1}{3}$ *or* $B = -1$.

Proof. If $|B| \leq \dfrac{1}{3}$ and $|z| < 1$, then

$$\operatorname{Re}\left\{\frac{zF_B'(z)}{F_B(z)}\right\} = \operatorname{Re}\left\{1 + \frac{1}{1-z} - \frac{1}{1+Bz}\right\} > \frac{3}{2} - \frac{1}{1-|B|} \geq 0\ ,$$

while $F_{-1}(z) = z$. To prove the necessity, suppose $B = \left(\dfrac{1}{3} + \varepsilon\right) e^{i\beta}$ for $0 < \varepsilon \leq \dfrac{2}{3}$ and $-\pi < \beta < \pi$. Let $z = re^{i\theta}$ with $0 < r < 1$ and $\theta = \pi - \beta$. For $0 < \varepsilon < \dfrac{2}{3}$,

$$\begin{aligned}\operatorname{Re}\left\{\frac{zF_B'(z)}{F_B(z)}\right\} &= \operatorname{Re}\left\{\frac{1}{1+re^{-i\beta}} - \frac{\left(\frac{1}{3}+\varepsilon\right)r}{1-\left(\frac{1}{3}+\varepsilon\right)r}\right\} \\ &\longrightarrow \frac{1}{2} - \frac{1+3\varepsilon}{2-3\varepsilon} = \frac{-9\varepsilon}{2(2-3\varepsilon)} < 0\end{aligned}$$

as $r \to 1^-$; for $\varepsilon = \dfrac{2}{3}$,

$$\begin{aligned}\operatorname{Re}\left\{\frac{zF_B'(z)}{F_B(z)}\right\} &= \operatorname{Re}\left\{\frac{1}{1+re^{-i\beta}} - \frac{r}{1-r}\right\} \\ &= \frac{1+r\cos\beta}{1+r^2+2r\cos\beta} - \frac{r}{1-r} \longrightarrow -\infty\end{aligned}$$

as $r \to 1^-$. Finally, suppose that $\beta = \pi$ and $0 < \varepsilon < \dfrac{2}{3}$; i.e., $-1 < B < -\dfrac{1}{3}$. Then $(1-3|B|)(1-|B|) < 0$ so that $\dfrac{1+3B^2}{4|B|} < 1$; for $z = e^{i\theta}$, we have that

$$\operatorname{Re}\left\{\frac{zF_B'(z)}{F_B(z)}\right\} = \frac{3}{2} - \frac{1-|B|\cos\theta}{1+B^2-2|B|\cos\theta} < 0$$

for $\dfrac{1+3B^2}{4|B|} < \cos\theta < 1$. If $|B| > 1$, then F_B is not univalent in Δ because $F_B(0) = F_B\left(-\dfrac{1}{B}\right) = 0$ and $-\dfrac{1}{B} \in \Delta$. It follows that, for $B \neq -1$, $F_B \notin S^*$ when $|B| > \dfrac{1}{3}$. $\square$

Lemma 1 immediately yields a subclass of starlike functions whose inverses with respect to convolution are also starlike.

Theorem 1. *Let* $\Omega = \left\{\zeta \in \mathbb{C} : |\zeta - 1| \le \frac{1}{3}\right\} \cap \left\{\zeta \in \mathbb{C} : \left|\zeta - \frac{9}{8}\right| \le \frac{3}{8}\right\}$. *Then*

$$z + A\sum_{n=2}^{\infty} z^n \in S^* \qquad \text{and} \qquad z + \frac{1}{A}\sum_{n=2}^{\infty} z^n \in S^*$$

if and only if $A \in \Omega$.

Proof. Since $z + A\sum_{n=2}^{\infty} z^n = z + \dfrac{Az^2}{1-z} = F_{(A-1)}$, taking $B = A - 1$ in Lemma 1 yields that $z + A\sum_{n=2}^{\infty} z^n \in S^*$ if and only if A satisfies $|A-1| \le \dfrac{1}{3}$ or $A = 0$. Similarly, taking $B = \dfrac{1}{A} - 1$ yields that $z + \dfrac{1}{A}\sum_{n=2}^{\infty} z^n = F_{(A^{-1}-1)}$ is starlike if and only if A satisfies $\left|\dfrac{1}{A} - 1\right| \le \dfrac{1}{3}$ which is equivalent to A satisfying $\left|A - \dfrac{9}{8}\right| \le \dfrac{3}{8}$. Combining the conditions leads to the desired conclusion. $\square$

Remark 1. The circles that form the boundary of Ω intersect at $\dfrac{17 \pm i\sqrt{35}}{18}$ which are on $\partial\Delta$.

Corollary 1. *For* $|\varepsilon| \le \dfrac{1}{4}$, $f_\varepsilon(z) = \dfrac{\frac{z}{1-z} + \varepsilon z}{1+\varepsilon} = z + \dfrac{1}{1+\varepsilon}\sum_{n=2}^{\infty} z^n \in S^*$ *and* $g_\varepsilon = z + (1+\varepsilon)\sum_{n=2}^{\infty} z^n \in S^*$. *Note that* $f_\varepsilon * g_\varepsilon = \dfrac{z}{1-z}$.

Proof. Taking $A = 1 + \varepsilon$ in Theorem 1 yields that $f_\varepsilon \in S^*$ and $g_\varepsilon \in S^*$ if and only if $\varepsilon \in \Omega = \left\{\zeta \in \mathbb{C} : |\zeta| \le \frac{1}{3}\right\} \cap \left\{\zeta \in \mathbb{C} : \left|\zeta - \frac{1}{8}\right| \le \frac{3}{8}\right\}$. The largest disk centered at the origin that is contained in Ω has radius $\dfrac{1}{4}$. $\square$

Theorem 2. *For n a fixed integer, $n \ge 2$, let*

$$\Omega_n = \left\{\zeta \in \mathbb{C} : |\zeta| \le \frac{1}{4n}\right\} \cap \left\{\zeta \in \mathbb{C} : \left|\zeta - \frac{1}{16n^2 - 1}\right| \le \frac{4n}{16n^2 - 1}\right\}.$$

If $\varepsilon \in \Omega_n$, *then* $F_\varepsilon(z) = \dfrac{z}{1-z} + \varepsilon z^n \in S^*$ *and its inverse* $G_\varepsilon(z) = \dfrac{z}{1-z} - \dfrac{\varepsilon}{1+\varepsilon} z^n$ *is also in* S^*.

Proof. For fixed $n \ge 2$, suppose that $\varepsilon \in \Omega_n$ and $F_\varepsilon(z) = \dfrac{z}{1-z} + \varepsilon z^n = z + \sum_{k=2}^{\infty} b_k z^k$. Since $\sum_{k=2}^{\infty} k|1 - b_k| = n|\varepsilon| \le \dfrac{1}{4}$, $F_\varepsilon \in N_{1/4}\left(\dfrac{z}{1-z}\right)$. Consequently, by Theorem B, $F_\varepsilon \in S^*$. Now $G_\varepsilon = F_{-\varepsilon/(1+\varepsilon)} \in N_{1/4}\left(\dfrac{z}{1-z}\right)$

whenever $\left|\dfrac{\varepsilon}{1+\varepsilon}\right| \leq \dfrac{1}{4n}$ which is equivalent to $\left|\varepsilon - \dfrac{1}{16n^2-1}\right| \leq \dfrac{4n}{16n^2-1}$. The latter holds since $\varepsilon \in \Omega_n$. $\square$

Remark 2. To see that the result of Theorem 2 is best possible, suppose that $\varepsilon = (-1)^n \dfrac{1+\gamma}{4n}$ for some $\gamma > 0$ and fixed n. Then, for F_ε as given in Theorem 2, $F'_\varepsilon(z) = \dfrac{1}{(1-z)^2} + \dfrac{(-1)^n(1+\gamma)}{4} z^{n-1}$. Since $F'_\varepsilon(-1) = \dfrac{1}{4} + \dfrac{(-1)^{2n-1}(1+\gamma)}{4} = -\dfrac{\gamma}{4} < 0$ and $F'_\varepsilon(0) = 1 > 0$, F'_ε has a zero in Δ from which we conclude that $F_\varepsilon \notin S$.

Remark 3. If $|\varepsilon| \leq \dfrac{1}{4n+1}$, then $\varepsilon \in \Omega_n$.

The last example given in this section makes use of the following result that is due to J. Lewis [7].

Theorem C. *The function $f_\lambda(z) = \sum_{n=1}^{\infty} n^{-\lambda} z^n \in K$ when $\lambda \geq 0$.*

In view of Theorem C, if $0 \leq \delta \leq 1$, $\phi_\delta(z) = z + \sum_{n=2}^{\infty} n^\delta z^n \in S^*$ because $\int_0^z \zeta^{-1}\phi_\delta(\zeta)\, d\zeta = z + \sum_{n=2}^{\infty} \dfrac{z^n}{n^{1-\delta}} \in K$. Hence, if $0 \leq \delta \leq 1$, $\phi_\delta \in S^*$ and $\phi_\delta * \phi_{-\delta} = \dfrac{z}{1-z}$ with $\phi_{-\delta} \in K$.

Remark 4. Note that $\phi_1(z) = z + \sum_{n=2}^{\infty} n z^n = \dfrac{z}{(1-z)^2}$, the well-known Koebe function, has the convex function $\phi_{-1}(z) = z + \sum_{n=2}^{\infty} \dfrac{1}{n} z^n = -\log(1-z)$ as its inverse.

3. Convex functions. Next we illustrate the important role played by the identity function under convolutions when determining if neighborhoods must contain starlike functions.

Theorem 3. *Suppose $H(z) = \dfrac{z}{1-z}$ and $h(z) = z + \sum_{n=2}^{\infty} a_n z^n = e^{-i\beta} H\left(e^{i\beta} z\right)$ for β real. Then for each $f \in N_{2/9}(h)$ the function g such that $f * g = h$ is starlike. The result is sharp.*

Proof. If $f \in N_{2/9}(h)$, we may set $f(z) = z + \sum_{n=2}^{\infty} (a_n + \varepsilon_n) z^n$ where $\{\varepsilon_n\}_{n=2}^{\infty}$ satisfies $\sum_{n=2}^{\infty} n|\varepsilon_n| \leq \dfrac{2}{9}$. For each $n \geq 2$, $|a_n| = 1$ and $|a_n + \varepsilon_n| \geq 1 - |\varepsilon_n| \geq 1 - \dfrac{1}{9} = \dfrac{8}{9}$. If $f * g = h$, then $g(z) = \dfrac{z}{1-z} - \sum_{n=2}^{\infty} \dfrac{\varepsilon_n}{a_n + \varepsilon_n} z^n = z + \sum_{n=2}^{\infty} c_n z^n$. It follows that

$$\sum_{n=2}^{\infty} n|1-c_n| = \sum_{n=2}^{\infty} n\left|\frac{\varepsilon_n}{a_n+\varepsilon_n}\right| \leq \frac{9}{8}\sum_{n=2}^{\infty} n|\varepsilon_n| \leq \left(\frac{9}{8}\right)\left(\frac{2}{9}\right) = \frac{1}{4};$$

i.e., $g \in N_{1/4}\left(\dfrac{z}{1-z}\right)$. By Theorem B, $g \in S^*$ as needed.

To see that this is best possible, let $f(z) = \dfrac{z}{1-z} - \left(\dfrac{1}{9}+\dfrac{\varepsilon}{2}\right) z^2 = z + \sum_{n=2}^{\infty} b_n z^n$ for some ε, $0 < \varepsilon < \dfrac{1}{36}$. Since

$$\sum_{n=2}^{\infty} n\,|1-b_n| = 2\left|1 - \left(1 - \frac{1}{9} - \frac{\varepsilon}{2}\right)\right| = \frac{2}{9} + \varepsilon < \frac{1}{4}\ ,$$

$f \in N_{1/4}\left(\dfrac{z}{1-z}\right)$ which, by Theorem B, yields that f is starlike. The inverse of this f is given by $g(z) = \dfrac{z}{1-z} + \dfrac{\left(\frac{1}{9}+\frac{\varepsilon}{2}\right)}{\left(\frac{8}{9}-\frac{\varepsilon}{2}\right)} z^2$. For $g'(z) = \dfrac{1}{(1-z)^2} + 2\dfrac{\left(\frac{1}{9}+\frac{\varepsilon}{2}\right)}{\left(\frac{8}{9}-\frac{\varepsilon}{2}\right)} z$, $g'(0) = 1$ while $g'(-1) = \dfrac{1}{4} - 2\dfrac{\left(\frac{1}{9}+\frac{\varepsilon}{2}\right)}{\left(\frac{8}{9}-\frac{\varepsilon}{2}\right)} < \dfrac{1}{4} - 2\left(\dfrac{1}{8}\right) = 0$. Thus, g' has a zero in Δ from which we conclude that g is not even univalent in Δ. $\square$

For functions in K other than rotations of $\dfrac{z}{1-z}$ the question arises as to whether the result of Theorem 3 will remain valid for perhaps a smaller than 2/9-neighborhood. We will show that a theorem of P. J. Eenigenburg and F. R. Keogh (see Theorem 4, in [3]) answers this question in the negative.

Theorem D. *If $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n \in K$ and is not a rotation of $H(z) = \dfrac{z}{1-z}$, then $|a_n| \to 0$ as $n \to \infty$.*

Theorem 4. *Suppose that $F(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ is such that $\inf\limits_{n} |a_n| = 0$. Then, for every $\delta > 0$, there exists an $f \in N_\delta(F)$ and a $g \in \mathcal{A} - S$ for which $g * f = F$.*

Proof. Either there exists an integer k such that $a_k = 0$ or $a_k \neq 0$ for each $k \geq 2$ and there exists a nonvanishing subsequence $\{a_{n_j}\}_{j=1}^{\infty}$ such that $\lim\limits_{j\to\infty} a_{n_j} = 0$.

Suppose that $a_k = 0$ for some $k \geq 2$. For $\delta > 0$, let $f(z) = z + \sum_{n=2}^{\infty} b_n z^n$ with $b_n = a_n$ for $n \neq k$ and $b_k = \dfrac{\delta}{k}$. Then $f \in N_\delta(F)$ and $g(z) =$

$\dfrac{z}{1-z}-z^k \in \mathcal{A}$ satisfies $f*g=F$. Since $g'(z)=\dfrac{1-kz^{k-1}+2kz^k-kz^{k+1}}{(1-z)^2}$, if $\zeta_1,\zeta_2,...,\zeta_{k+1}$ are the $k+1$ roots of g', then $\prod_{j=1}^{k+1}|\zeta_j|=\dfrac{1}{k}<1$. Thus, at least one of the ζ_j is in Δ from which we conclude that $g\notin S$.

Suppose that there exists a nonvanishing subsequence $\{a_{n_j}\}_{j=1}^{\infty}$ with $\lim_{j\to\infty} a_{n_j}=0$. Then, for $\delta>0$, we can choose fixed k large enough so that $|a_k|+\dfrac{\delta}{k}<\dfrac{\delta}{2}$ and set $\varepsilon_k=\dfrac{\delta}{k}\exp(i\arg a_k)$. Let $f(z)=z+\sum_{n=2}^{\infty}b_nz^n$ with $b_n=a_n$ for $n\neq k$ and $b_k=a_k+\varepsilon_k$. Then $f\in N_\delta(F)$ and $g(z)=\dfrac{z}{1-z}-\dfrac{\varepsilon_k}{a_k+\varepsilon_k}z^k\in\mathcal{A}$ is the only function for which $f*g=F$. Now $g'(z)=\dfrac{1}{(1-z)^2}-\dfrac{k\varepsilon_k}{a_k+\varepsilon_k}z^{k-1}=\dfrac{1}{(1-z)^2}-\dfrac{\delta}{|a_k|+\dfrac{\delta}{k}}z^{k-1}=\dfrac{1}{(1-z)^2}-Az^{k-1}$ where $A\geq 2$. In this case, $g'(z)=\dfrac{P_{k+1}(z)}{(1-z)^2}$ where P_{k+1} is a polynomial of degree $k+1$ and, if $\zeta_1,\zeta_2,...,\zeta_{k+1}$ are the $k+1$ roots of P_{k+1}, then $\prod_{j=1}^{k+1}|\zeta_j|=\dfrac{1}{A}<1$. Again, we conclude that g' has at least one root in Δ. Hence. $g\notin S$ as needed. $\square$

Remark 5. Note that, in Theorem 4, we did not require that the function F be convex. From Theorem D, we have that Theorem 4 holds for any convex function that is not a rotation of $\dfrac{z}{1-z}$.

4. Some open questions. In Section 2, we looked at $f,g\in S^*$ for which $f*g=\dfrac{z}{1-z}$. Are there characterizing conditions for a subclass of S^* that consists of functions and their inverses by convolution?

In Section 3, we gave a characterizing condition for the existence of a neighborhood N_δ of a convex functions h such that for $f\in N_\delta$, there exists a unique $g\in S^*$ such that $f*g=h$. Our next example gives an $h\in S^*-K$ for which such a neighborhood exists.

Example 1. Let $h(z)=z+(1.01)\sum_{n=2}^{\infty}z^n$. Then $h\in S^*-K$ in view of Lemma 1 and the well-known coefficient bound for convex functions. From Lemma 1, $\dfrac{h(z)+\varepsilon z}{1+\varepsilon}=z+\dfrac{1.01}{1+\varepsilon}\sum_{n=2}^{\infty}z^n\in S^*$ for $|\varepsilon|\leq .2$. Hence, from Theorem A, $N_\delta(h)\subset S^*$ for $\delta=.2$. For $f\in N_\delta(h)$, we can set $f(z)=z+\sum_{n=2}^{\infty}(1.01+\varepsilon_n)z^n$ where $\{\varepsilon_n\}_{n=2}^{\infty}$ is such that $\sum_{n=2}^{\infty}n|\varepsilon_n|\leq .2$. If

$g(z) = \dfrac{z}{1-z} - \sum_{n=2}^{\infty} \dfrac{\varepsilon_n}{1.01+\varepsilon_n}$, then $f * g = h$ and

$$\sum_{n=2}^{\infty} n \left| \frac{\varepsilon_n}{1.01+\varepsilon_n} \right| \le \sum_{n=2}^{\infty} n \frac{|\varepsilon_n|}{1.01 - \left|\dfrac{\delta}{2}\right|} \le \frac{.2}{1.01-.1} = \frac{20}{91} < \frac{1}{4}.$$

Therefore, $g \in S^*$.

Is there some characterizing condition for a subclass $\mathcal{F}$ of functions in S^* such that $h \in \mathcal{F}$ admits a δ-neighborhood for which $f \in N_\delta(h)$ implies that there exists $g \in S^*$ such that $f * g = h$?

Given $h \in S$, what can we say about $f, g \in S$ for which $f * g = h$? Since $f(z) * g(z) = \dfrac{f(xz)}{x} * \dfrac{g(\overline{x}z)}{\overline{x}}$, $|x| = 1$, if a condition holds for f and g, then it holds for rotations.

For every $h \in S$, $h * \dfrac{z}{1-z} = h$. To what extent does the identity function $\dfrac{z}{1-z}$ play a unique role? That is, are there any functions $h \in S$ for which the univalent convolution $f * g = h$ can occur only if f or g is the identity? The Koebe function can be written as $z + \sum_{n=2}^{\infty} nz^n = \left(z + \sum_{n=2}^{\infty} n^\lambda z^n\right) * \left(z + \sum_{n=2}^{\infty} n^{1-\lambda} z^n\right)$ for $\lambda \in [0,1]$. The next result gives a class of functions that can be realized as the convolution of two functions in S neither of which is the identity $\dfrac{z}{1-z}$.

Theorem 5. *Let $h(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$. If (i) there exists $k \ge 2$ for which $a_k = 0$ or (ii) there exists a $\delta > 0$ for which $N_\delta(h) \subset S$, then there exist functions f and g in $S - \left\{\dfrac{z}{1-z}\right\}$ such that $f * g = h$.*

Proof. If $a_k = 0$ for some k, set $f(z) = \dfrac{z}{1-z} + \dfrac{1}{4k} z^k$ and $g = h$. Since $f \in N_{1/4}\left(\dfrac{z}{1-z}\right) \subset S$, it follows that $f, g \in S - \left\{\dfrac{z}{1-z}\right\}$ and $f * g = h$ as needed. Now, suppose that $N_\delta(h) \subset S$ for some $\delta > 0$. In view of the last example, we may assume that a_n never vanishes. Then for a fixed $k \ge 2$, choose $\varepsilon > 0$ small enough so that $|\varepsilon| < \dfrac{\delta}{k}$ and $\left|\dfrac{\varepsilon}{a_k + \varepsilon}\right| \le \dfrac{1}{4k}$. Then $f = h + \varepsilon z^k \in S$ from the hypothesis, $g = \left(\dfrac{z}{1-z} - \dfrac{\varepsilon}{a_k+\varepsilon} z^k\right) \in S^*$, by Theorem B, and $f * g = h$. $\square$

It is known [12] that, for $h \in S^*\left(\dfrac{1}{2}\right)$, the class of functions that are starlike of order $1/2$, $N_{1/4}(h)$ consists of close-to-convex functions. This leads to the following

Corollary 2. *If* $h \in S^*\left(\frac{1}{2}\right)$, *then there exist functions* f *and* g *in* $S - \left\{\frac{z}{1-z}\right\}$ *such that* $f * g = h$.

References

[1] Barnard, R.W., C. Kellogg, *Applications of convolution operators to problems in univalent function theory*, Michigan Math. J. **27** (1980), 81–94.

[2] Brown, J.E., *Some sharp neighborhoods of univalent functions*, Trans. Amer. Math. Soc. **287** (1985), 475–482.

[3] Eenigenburg, P.J., F.R. Keogh, *The Hardy class of some univalent functions and their derivatives*, Michigan Math. J. **17** (1970), 335–346.

[4] Fournier, R., *A note on neighborhoods of univalent functions*, Proc. Amer. Math. Soc. **87** (1983), 117–120.

[5] Fournier, R., *On neighborhoods of univalent convex functions*, Rocky Mountain J. Math. **16** (1986), 579–589.

[6] Goodman, A.W., *Univalent functions and nonanalytic curves*, Proc. Amer. Math. Soc. **8** (1957), 598–601.

[7] Lewis, John L., *Convexity of certain series*, J. London Math. Soc. **27** (1983), 435–446.

[8] Pólya, G., I. J. Schoenberg, *Remarks on the de la Valle Poussin means and convex conformal mapping of the circle*, Pacific J. Math. **8** (1958), 295–334.

[9] Ruscheweyh, St., T. Sheil-Small, *Hadamard products of schlicht functions and the Pólya-Schoenberg conjecture*, Comment. Math. Helv. **48** (1973), 119–135.

[10] Ruscheweyh, St., *New criteria for univalent functions*, Proc. Amer. Math. Soc. **49** (1975), 109–115.

[11] Ruscheweyh, St., *Linear operators between classes of prestarlike functions*, Comment. Math. Helv. **52** (1977), 497–509.

[12] Ruscheweyh, St., *Neighborhoods of univalent functions*, Proc. Amer. Math. Soc. **81** (1981), 521–527.

[13] Sheil-Small, T., E.M. Silvia, *Neighborhoods of analytic functions*, J. Anal. Math. **52** (1989), 210–240.

[14] Silverman, H., E.M. Silvia, *The influence of the second coefficient on prestarlike functions*, Rocky Mountain J. Math. **10** (1980), 469–474.

[15] Silverman, H., E.M. Silvia, *Subclasses of starlike functions subordinate to convex functions*, Canad. J. Math. **37** (1985), 48–61.

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Received October 9, 2003