HIDETAKA HAMADA and GABRIELA KOHR

Subordination chains
and univalence of holomorphic mappings
on bounded balanced pseudoconvex domains

ABSTRACT. Let Ω be a bounded balanced pseudoconvex domain with C^1 plurisubharmonic defining functions in $\mathbf{C}^n$. We introduce a subclass of univalent mappings on Ω, called the class of mappings which have the parametric representation and we study several properties of these mappings, concerning the growth, covering and distortion results. We give some consequences, examples and conjectures.

1. Introduction. Let $\mathbf{C}^n$ be the space of n complex variables $z = (z_1,\ldots,z_n)'$ with the Euclidean inner product $\langle z, w \rangle = \sum_{j=1}^n z_j \overline{w}_j$ and the norm $\|z\| = \langle z, z \rangle^{1/2}$, $z \in \mathbf{C}^n$. The symbol $'$ means the transpose of vectors and matrices. The origin $(0, 0, \ldots, 0)'$ is denoted by 0 and by $L(\mathbf{C}^n, \mathbf{C}^m)$ we denote the space of all continuous, linear operators from $\mathbf{C}^n$ into $\mathbf{C}^m$ with the standard operator norm. By I we denote the identity in $L(\mathbf{C}^n, \mathbf{C}^n)$. Let $H(G)$ be the set of holomorphic mappings from a domain $G \subset \mathbf{C}^n$ into $\mathbf{C}^n$.

A mapping $f \in H(G)$ is said to be locally biholomorphic on G if its Fréchet derivative

$$Df(z) = \left[\frac{\partial f_j(z)}{\partial z_k} \right]_{1 \leq j,k \leq n}$$

as an element of $L(\mathbf{C}^n, \mathbf{C}^n)$ is nonsingular at each point $z \in G$. If $f \in H(G)$, we say that f is biholomorphic on G if the inverse f^{-1} exists, is holomorphic on a domain Ω and $f^{-1}(\Omega) = G$.

Let D be a balanced pseudoconvex domain in $\mathbf{C}^n$. The Minkowski function h of D is defined as follows

$$h(z) = \inf \left\{ t > 0 : \ \frac{z}{t} \in D \right\}.$$

Then $D = \{z \in \mathbf{C}^n : \ h(z) < 1\}$. The set $\{z \in \mathbf{C}^n : \ h(z) < r\}$ is denoted by D_r for $0 < r \leq 1$. A mapping $v \in H(D)$ is called a Schwarz mapping if $h(v(z)) \leq h(z)$ for all $z \in D$.

If $f, g \in H(D)$, we say that f is subordinate to g ($f \prec g$) if there exists a Schwarz mapping $v \in H(D)$ such that $f(z) = g(v(z))$, for all $z \in D$.

Let $\{f(z,t)\}_{t \geq 0}$ be a family of mappings such that $f_t(z) = f(z,t) \in H(D)$ and $f_t(0) = 0$ for each $t \geq 0$. We call $\{f(z,t)\}$ a subordination chain if $f(z,s) \prec f(z,t)$ for all $z \in D$ and $0 \leq s \leq t$. Moreover, $f(z,t)$ is called univalent if $f(\cdot, t)$ is univalent on D for each $t \geq 0$.

Let Ω be a domain in $\mathbf{C}^n$. We say that Ω has C^k ($k \geq 1$) plurisubharmonic defining functions, if for any $\zeta \in \partial\Omega$, there exist a neighborhood V of ζ in $\mathbf{C}^n$ and a C^k plurisubharmonic function r on V such that $\Omega \cap V = \{z \in V : \ r(z) < 0\}$. Then Ω is pseudoconvex. For example, a bounded pseudoconvex Reinhardt domain with C^k-boundary ($k \geq 2$) has C^k plurisubharmonic defining functions ([2, Lemma 2]) and the complex ellipsoids

$$B(p_1, \ldots, p_n) = \left\{ z \in \mathbf{C}^n : \ \sum_{j=1}^{n} |z_j|^{p_j} < 1 \right\},$$

with $p_1, \ldots, p_n > 1$, have C^1 plurisubharmonic defining functions.

From now on, let Ω be a bounded balanced pseudoconvex domain in $\mathbf{C}^n$ with C^1 plurisubharmonic defining functions and let h be the Minkowski function of Ω.

Then we have the following proposition.

Proposition 1.1 ([5]). *Let h be the Minkowski function of Ω, where Ω is a bounded balanced pseudoconvex domain in $\mathbf{C}^n$ with C^1 plurisubharmonic defining functions. Then h is C^1 on $\mathbf{C}^n \setminus \{0\}$ and continuous on $\mathbf{C}^n$. Also, $h(z) = 0$ iff $z = 0$ and $\overline{\Omega}_r = \{z \in \Omega : \ h(z) \leq r\}$ for any r, $0 < r < 1$.*

By using the method of subordination chains, we will introduce a proper subclass of $S(\Omega)$, where $S(\Omega)$ denotes the class of biholomorphic mappings

on Ω, normalized by $f(0) = 0$ and $Df(0) = I$, for $f \in S(\Omega)$, and we will investigate several properties of this class. A similar problem was recently solved by the second author on the unit Euclidean ball of $\mathbf{C}^n$ (see [12]) and by T. Poreda on the unit polydisc of $\mathbf{C}^n$ (see [17], [18]).

Let

$$\mathcal{M} = \left\{ g \in H(\Omega) : \ g(0) = 0, \ Dg(0) = I, \ \mathrm{Re} \left\langle g(z), \frac{\partial h^2}{\partial \overline{z}}(z) \right\rangle > 0, \right.$$

$$\left. z \in \Omega \setminus \{0\} \right\},$$

where

$$\frac{\partial h^2}{\partial \overline{z}} = \left(\frac{\partial h^2}{\partial \overline{z}_1}, \ldots, \frac{\partial h^2}{\partial \overline{z}_n} \right)'.$$

Then we have the following proposition.

Proposition 1.2 ([6]). *Let $g_t(z) = g(z, t) : \ \Omega \times [0, \infty) \to \mathbf{C}^n$ such that*
  (i) *for each $t \geq 0$, $g_t(z) \in \mathcal{M}$;*
  (ii) *for each $z \in \Omega$, $g(z, t)$ is a measurable function of t on $[0, \infty)$;*
  (iii) *for each $T > 0$ and $r \in (0, 1)$, there exists a constant $K(r, T)$ such that $\|g(z, t)\| \leq K(r, T)$, for $z \in \overline{\Omega}_r$ and $t \in [0, T]$.*
Then, for each $s \geq 0$ and $z \in \Omega$, there exists a unique locally absolutely continuous solution $v_{s,t}(z) = v(z, s, t)$ of the initial problem

$$(1.1) \qquad \frac{\partial v}{\partial t} = -g(v, t) \quad a.e. \quad t \geq s, \ v(z, s, s) = z.$$

Furthermore, $v_{s,t}(z)$ is a univalent Schwarz mapping on Ω, is a locally absolutely continuous function of t locally uniformly with respect to $z \in \Omega$ and $Dv_{s,t}(0) = e^{s-t}I$, for each s and t with $0 \leq s \leq t$.

Under the assumption of Proposition 1.2 the following proposition holds.

Proposition 1.3 ([6]). *Let $g(z, t)$ satisfy the assumptions of* Proposition *1.2, and let $v(z, s, t)$ be the locally absolutely continuous solution of the initial value problem (1.1). Then*

$$(1.2) \qquad \frac{e^t h(v(z, s, t))}{(1 - h(v(z, s, t)))^2} \leq e^s \frac{h(z)}{(1 - h(z))^2}$$

and

$$(1.3) \qquad \frac{e^s h(z)}{(1 + h(z))^2} \leq \frac{e^t h(v(z, s, t))}{(1 + h(v(z, s, t)))^2}$$

hold for all $z \in \Omega$ and $0 \leq s \leq t$.

2. Main results. We begin this section with the following lemma.

Lemma 2.1. *Let $g(z, t)$ satisfy the assumptions of* Proposition 1.2. *Then there exists the limit* $\lim_{t \to \infty} e^t v(z, s, t) = f(z, s)$, *locally uniformly in Ω as t increases to ∞ through a suitable sequence $\{t_m\}$, for $s \geq 0$ fixed, where $v = v(z, s, t)$ is the solution of the equation* (1.1). *Moreover, $f(\cdot, s)$ is univalent on Ω and $Df(0, s) = e^s I$.*

Proof. Fix $s \geq 0$ and consider $\varphi(z, t) = e^t v(z, s, t)$. By (1.2), we have

$$h(\varphi(z, t)) \leq e^s \frac{h(z)}{(1 - h(z))^2}, \quad z \in \Omega, \ t \geq s.$$

Since $\Omega = \{z \in \mathbf{C}^n : h(z) < 1\}$ is a bounded set with respect to h, it is also bounded with respect to the Euclidean distance, hence $\{\varphi(z, t)\}_{t \geq s}$ forms a normal family on Ω. Thus, there exists a sequence $\{t_m\}$, $t_m \to \infty$ such that $\lim_{m \to \infty} \varphi(z, t_m) = f(z, s)$ locally uniformly on Ω, where $f(z, s)$ is a holomorphic mapping on Ω. Since $\varphi(z, t_m)$ is univalent on Ω for each $t_m \geq s$, by taking into account Proposition 1.2, $D\varphi(0, t_m) = e^s I$, the mapping $f(z, s)$ cannot have the Jacobian identically zero. Therefore, $f(z, s)$ is univalent on Ω and also, $Df(0, s) = e^s I$. This completes the proof. $\square$

Taking into account the result of Lemma 2.1 we can introduce the following definition.

Definition 2.2. Let $f : \Omega \to \mathbf{C}^n$. We say that $f \in S^0(\Omega)$ if there exists a mapping $g(z, t) : \Omega \times [0, \infty) \to \mathbf{C}^n$ which satisfies the assumptions of Proposition 1.2, such that $\lim_{t \to \infty} e^t v(z, t) = f(z)$, locally uniformly on Ω, through a suitable sequence $\{t_m\}$, $t_m > 0$, $t_m \to \infty$, where $v = v(z, \cdot)$ is the solution of the equation

$$\frac{\partial v}{\partial t} = -g(v, t), \quad \text{a.e.} \ \ t \geq 0, \ v(z, 0) = z,$$

such that for each $z \in \Omega$, $v(z, \cdot)$ is a locally absolutely continuous function on $[0, \infty)$, locally uniformly with respect to z.

The class $S^0(\Omega)$ is called the class of mappings which have the parametric representation. This class was studied by T. Poreda when Ω is the unit polydisc of $\mathbf{C}^n$ with the maximum norm (see [17], [18]) and by the second author when Ω is the unit Euclidean ball of $\mathbf{C}^n$ (see [12]). Clearly, $S^0(\Omega) \subseteq S(\Omega)$.

Taking into account Theorems 6.1 and 6.3 [16] for $n = 1$, we have the following well known result:

$$S^0(U) = S(U),$$

where U denotes the unit disc in $\mathbf{C}$. It is natural to ask if an analogous result holds in several complex variables. Poreda and Kohr showed that the answer is negative on the unit polydisc and the unit Euclidean ball of $\mathbf{C}^n$, $n \geq 2$ ([17], [18], [12]), thus we expect to have a negative answer on Ω.

Before giving the main result of this paper, we give the following result that provides an example of a mapping from $S^0(\Omega)$.

Lemma 2.3 ([6]). *Let $f(z,t) : \Omega \times [0,\infty) \to \mathbf{C}^n$ such that $f_t(z) = f(z,t) \in H(\Omega)$, $Df_t(0) = e^t I$ and assume $f(z,t)$ is a locally absolutely continuous function of $t \in [0,\infty)$ locally uniformly with respect to $z \in \Omega$. Let $g(z,t)$ be as in* Proposition 1.2. *Suppose that $f(z,t)$ satisfies the following differential equation*

$$\frac{\partial}{\partial t} f(z,t) = Df(z,t)g(z,t), \quad a.e. \quad t \geq 0, \ z \in \Omega.$$

Moreover, suppose that there exists a sequence $\{t_m\}$ such that $t_m \to \infty$ and

$$\lim_{m \to \infty} e^{-t_m} f(z,t_m) = F(z)$$

locally uniformly in Ω. Then $\{f(z,t)\}$ is a univalent subordination chain and the limit $\lim_{t \to \infty} e^t v(z,s,t) = f(z,s)$, exists locally uniformly on Ω, as t increases to ∞ through a suitable subsequence of $\{t_m\}$ with $s \geq 0$ fixed, where $v = v(z,s,t)$ is the solution of the initial value problem (1.1). Thus, $f \in S^0(\Omega)$, where $f(z) = f(z,0)$, $z \in \Omega$.

Theorem 2.4. *If $f \in S^0(\Omega)$, then*

$$\frac{h(z)}{(1+h(z))^2} \leq h(f(z)) \leq \frac{h(z)}{(1-h(z))^2}, \quad z \in \Omega.$$

Consequently, $f(\Omega) \supset \frac{1}{4}\Omega$.

Proof. If $f \in S^0(\Omega)$, there exist a mapping $g = g(z,t) : \Omega \times [0,\infty) \to \mathbf{C}^n$ satisfying the conditions of Proposition 1.2 and a sequence $\{t_m\}$, $t_m > 0$, increasing to ∞ such that $f(z) = \lim_{m \to \infty} e^{t_m} v(z,t_m)$ locally uniformly on Ω, where $v = v(z,t)$ is the solution of the equation

$$\frac{\partial v}{\partial t} = -g(v,t), \quad a.e. \ t \geq 0, \ v(z,0) = z.$$

From Proposition 1.3 we deduce the following inequalities

$$\frac{e^t h(v(z,t))}{(1-h(v(z,t)))^2} \leq \frac{h(z)}{(1-h(z))^2}$$

and

$$\frac{e^t h(v(z,t))}{(1+h(v(z,t)))^2} \geq \frac{h(z)}{(1+h(z))^2},$$

for $z \in \Omega$ and $t \geq 0$. Since $\lim_{m \to \infty} h(e^{t_m} v(z,t_m)) = h(f(z)) < \infty$ for $z \in \Omega$, $\lim_{m \to \infty} h(v(z,t_m)) = \lim_{m \to \infty} e^{-t_m} h(e^{t_m} v(z,t_m)) = 0$. Substituting $t = t_m$ into the above inequalities and letting $m \to \infty$, we have

$$\frac{h(z)}{(1+h(z))^2} \leq h(f(z)) \leq \frac{h(z)}{(1-h(z))^2}. \quad \square$$

Remark 2.5. Let $f : \Omega \to \mathbf{C}^n$ be a starlike mapping, normalized by $f(0) = 0$, $Df(0) = I$. Then, as in [5] we can show that $f(z,t) = e^t f(z)$ is a univalent subordination chain that satisfies the assumptions of Lemma 2.3, where $g(z,t) = [Df(z)]^{-1} f(z)$, $z \in \Omega$, $t \geq 0$. Therefore $f \in S^0(\Omega)$.

Next, we recall the notion of spirallikeness, due to Gurganus [4], Suffridge [22] and the authors [9]. Let $A \in L(\mathbf{C}^n, \mathbf{C}^n)$ be such that $m(A) > 0$, where

$$m(A) = \min \left\{ \mathrm{Re} \left\langle A(z), \frac{\partial h^2}{\partial \overline{z}}(z) \right\rangle : z \in \partial \Omega \right\}.$$

Also, let $f \in H(\Omega)$ be normalized by $f(0) = 0$ and $Df(0) = I$. We say that f is spirallike relative to A, if f is univalent on Ω and $f(\Omega)$ is a spirallike domain relative to A, i.e.

$$e^{-sA} f(\Omega) \subset f(\Omega), \ s \geq 0,$$

where $e^{-sA} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} s^k A^k$.

Gurganus [4] gave a characterization of spirallikeness for locally biholomorphic mappings on the unit Euclidean ball B and Suffridge [22] extended the result to locally biholomorphic mappings on the unit ball of a Banach space. They showed that if f is locally biholomorphic on B, normalized by $f(0) = 0$ and $Df(0) = I$, then f is spirallike relative to $A \in L(\mathbf{C}^n, \mathbf{C}^n)$, with $m(A) > 0$, if and only if

$$Re \left\langle [Df(z)]^{-1} Af(z), z \right\rangle > 0, \ z \in B \setminus \{0\}.$$

The authors [9] extended the above result to bounded balanced pseudoconvex domains Ω with C^1 plurisubharmonic defining functions. Namely, if f is locally biholomorphic on Ω, normalized by $f(0) = 0$ and $Df(0) = I$, then f is spirallike relative to $A \in L(\mathbf{C}^n, \mathbf{C}^n)$, with $m(A) > 0$, if and only if

$$\mathrm{Re} \left\langle [Df(z)]^{-1} Af(z), \frac{\partial h^2}{\partial \overline{z}}(z) \right\rangle > 0, \ z \in \Omega \setminus \{0\}.$$

Now, let $f : \Omega \to \mathbf{C}^n$ be locally biholomorphic on Ω, normalized by $f(0) = 0$, $Df(0) = I$. Let $\alpha \in \mathbb{R}$ with $|\alpha| < \pi/2$. We say that f is spirallike of type α if

$$\mathrm{Re} \left\langle e^{-i\alpha} [Df(z)]^{-1} f(z), \frac{\partial h^2}{\partial \overline{z}} (z) \right\rangle > 0,$$

for $z \in \Omega \setminus \{0\}$ (see for details [7], [8]).

Clearly, spirallike mappings of type α, $\alpha \in \mathbb{R}$, $|\alpha| < \pi/2$, are also, spirallike relative to $A = e^{-i\alpha} I$.

Recently the authors showed that if f is locally biholomorphic on Ω and normalized, then f is spirallike of type α iff $\{f(z, t)\}$ is a univalent subordination chain, where $f(z, t) = e^{(1-ia)t} f(e^{iat} z)$, and $a = \tan \alpha$ (see [7], [8]).

This chain satisfies the assumptions of Lemma 2.3, where

$$g(z, t) = iaz + (1 - ia) e^{-iat} [Df(e^{iat} z)]^{-1} f(e^{iat} z).$$

Thus $f \in S^0(\Omega)$.

Therefore, we conclude that both spirallike mappings of type α, with $|\alpha| < \pi/2$, and starlike mappings, normalized, satisfy the growth and $\frac{1}{4}$-theorem, given in Theorem 2.4 (see also [8]).

Next, we will show that, contrary to the case of spirallike mappings of type α, with $\alpha \in \mathbb{R}$, $|\alpha| < \pi/2$, in general a spirallike mapping relative to a fixed linear operator , does not have parametric representation in $\mathbf{C}^n$, with $n \geq 2$. To this end, let $A \in L(\mathbf{C}^2, \mathbf{C}^2)$, given by $A(z_1, z_2) = (2z_1, z_2)'$, for all $(z_1, z_2)' \in \mathbf{C}^2$. Also, let $f(z_1, z_2) = (z_1 + az_2^2, z_2)'$, $(z_1, z_2)' \in B$, where $a \in \mathbf{C}$. Then $m(A) > 0$ and f is spirallike relative to A, for all $a \in \mathbf{C}$.

Now, let $a \in \mathbb{R}$, $a > 2\sqrt{15}$ and $z_0 = \left(0, \frac{1}{2}\right)'$. Then $\|f(z_0)\| > 2 = \|z_0\|/(1 - \|z_0\|)^2$ and taking into account the result of Theorem 2.4, we deduce that $f \notin S^0(B)$.

It is interesting to note that if we consider the complex ellipsoid $B(p_1, \ldots, p_n)$, where $p_1, \ldots, p_n > 1$, and if $f : B(p_1, \ldots, p_n) \to \mathbf{C}^n$, where $f(z) = (f_1(z_1), \ldots, f(z_n))'$, for $z = (z_1, \ldots, z_n)' \in B(p_1, \ldots, p_n)$, then it is not difficult to show that $f \in S^0(B(p_1, \ldots, p_n))$ if and only if $f_j \in S(U)$, for all $j \in \{1, \ldots, n\}$. This remark may provide many examples of mappings which have parametric representation on $B(p_1, \ldots, p_n)$, with $p_1, \ldots, p_n > 1$.

Now, we give the following estimate of coefficients of univalent mappings in Ω which can be imbedded in subordination chains.

Theorem 2.6. *Let $f(z, s) : \Omega \times [0, \infty) \to \mathbf{C}^n$ be a univalent subordination chain which satisfies the assumptions of* Lemma 2.3, *where Ω is a bounded balanced convex domain in $\mathbf{C}^n$ with C^1 plurisubharmonic defining functions. If h is the Minkowski function of Ω, then*

$$\left| \frac{1}{2!} \left\langle D^2 f(0)(z, z), \frac{\partial h^2}{\partial \overline{z}} (z) \right\rangle \right| \leq 2h^3(z), \quad z \in \Omega \setminus \{0\},$$

where $f(z) = f(z, 0)$, $z \in \Omega$.

In addition, if we denote by $D^k f(0)$ the kth Fréchet derivative of f at zero, then

$$h\left(\frac{1}{k!}D^k f(0)(z, \ldots, z)\right) \le \left[\frac{e(k+1)}{2}\right]^2,$$

for all $z \in \mathbf{C}^n$ with $h(z) = 1$ and $k \ge 2$.

Proof. Since $f(z, s)$ is a univalent subordination chain that satisfies the assumptions of Lemma 2.3, there exists a sequence $\{t_m\}$, increasing to ∞, such that $\lim_{m \to \infty} e^{t_m} v(z, s, t_m) = f(z, s)$ locally uniformly on Ω with $s \ge 0$ fixed, where v is the solution of the equation

$$\frac{\partial v}{\partial t} = -g(v, t), \quad \text{a.e. } t \ge s, \; v(z, s, s) = z.$$

Next, let $z \in \Omega \setminus \{0\}$ and $t \ge 0$ fixed. Also, let $p_t : U \to \mathbf{C}$,

$$p_t(\zeta) = \begin{cases} \frac{1}{\zeta} \left\langle g\left(\zeta \frac{z}{h(z)}, t\right), \frac{\partial h^2}{\partial \bar{z}}\left(\frac{z}{h(z)}\right)\right\rangle, & \zeta \ne 0 \\ 1, & \zeta = 0. \end{cases}$$

Then $p_t \in H(U)$. Since $g_t(z) \in \mathcal{M}$, $\operatorname{Re} p_t(\zeta) > 0$ for $\zeta \in U$. Hence

$$\frac{1 - |\zeta|}{1 + |\zeta|} \le \operatorname{Re} p_t(\zeta) \le \frac{1 + |\zeta|}{1 - |\zeta|}$$

on U. Substituting $\zeta = h(z)$, we obtain

$$(2.1) \qquad h^2(z)\frac{1 - h(z)}{1 + h(z)} \le \operatorname{Re}\left\langle g(z, t), \frac{\partial h^2}{\partial \bar{z}}(z)\right\rangle \le h^2(z)\frac{1 + h(z)}{1 - h(z)}.$$

Also, since $\operatorname{Re} p_t(\zeta) > 0$ on U, we have $|p_t'(0)| \le 2$. Since

$$p_t'(0) = \lim_{\zeta \to 0} p_t'(\zeta) = \frac{1}{2h^3(z)}\left\langle D^2 g(0, t)(z, z), \frac{\partial h^2}{\partial \bar{z}}(z)\right\rangle,$$

$$(2.2) \qquad \left|\frac{1}{2!}\left\langle D^2 g(0, t)(z, z), \frac{\partial h^2}{\partial \bar{z}}(z)\right\rangle\right| \le 2h^3(z).$$

On the other hand, by Lemma 2.3 $f(z, t)$ satisfies the following differential equation

$$\frac{\partial}{\partial t} f(z, t) = Df(z, t)g(z, t), \quad \text{a.e. } t \ge 0, \; z \in \Omega.$$

Fix $T > 0$. Then we can write the above equality in the form

$$f(z, T) - f(z, 0) = \int_0^T Df(z, t) g(z, t) dt, \quad z \in \Omega.$$

Now, fix $z \in \Omega$ and let $G_z(\zeta) = f(\zeta z, T) - f(\zeta z, 0)$ and

$$H_z(\zeta) = \int_0^T Df(\zeta z, t) g(\zeta z, t) dt,$$

for $\zeta \in U$. Then, clearly $G_z(\zeta) = H_z(\zeta)$, $\zeta \in U$.

Considering the result of Lemma 2.3, by differentiation we deduce that

$$H_z''(0) = \int_0^T [2D^2 f(0, t)(z, z) + e^t D^2 g(0, t)(z, z)] dt,$$

hence, we get at once

$$D^2 f(0, T)(z, z) - D^2 f(0, 0)(z, z)$$
$$= \int_0^T [2D^2 f(0, t)(z, z) + e^t D^2 g(0, t)(z, z)] dt.$$

By simple transformations this equality is equivalent to the following

$$e^{-2T} D^2 f(0, T)(z, z) - D^2 f(0, 0)(z, z) = \int_0^T e^{-t} D^2 g(0, t)(z, z) dt,$$

hence,

(2.3)
$$e^{-2T} \left\langle D^2 f(0, T)(z, z), \frac{\partial h^2}{\partial \overline{z}}(z) \right\rangle - \left\langle D^2 f(0, 0)(z, z), \frac{\partial h^2}{\partial \overline{z}}(z) \right\rangle$$
$$= \int_0^T e^{-t} \left\langle D^2 g(0, t)(z, z), \frac{\partial h^2}{\partial \overline{z}}(z) \right\rangle dt.$$

As in the proof of Theorem 2.1 we can show that

(2.4)
$$h(f(z, T)) \le e^T \frac{h(z)}{(1 - h(z))^2}.$$

It is well known that, due to convexity of Ω, h is a norm in $\mathbf{C}^n$ and Ω is the unit ball of $\mathbf{C}^n$ with respect to h. By using the Cauchy formula

$$\frac{1}{2!} D^2 f(0, T)(z, z) = \frac{1}{2\pi i} \int_{|\zeta| = r} \frac{f(\zeta z, T)}{\zeta^3} d\zeta,$$

for $0 < r < 1$, and taking into account the relation (2.4), we easily obtain $\lim_{T \to \infty} h(e^{-2T} D^2 f(0, T)(z, z)) = 0$, hence at once, we get

$$(2.5) \qquad \lim_{T \to \infty} e^{-2T} D^2 f(0, T)(z, z) = 0.$$

Next, making use of the relations (2.2), (2.3) and (2.5), we deduce that

$$\left| \frac{1}{2!} \left\langle D^2 f(0, 0)(z, z), \frac{\partial h^2}{\partial \bar{z}}(z) \right\rangle \right| \leq 2 h^3(z).$$

Since $f(z) = f(z, 0)$ for $z \in \Omega$, considering the above inequality we obtain the desired conclusion. This completes the proof of the first part of our result.

Now, let $z \in \mathbf{C}^n$, $h(z) = 1$ and $k \geq 2$ be fixed. The last inequalities from our result follow from the Cauchy formula

$$\frac{1}{k!} D^k f(0)(z, \ldots, z) = \frac{1}{2\pi i} \int\limits_{|\zeta| = r} \frac{f(\zeta z)}{\zeta^{k+1}} d\zeta,$$

for all $0 < r < 1$.

Since h is a norm on $\mathbf{C}^n$, using the result of Theorem 2.4 we easily obtain

$$h\left(\frac{1}{k!} D^k f(0)(z, \ldots, z)\right) \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{h(f(re^{i\theta}z))}{r^k} d\theta$$

$$\leq \frac{1}{r^{k-1}(1-r)^2}, \quad 0 < r < 1.$$

On the other hand, since

$$\min_{0 < r < 1} \frac{1}{r^{k-1}(1-r)^2} \leq \left[\frac{e(k+1)}{2}\right]^2,$$

combining the above inequalities we obtain the desired relation

$$h\left(\frac{1}{k!} D^k f(0)(z, \ldots, z)\right) \leq \left[\frac{e(k+1)}{2}\right]^2.$$

This ends the proof. $\square$

Of a particular interest is the case of the complex ellipsoid $B(p_1, \ldots, p_n)$, with $p_1, \ldots, p_n > 1$. Then we can show that the results presented in Theorems 2.4 and 2.6 are sharp.

Corollary 2.7. *If $f \in S^0(B(p_1, \ldots, p_n))$, with $p_1, \ldots, p_n > 1$, then*

$$\frac{h(z)}{(1 + h(z))^2} \le h(f(z)) \le \frac{h(z)}{(1 - h(z))^2}, \quad z \in \Omega,$$

where h is the Minkowski function of $B(p_1, \ldots, p_n)$. Furthermore, the result is sharp.

We note that this result was recently obtained by Barnard-FitzGerald-Gong [1] for starlike and normalized mappings on the unit Euclidean ball and by Pfaltzgraff [14] on $B(p) = B(p_1, \ldots, p_n)$, with $p_1 = \cdots = p_n = p > 1$.

Proof of Corollary 2.7. It suffices to show that the estimates are sharp. To this end, let

$$f(z) = \left(\frac{z_1}{(1 - z_1)^2}, \ldots, \frac{z_n}{(1 - z_n)^2} \right)', \quad z = (z_1, \ldots, z_n)' \in B(p_1, \ldots, p_n).$$

Then f is a normalized starlike mapping on $B(p_1, \ldots, p_n)$ and $f(z, t) = e^t f(z)$ is a univalent subordination chain that satisfies the assumptions of Lemma 2.3 (see for details [6]). Thus, $f \in S^0(B(p))$.

If $z = (r, 0, \ldots, 0)'$, where $r \in [0, 1)$, then $h(z) = r$ and $h(f(z)) = r/(1 - r)^2 = h(z)/(1 - h(z))^2$. Also, if $z = (-r, 0, \ldots, 0)'$, where $r \in [0, 1)$, then $h(f(z)) = \frac{r}{(1+r)^2} = \frac{h(z)}{(1+h(z))^2}$. This completes the proof. $\square$

Now, consider $p_1 = \cdots = p_n = p > 1$ and denote $B(p_1, \ldots, p_n)$ by $B(p)$. In this case

$$h(z) = \|z\|_p = \left[\sum_{j=1}^n |z_j|^p \right]^{1/p},$$

for $z \in \mathbf{C}^n$, and we have the following

Corollary 2.8. *If $f(z, s) : B(p) \times [0, \infty) \to \mathbf{C}^n$ is a univalent subordination chain that satisfies the assumptions of* Lemma 2.3, *then*

$$\left| \sum_{j=1}^n u_j(z) \frac{|z_j|^p}{z_j} \right| \le 4\|z\|_p^{p+1},$$

for $z = (z_1, \ldots, z_n)' \in B(p) \setminus \{0\}$, where $u(z) = (u_1(z), \ldots, u_n(z))' = D^2 f(0)(z, z)$. This estimate is sharp.

Proof. We will show that the inequality is sharp. To this end, consider

$$f(z) = \left(\frac{z_1}{(1 - z_1)^2}, \ldots, \frac{z_n}{(1 - z_n)^2} \right)',$$

for $z \in B(p)$. We have just seen that $f \in S^0(B(p))$ and since

$$D^2 f(0)(z, \cdot) = \left(\sum_{m=1}^n \frac{\partial^2 f_k}{\partial z_j \partial z_m}(0) z_m \right)_{1 \leq j, k \leq n},$$

after short computations we obtain $D^2 f(0)(z, \cdot) = (a_{jk})_{1 \leq j, k \leq n}$, where

$$a_{jk} = \begin{cases} 4z_k, & j = k \\ 0, & j \neq k. \end{cases}$$

Hence,

$$\sum_{j=1}^n u_j(z) \frac{|z_j|^p}{z_j} = 4 \sum_{j=1}^n |z_j|^p z_j, \quad z \in B(p) \setminus \{0\}.$$

Next, let $z = (r, 0, \ldots, 0)'$, with $r \in [0, 1)$. Then $\|z\|_p = r$ and

$$\left| \sum_{j=1}^n u_j(z) \frac{|z_j|^p}{z_j} \right| = 4r^{p+1} = 4\|z\|_p^{p+1}.$$

This ends the proof. $\quad \square$

Now, we come back to the result of Theorem 2.6. It is natural to ask if the inequality

$$h\left(\frac{1}{2!} D^2 f(0)(z, z) \right) \leq 2h^2(z)$$

holds on $\Omega \setminus \{0\}$ for f satisfying the assumptions of Theorem 2.5. We give a negative answer if $\Omega = B(p_1, \ldots, p_n)$, where $p_1, \ldots, p_n > 1$.

To this end, consider the following example, in the case of $n = 2$. We note that this example was considered by T. J. Suffridge [22] and by K. Roper and T.J. Suffridge [19] on the unit ball $B(p)$, where $p \geq 1$.

Example 2.9. Let $f : B(p_1, p_2) \to \mathbf{C}^2, p_1 \geq p_2 > 1$, and $f(z) = (z_1 + az_2^2, z_2)'$, for $z = (z_1, z_2)' \in B(p_1, p_2) \setminus \{0\}$, where

$$a \in \mathbf{C}, \ |a| \leq \frac{p_2}{p_1} \left(1 + \frac{2p_1}{p_2(p_1 - 1)} \right)^{\frac{p_1 - 1}{p_1}} \left(1 + \frac{p_2(p_1 - 1)}{2p_1} \right)^{\frac{2}{p_2}}.$$

Then $f \in H(B(p_1, p_2))$, $f(0) = 0$, $Df(0) = I$, and by straightforward calculations we deduce that $\det Df(z) = 1$. Hence f is locally biholomorphic on $B(p_1, p_2)$.

On the other hand, $[Df(z)]^{-1} f(z) = (z_1 - az_2^2, z_2)'$. In order to show that f is starlike it suffices to prove that

$$\text{Re} \left\{ p_1 w_1(z) \frac{|z_1|^{p_1}}{z_1 h^{p_1}(z)} + p_2 w_2(z) \frac{|z_2|^{p_2}}{z_2 h^{p_2}(z)} \right\} \geq 0,$$

for all $z = (z_1, z_2)' \in B(p_1, p_2) \setminus \{0\}$, where $w(z) = [Df(z)]^{-1} f(z)$, see Theorem 3 [5].

Since $h(\zeta z) = |\zeta| h(z)$ for $\zeta \in \mathbf{C}$, $z \in \mathbf{C}^n$, this condition is equivalent to

$$\mathrm{Re} \left\{ p_1 w_1(\zeta \tilde{z}) \frac{|\tilde{z}_1|^{p_1}}{\zeta \tilde{z}_1 h^{p_1}(\tilde{z})} + p_2 w_2(\zeta \tilde{z}) \frac{|\tilde{z}_2|^{p_2}}{\zeta \tilde{z}_2 h^{p_2}(\tilde{z})} \right\} \geq 0,$$

for all $\tilde{z} = (\tilde{z}_1, \tilde{z}_2)' \in \partial B(p_1, p_2)$ and $\zeta \in U \setminus \{0\}$. On the other hand, applying the minimum principle to the harmonic function

$$\mathrm{Re} \sum_{j=1}^{2} p_j w_j(\zeta \tilde{z}) \frac{|\tilde{z}_j|^{p_j}}{\zeta \tilde{z}_j h^{p_j}(\tilde{z})}, \quad \zeta \in \overline{U},$$

we may assume that $\sum_{j=1}^{2} |z_j|^{p_j} = 1$. Then we have,

$$\mathrm{Re} \sum_{j=1}^{2} p_j w_j(z) \frac{|z_j|^{p_j}}{z_j h^{p_j}(z)} = p_1 \frac{|z_1|^{p_1}}{h^{p_1}(z)} + p_2 \frac{|z_2|^{p_2}}{h^{p_2}(z)} - p_1 \mathrm{Re} \left[a z_2^2 \frac{|z_1|^{p_1}}{z_1 h^{p_1}(z)} \right]$$

$$\geq p_2 - p_1 |a| (1 - r^{p_1})^{2/p_2} r^{p_1 - 1},$$

where $|z_1| = r$. By an elementary computation we can see that the function $g(r) = p_2 - p_1 |a| (1 - r^{p_1})^{2/p_2} r^{p_1 - 1}$ has the minimum value

$$p_2 - p_1 |a| \left(\frac{p_1 - 1}{2 p_1 / p_2 + p_1 - 1} \right)^{(p_1 - 1)/p_1} \left(\frac{2 p_1 / p_2}{2 p_1 / p_2 + p_1 - 1} \right)^{2/p_2},$$

therefore, for

$$|a| \leq \frac{p_2}{p_1} \left(1 + \frac{2 p_1}{p_2 (p_1 - 1)} \right)^{(p_1 - 1)/p_1} \left(1 + \frac{p_2 (p_1 - 1)}{2 p_1} \right)^{2/p_2},$$

we deduce that f is starlike and consequently $f \in S^0(B(p_1, p_2))$.

On the other hand, an easy computation shows that

$$D^2 f(0)(z, \cdot) = (a_{jk})_{1 \leq j, k \leq 2},$$

where

$$a_{jk} = \begin{cases} 2 a z_2, & j = 1, \ k = 2 \\ 0, & \text{otherwise,} \end{cases}$$

hence $D^2 f(0)(z, z) = (2 a z_2^2, 0)'$.

Now, let $z = (0, r)'$, where $r \in (0, 1]$. Then $h(z) = r$ and

$$h\left(\frac{1}{2!}D^2 f(0)(z, z)\right) = |a|r^2 = |a|h^2(z) > 2h^2(z)$$

for $|a| > 2$.

Finally, we conclude that if $p_1 \geq p_2 > 1$ and

$$2 < |a| \leq \frac{p_2}{p_1}\left(1 + \frac{2p_1}{p_2(p_1 - 1)}\right)^{(p_1 - 1)/p_1}\left(1 + \frac{p_2(p_1 - 1)}{2p_1}\right)^{2/p_2},$$

then

$$h\left(\frac{1}{2!}D^2 f(0)(z, z)\right) > 2h^2(z).$$

For example, let $p_1 = \ldots = p_n = p > 1$. Then the above inequality becomes

$$2 < |a| \leq \frac{p + 1}{p - 1}\left(\frac{p^2 - 1}{4}\right)^{1/p}.$$

Now, consider the function $g(x) = \frac{x+1}{x-1}\left(\frac{x^2-1}{4}\right)^{1/x}$, for $x > 1$. Obviously g is strictly increasing on $(1, \sqrt{5})$ and strictly decreasing on $(\sqrt{5}, \infty)$.

On the other hand, since $\lim_{x \to \infty} g(x) = \lim_{x \to 1} g(x) = 1$ and $g(\sqrt{5}) = (3 + \sqrt{5})/2 > 2$, there exists a unique $x_1 \in (\sqrt{5}, \infty)$, such that $g(x_1) = 2$. Also, since $g(2) = 3\sqrt{3}/2 > 2$, there exists a unique $x_0 \in (1, 2)$, such that $g(x_0) = 2$. Thus, we conclude that $g(x) > 2$ for all $x \in (x_0, x_1)$. Hence, if we choose

$$|a| = \frac{p + 1}{p - 1}\left(\frac{p^2 - 1}{4}\right)^{1/p},$$

where $p \in (x_0, x_1)$, then $|a| > 2$ and

$$\left\|\frac{1}{2}D^2 f(0)(z, z)\right\|_p > 2\|z\|_p^2.$$

This ends the proof. $\square$

We finish this section with an example of a mapping $f \in S^0(B(p))$, with $p > 1$, which cannot be imbedded in a univalent subordination chain that satisfies the assumptions of Lemma 2.3, in the case $n \geq 2$.

Example 2.10. Let $f : B(p) \to \mathbf{C}^n$, where $p > 1$ and let

$$f(z) = (z_1 + a z_2^2, z_2 \ldots, z_n)', \quad z = (z_1, \ldots, z_n)' \in B(p),$$

where $a \in \mathbf{C}$, $|a| > 2^{2+1/p}$. Then, as in the proof of Example 2.9, we deduce that f being biholomorphic on $B(p)$, is normalized, hence $f \in S(B(p))$.

From Example 2.9, we deduce that

$$\left| \frac{1}{2!} \left\langle D^2 f(0)(z,z), \frac{\partial h^2}{\partial \bar{z}}(z) \right\rangle \right| = \|z\|_p^{2-p} \cdot |a| \cdot |z_1|^{p-1} |z_2|^2,$$

for $z = (z_1, \ldots, z_n)' \in B(p) \setminus \{0\}$.

For $z = (r, r, 0, \ldots, 0)'$, where $0 < r < \frac{1}{2^{1/p}}$, we obtain

$$\left| \frac{1}{2!} \left\langle D^2 f(0)(z,z), \frac{\partial h^2}{\partial \bar{z}}(z) \right\rangle \right| = \frac{|a|}{2^{1+1/p}} \|z\|_p^3 > 2\|z\|_p^3,$$

for $|a| > 2^{2+1/p}$. Now, taking into account the result of Theorem 2.6, we deduce that f cannot be imbedded in a univalent subordination chain $f(z,t)$ that satisfies the assumptions of Lemma 2.3.

Hence, for $n \geq 2$ the class $S(B(p))$, $p > 1$, is essentially wider than the class of those biholomorphic mappings on $B(p)$, normalized, which can be imbedded in univalent subordination chains and which satisfy the assumptions of Lemma 2.3.

Finally, we would like to point out that similar results as in this section can be also obtained on the unit ball of $\mathbf{C}^n$, with an arbitrary norm.

3. On some estimates of Jacobian determinant for mappings with parametric representation on $B(p_1, ..., p_n)$, where $p_1, ..., p_n > 1$.

The aim of this section is to investigate the Jacobian determinant for a mapping from $S^0(B(p_1, \ldots, p_n))$ with $p_1, \ldots, p_n > 1$. A similar problem was recently studied by Pfaltzgraff and Suffridge for starlike mappings on the unit Euclidean ball of $\mathbf{C}^n$, see [15]. First, we give the following result that is essential in this work and which provides a large number of examples of mappings from $S^0(B(p_1, \ldots, p_n))$.

We remark that the results of Theorems 3.1, 3.3 and 3.5 remain true on the unit ball of $\mathbf{C}^n$ with an arbitrary norm, but in this case we don't know whether our estimations remain sharp.

Theorem 3.1. *For each $j \in \{1, \ldots, n\}$, let $f_j \in S(U)$. If $\lambda_j \geq 0$ and $\sum_{j=1}^n \lambda_j = 1$, then $F \in S^0(B(p_1, \ldots, p_n))$, where $p_1, \ldots, p_n > 1$ and*

$$(3.1) \qquad F(z) = z \prod_{j=1}^n \left(\frac{f_j(z_j)}{z_j} \right)^{\lambda_j}, \ z \in B(p_1, \ldots, p_n).$$

Proof. Since $f_j \in S(U)$, by Theorem 6.3 [16] $f_j(\zeta) = \lim_{t \to \infty} e^t v_j(\zeta, t)$ locally uniformly on U, for each $j \in \{1, \ldots, n\}$, where $v_j = v_j(\zeta, \cdot)$ is the solution

of the equation

$$\frac{\partial v_j}{\partial t}(\zeta, t) = -v_j(\zeta, t)p_j(v_j(\zeta, t), t), \quad \text{a.e. } t \geq 0, \ v_j(\zeta, 0) = \zeta.$$

This holds for all $\zeta \in U$, where $p_j(\cdot, t) \in H(U)$, $p_j(0, t) = 1$ and $\operatorname{Re} p_j(\zeta, t) > 0$ on U.

Now, let

$$V(z, t) = z \prod_{j=1}^{n} (v_j(z_j, t)/z_j)^{\lambda_j},$$

for all $z = (z_1, \ldots, z_n)' \in B(p_1, \ldots, p_n)$. Then obviously

$$F(z) = \lim_{t \to \infty} e^t V(z, t)$$

locally uniformly on $B(p_1, \ldots, p_n)$. Also, $V(z, 0) = z$ and after simple computations we obtain

$$\frac{\partial V}{\partial t}(z, t) = -V(z, t) \sum_{j=1}^{n} \lambda_j p_j(v(z_j, t), t), \quad \text{a.e. } t \geq 0,$$

for all $z \in B(p_1, \ldots, p_n)$.

Let $g : B(p_1, \ldots, p_n) \times [0, \infty) \to \mathbf{C}^n$ be given by $g(z, t) = z \sum_{j=1}^{n} \lambda_j p_j(z_j, t)$, for all $z = (z_1, \ldots, z_n)' \in B(p_1, \ldots, p_n)$ and $t \geq 0$. Then it is easy to see that $g(0, t) = 0$, $Dg(0, t) = I$ and

$$\operatorname{Re}\langle g(z, t), z \rangle = \|z\|^2 \sum_{j=1}^{n} \lambda_j \operatorname{Re} p_j(z_j, t) > 0,$$

for $z \in B(p_1, \ldots, p_n) \setminus \{0\}$, and $t \geq 0$.

On the other hand, using the above notations, we deduce that $V(z, t)$ satisfies the following equation

$$\frac{\partial V}{\partial t}(z, t) = -g(V(z, t), t), \quad \text{a. e. } t \geq 0, \ V(z, 0) = z,$$

for all $z \in B(p_1, \ldots, p_n)$. It is clear that the mapping g satisfies the assumptions of Proposition 1.2, therefore we deduce that $F \in S^0(B(p_1, \ldots, p_n))$, as the claimed conclusion. $\square$

Remark 3.2. It is interesting to note that if in particular each function f_j is spirallike of type α, where $\alpha \in \mathbb{R}$ and $|\alpha| < \pi/2$, then the mapping F is also spirallike of type α. To this end it suffices to see that $f_j(\zeta, t) =$

$e^{(1-ia)t}f_j(e^{iat}\zeta)$ is a univalent subordination chain, where $a = \tan\alpha$. Also, if we form the chain $F(z,t) = e^{(1-ia)t}F(e^{iat}z)$, for $z \in B(p_1, \ldots, p_n)$ and $t \geq 0$, then it is not difficult to show that $F(z,t)$ is a univalent subordination chain, that means F is spirallike of type α.

Indeed,

$$F(z,s) = z\prod_{j=1}^{n}\left(\frac{f_j(z_j,s)}{z_j}\right)^{\lambda_j} = \lim_{t\to\infty}e^tV(z,s,t),$$

locally uniformly on $B(p_1, \ldots, p_n)$, for each $s \geq 0$, where $V(z,s,t) = z\prod_{j=1}^{n}\left(\frac{v_j(z_j,s,t)}{z_j}\right)^{\lambda_j}$, and $v_j(\zeta,s,t)$ is the unique solution of the following differential equation

$$\frac{\partial}{\partial t}v_j(\zeta,s,t) = -v_j(\zeta,s,t)p_j(v_j(\zeta,s,t),s,t), \quad \text{a.e. } t \geq s, \; v_j(\zeta,s,s) = \zeta,$$

for all $\zeta \in U$, $s \geq 0$ and $j \in \{1, \ldots, n\}$. Next, it suffices to use the same kind of arguments as in the proof of Theorem 3.1, in order to show that $V(z,s,t)$ satisfies the following differential equation

$$\frac{\partial}{\partial t}V(z,s,t) = -g(V(z,s,t),s,t), \quad \text{a.e. } t \geq s, \; V(z,s,s) = z,$$

for all $z \in B(p_1, \ldots, p_n)$ and $s \geq 0$, where $g(z,t) = z\sum_{j=1}^{n}\lambda_j p_j(z_j,t)$. Finally, as in the proof of Theorem 3.1, we deduce that $g(\cdot,t) \in \mathcal{M}$ for each $t \geq 0$ and $g(z,t)$ satisfies the assumptions of Proposition 1.2. Hence, from Lemma 2.1, we deduce that $F(z,s)$ is a univalent subordination chain, as desired conclusion.

In particular, if $f_1, \ldots, f_n$ are starlike and normalized by $f_j(0) = f'_j(0) - 1 = 0$, then F is starlike on $B(p_1, \ldots, p_n)$. This result was recently obtained by Pfaltzgraff and Suffridge in the case of the unit Euclidean ball of $\mathbf{C}^n$, see [15]. Their proof is based on the well known analytic property of starlike mappings on the unit ball, e.g. $\text{Re}\langle[Df(z)]^{-1}f(z), z\rangle > 0$ on $B \setminus \{0\}$.

Now, we give the following distortion result for the Jacobian determinant of the mapping F that we have just constructed in the above theorem. To this end we introduce the following notations. Let $S_\pi^0(B(p_1, \ldots, p_n))$ be the class of mappings F defined by (3.1) and $J_F(z) = \det DF(z)$ for $z \in B(p_1, \ldots, p_n)$. Clearly $S_\pi^0(U) = S(U)$, but in several variables this is not true, because the mapping $F(z_1, z_2) = (z_1 + az_2^2, z_2)'$, $(z_1, z_2)' \in B(p_1, p_2)$ where

$$|a| \leq \frac{p_2}{p_1}\left(1 + \frac{2p_1}{p_2(p_1-1)}\right)^{(p_1-1)/p_1}\left(1 + \frac{p_2(p_1-1)}{2p_1}\right)^{2/p_2},$$

is an example of a mapping from $S^0(B(p_1, p_2))$, with $p_1 \geq p_2 > 1$, but not from $S_\pi^0(B(p_1, p_2))$.

Theorem 3.3. *If $F \in S_\pi^0(B(p_1, \ldots, p_n))$, where $p_1, \ldots, p_n > 1$, then*

$$(3.2) \qquad |J_F(z)| \leq \frac{1 + h(z)}{(1 - h(z))^{2n+1}}, \; z \in B(p_1, \ldots, p_n),$$

where h is the Minkowski function of $B(p_1, \ldots, p_n)$. This estimate is sharp.

Proof. Since $F \in S_\pi^0(B(p_1, \ldots, p_n))$, $F(z) = z \prod_{j=1}^n (\frac{f_j(z_j)}{z_j})^{\lambda_j}$, where each $f_j \in S(U)$ and $\lambda_j \geq 0$, $\sum_{j=1}^n \lambda_j = 1$. As in the proof of Theorem 3 [15], we can easily deduce that

$$J_F(z) = \prod_{j=1}^n \left(\frac{f_j(z_j)}{z_j} \right)^{n\lambda_j} \sum_{j=1}^n \lambda_j \frac{z_j f_j'(z_j)}{f_j(z_j)}.$$

Since f_j are normalized and univalent on U, by the well known Koebe-Bieberbach growth and distortion theorem we obtain the following relations, see [16],

$$\left| \frac{z_j f_j'(z_j)}{f_j(z_j)} \right| \leq \frac{1 + |z_j|}{1 - |z_j|} \leq \frac{1 + h(z)}{1 - h(z)},$$

and

$$\left| \frac{f_j(z_j)}{z_j} \right| \leq \frac{1}{(1 - |z_j|)^2} \leq \frac{1}{(1 - h(z))^2},$$

for all $z = (z_1, \ldots, z_n)' \in B(p_1, \ldots, p_n)$ and $j \in \{1, \ldots, n\}$.

Therefore, we have

$$|J_F(z)| \leq \frac{1 + h(z)}{(1 - h(z))^{2n+1}},$$

for $z \in B(p_1, \ldots, p_n)$.

In order to show that the result is sharp, let $f_1(\zeta) = \zeta/(1 - \zeta)^2$, for $\zeta \in U$. Also, let $\lambda_1 = 1$ and $\lambda_j = 0$ for $j \neq 1$. Then $f_1 \in S(U)$ and

$$J_F(z) = \frac{1}{(1 - z_1)^{2n}} \left(\frac{1 + z_1}{1 - z_1} \right),$$

for all $z = (z_1, \ldots, z_n)' \in B(p_1, \ldots, p_n)$. If $z = (r, \ldots, 0)'$, where $r \in [0, 1)$, then $h(z) = r$ and $|J_F(z)| = (1 + h(z))/(1 - h(z))^{2n+1}$. $\square$

Conjecture 3.4. *If $f \in S^0(B(p_1, \ldots, p_n))$, then J_f satisfies the relation* (3.2).

If in addition, we assume that each function f_j is starlike and normalized for $j \in \{1, \ldots, n\}$, then we can give the following better result. This result

was obtained by Pfaltzgraff and Suffridge on the unit Euclidean ball of $\mathbf{C}^n$. In order to obtain the lower estimate, denote by $S_\pi^*(B(p_1,\dots,p_n))$ the class of those mappings F given by (3.1), where each function f_j is starlike and normalized on the unit disc U. Clearly, this class is a subclass of starlike and normalized mappings on $B(p_1,\dots,p_n)$. Also, each function f_j has the property that $z_j f_j'(z_j)/f_j(z_j)$ is a point lying in a disc centered on the positive real axis whose diameter is the line segment $\left[\frac{1-r}{1+r}, \frac{1+r}{1-r}\right]$ for all $z = (z_1,\dots,z_n)' \in B(p_1,\dots,p_n)$ with $h(z) < r$ and $j \in \{1,\dots,n\}$.

Then, as in the proof of Theorem 3 [15], we can show the following

Theorem 3.5. *If $F \in S_\pi^*(B(p_1,\dots,p_n))$, where $p_1,\dots,p_n > 1$, then*

$$(3.3) \qquad \frac{1-h(z)}{(1+h(z))^{2n+1}} \le |J_F(z)| \le \frac{1+h(z)}{(1-h(z))^{2n+1}},$$

for all $z \in B(p_1,\dots,p_n)$. The result is sharp.

Conjecture 3.6. *If f is starlike on $B(p_1,\dots,p_n)$ and normalized by $f(0) = 0$ and $Df(0) = I$, then J_f satisfies the relation (3.3).*

We remark that the same conjecture as in Theorem 3.5, was also given for starlike and normalized mappings on the unit ball of $\mathbf{C}^n$ by Pfaltzgraff and Suffridge, see [15].

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Faculty of Engineering
Kyushy Kyoritsu University
1-8 Jiyugaoka, Yahatanishi-ku
Kitakyushu 807-8585, Japan
e-mail: hamada@kyukyo-u.ac.jp

Faculty of Mathematics
Babeş-Bolyai University
1 M. Kogălniceanu Str.
3400 Cluj-Napoca, Romania
e-mail: gkohr@math.ubbcluj.ro