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A. SWAMINATHAN

Polynomial approximation of outer functions

Abstract. We are interested in finding the polynomial approximants which retain the zero free property of a given analytic function in the unit disk. We show using convolution methods that the classical Cesáro means of order α, as an approximant, retains the zero free property of the derivatives of bounded convex functions in the unit disk. A cone-like conditions is also derived. These results generalizes the earlier results obtained in [1]. We extend this result for other source functions and suitable polynomial approximants.

1. Introduction. Let $\mathbb{D}$ denote the unit disk $\{z : |z| < 1\}$. An outer function for the class $\mathcal{H}^p$ $(p > 0)$ is a function of the form

$$F(z) = e^{i\gamma} \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log \psi(t)dt \right\}, \quad z \in \mathbb{D},$$

where $\gamma \in \mathbb{R}$, $\psi(t) \geq 0$, $\log \psi(t) \in L^1$ and $\psi(t) \in L^p$. From the above definition, it is not clear which functions are outer functions. One way of generating outer functions is by considering Smirnov domains. It can also be shown that the derivatives of bounded convex functions are outer functions. An extensive study of outer functions can be found in [4] and [8] and references therein.

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Problem 1. *If f is given as an infinite series, to find a suitable finite (polynomial) approximation so that the approximant retains the zero-free property of f.*

For the present problem, it is natural to consider the Taylor series approximation or partial sums. It was shown in [1] that the Taylor approximating polynomials to outer functions in general can vanish on $\mathbb{D}$, even if fairly restrictive geometric conditions such as convexity are imposed on the outer functions. This is only the case of lower order approximants whereas Hurwitz's theorem guarantees the existence of the solution in case of higher order Taylor approximating polynomials. We are interested in the zero-free property inherited by all approximants. So, a search is on for considering various polynomial (polynomial operators) so that they maintain the zero-free property of f. One of the polynomial best suited in the situation are Cesáro polynomials. The Cesáro sums (see [10, p. 142]) of order $\alpha \in \mathbb{N} \cup \{0\}$ of a series $\sum_{n=0}^{\infty} a_n z^n$ can be defined as

$$\sigma_n^{\alpha}(z, f) = \sum_{k=0}^{n} \frac{\binom{n-k+\alpha}{n-k}}{\binom{n+\alpha}{n}} a_k z^k,$$

where $\binom{a}{b} = \frac{a!}{b!(a-b)!}$. We write $\sigma_n^{\alpha}(z, f) = \sigma_n^{\alpha} * f(z)$ where $*$ denotes the Hadamard product or convolution given in the following way. For f, g analytic with $f(z) = a_0 + a_1 z + a_2 z^2 + \ldots$ and $g(z) = b_0 + b_1 z + b_2 z^2 + \ldots$, the (Hadamard) convolution of f and g is defined by $(f * g)(z) = a_0 b_0 + a_1 b_1 z + a_2 b_2 z^2 + \ldots$. It is natural to use the notation $f(z) * g(z)$ for $(f * g)(z)$ and vice versa frequently. In [1], the Cesáro means of order one are considered and the problem was solved for the derivatives of bounded convex functions.

So, we are interested in investigating the present problem with some other outer functions and Cesáro means of higher order and other polynomial approximants. The outer functions play a vital role in the discussion and a careful analysis is needed to find the outer functions. Jentzsch's (see [3, p. 352]) classical result shows that the circle of convergence for a Taylor series is a subset of the set of limit points of the zero sets of the sequence of Taylor approximants (partial sums). In [1], it was shown that Jentzsch's theorem can be extended to the first order Cesáro sums of order one. It is observed in [2] that the limit set to the zeros of Cesáro means of higher order contains much more than the circle of convergence and this is stated as

Theorem 1 ([2]). *Let σ_n^{α} be the Cesáro sums of order α of $f(z) = \sum_{k=0}^{\infty} a_n z^n$, $z \in \mathbb{D}$, where $\{a_n\}$ is a positive monotonically decreasing sequence such that $\frac{a_n}{a_{n+1}} \to 1$, and $\frac{a_0}{a_m} \leq a\, m^b$ for some $a, b \in \mathbb{R}$. Then*

$$\lim \frac{\sigma_n^\alpha(z)}{\frac{n!\alpha!}{(n+\alpha)!}a_n z^n} = \frac{1}{\left(1-\frac{1}{z}\right)^{\alpha+1}}$$

uniformly for $|z| \geq 1+\delta,\ \delta > 0.$

We require some basic facts and notations for further discussion.

Let $\mathcal{A}$ denote the class of functions analytic in $\mathbb{D}$ and $\mathcal{S}$ denote the class of functions in $\mathcal{A}$ such that f is one-to-one in $\mathbb{D}$ and $f(0) = 0 = f'(0) - 1$. We recall the following subclasses of $\mathcal{S}$.

A function $f \in \mathcal{A}$ is starlike of order α for some $0 < \alpha < 1$, if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, \quad z \in \mathbb{D}.$$

$\mathcal{S}^*(\alpha)$ denotes the class of these functions. By $\mathcal{C}(\alpha)$ we mean the class of convex functions of order $\alpha,\ 0 < \alpha < 1$, defined as

$$f \in \mathcal{C}(\alpha) \Longleftrightarrow zf' \in \mathcal{S}^*(\alpha), \quad z \in \mathbb{D}.$$

$\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{C}(0) = \mathcal{C}$ denote the well-known class of functions that map $\mathbb{D}$ onto starlike and convex domains respectively.

It is known that the class $\mathcal{K}(\alpha) \subset \mathcal{A}$ of close-to-convex functions f of type α satisfying

$$\exists\ g \in \mathcal{S}^*(\alpha), \phi \in \mathbb{R}: \quad \operatorname{Re} \frac{e^{i\phi} zf'(z)}{g(z)} > 0, \quad z \in \mathbb{D},$$

are univalent for $\alpha \geq 0$.

By $\mathcal{P}(\alpha)$, we mean the class of functions f satisfying the condition $\operatorname{Re} f(z) > \alpha$, $z \in \mathbb{D}$. For a detailed study of the above subclasses and various other classes we refer to [5].

A function $f \in \mathcal{A}$ is in the class of prestarlike functions of order α denoted by $\mathcal{R}^*(\alpha)$ (see [10, p. 48]), if and only if

$$\begin{cases} f * \frac{z}{(1-z)^{2-2\alpha}} \in \mathcal{S}^*(\alpha), & \alpha < 1 \\ \operatorname{Re} \frac{f(z)}{z} > \frac{1}{2}, & \alpha = 1 \end{cases}$$

for $z \in \mathbb{D}$.

Note that the factor $z/(1-z)^{2-2\alpha}$ itself is in $\mathcal{S}^*(\alpha)$. Thus we have $\mathcal{R}^*(1/2) = \mathcal{S}^*(1/2)$ and $\mathcal{R}^*(0) = \mathcal{C}$.

The Cesáro sums of order α, $\alpha \geq 1$ play an important role in geometric function theory (see e.g. [9], [10] and [11]). In particular, we give the following results.

Lemma 1 ([11]). *For* $n \in \mathbb{N}$, *we have*

(i) $z\sigma_n^\alpha \in \mathcal{R}^*\left(\frac{3-\alpha}{2}\right)$ *for* $\alpha \geq 1$, $z \in \mathbb{D}$.
(ii) $z\sigma_n^\alpha \in \mathcal{S}^*\left(\frac{1}{2}\right)$ *for* $\alpha \geq 2$, $z \in \mathbb{D}$.
(iii) $z\sigma_n^\alpha \in \mathcal{C}$ *for* $\alpha \geq 3$, $z \in \mathbb{D}$.

Lemma 2 ([10])**.**

(i) *For* $0 < \beta < \alpha \leq 1$ *we have* $\mathcal{R}^*(\beta) \subset \mathcal{R}^*(\alpha)$.
(ii) *Let* $0 \leq \alpha \leq 1$ *and* $f \in \mathcal{R}^*(\alpha)$, $g \in \mathcal{S}^*(\alpha)$. *Then* $f * g \in \mathcal{S}^*(\alpha)$. *A corresponding result holds with* $\mathcal{S}^*(\alpha)$ *replaced by either of* $\mathcal{C}(\alpha)$, $\mathcal{K}(\alpha)$, $\mathcal{R}^*(\alpha)$.

We now give a slight reformulation of a theorem given in [10].

Lemma 3 ([10])**.** *Let* $0 \leq \alpha \leq 1$. *Then for* $f \in \mathcal{R}^*(\alpha)$, $g \in \mathcal{S}^*(\alpha)$ *and* $p \in \mathcal{P}(\alpha)$, *there exist* $p_1 \in \mathcal{P}(\alpha)$ *such that* $f * gp = (f * g)p_1$, *where* $p_1(z) \in \mathcal{P}(\alpha)$.

2. Main results.

Theorem 2. *Let* $f \in \mathcal{A}$ *be such that* $f(\mathbb{D})$ *is convex. Then, the Cesáro means* $\sigma_n^\alpha(z, f')$ *of order* $\alpha \geq 1$ *of* f' *are zero-free on* $\mathbb{D}$ *for all* n.

Proof. Consider $\sigma_n^\alpha(z, f')$. Let k be defined as $k(z) = z/(1-z)^2$ and note that $zf'(z) = k(z) * f(z)$. Then,

$$\begin{aligned}\sigma_n^\alpha(z, f') &= \sigma_n^\alpha(z) * f'(z) \\ &= \frac{1}{z}\left\{zf'(z) * z\sigma_n^\alpha(z)\right\} \\ &= \frac{1}{z}\left\{f(z) * z\left(z\sigma_n^\alpha(z)\right)'\right\}.\end{aligned}$$

We know that $z\sigma_n^\alpha(z) \in \mathcal{K}(0)$. Therefore, there exists $g \in \mathcal{S}^*$ such that by the definition of $\mathcal{K}(0)$, $z\left(z\sigma_n^\alpha(z)\right)' = g(z)p(z)$, where $p(z) \in \mathcal{P}(0)$. So,

$$\sigma_n^\alpha(z, f') = \frac{f(z) * g(z)p(z)}{z}.$$

Now, by Lemma 3, $f \in \mathcal{C} = \mathcal{R}^*(0)$ and $g \in \mathcal{S}^*$ we have $f(z) * g(z)p(z) = (f(z) * g(z))p_1(z)$ to give

$$\sigma_n^\alpha(z, f') = \frac{[f(z) * g(z)]\, p_1(z)}{z}.$$

We know that $\operatorname{Re} p_1(z) > 0$ and $f * g = 0$ if and only if $z = 0$. Hence, $\sigma_n^\alpha(z, f') \neq 0$ and the proof is complete. $\square$

Recall that $h \in \mathcal{C}$ implies that $h(z)/z \in \mathcal{P}(1/2)$ and $h(z) \in \mathcal{S}^*(1/2)$. Now, we try to find some cone condition on the boundary for the approximants of Cesáro means of order α.

Theorem 3. *Let $f \in \mathcal{C}$. Then for $\alpha \geq 1$, $\sigma_n^{\alpha+1}(z, f')$ have their ranges contained in a cone (from 0) with opening $2\beta\pi$, where $\beta < 1$.*

Proof. We note that Marx–Strohhäcker theorem states that (see [5] for details) $f \in \mathcal{C} \Rightarrow f \in \mathcal{S}^*(1/2)$. We write,

$$\begin{aligned}
\sigma_n^{\alpha+1}(z, f') &= \sigma_n^{\alpha+1}(z) * f'(z) \\
&= \frac{1}{z}\left\{zf'(z) * z\sigma_n^{\alpha+1}(z)\right\} \\
&= \frac{1}{z}\left\{f(z) * z\left(z\sigma_n^{\alpha+1}(z)\right)'\right\} \\
&= \frac{1}{z}\left\{f(z) * z\sigma_n^{\alpha+1}(z)\frac{z\left(z\sigma_n^{\alpha+1}(z)\right)'}{z\sigma_n^{\alpha+1}(z)}\right\}.
\end{aligned}$$

Now, by Lemma 1, we have

$$z\sigma_n^{\alpha+1}(z) \in \mathcal{S}^*(1/2) \Longleftrightarrow \frac{z\left(z\sigma_n^{\alpha+1}(z)\right)'}{z\sigma_n^{\alpha+1}(z)} =: p(z)$$

where $p(z) \in \mathcal{P}(1/2)$. By using an argument similar to Theorem 2, we get

$$\begin{aligned}
\sigma_n^{\alpha+1}(z, f') &= \frac{f(z) * \left(z\sigma_n^{\alpha+1}(z)\right)p(z)}{z} \\
&= \frac{f(z) * z\sigma_n^{\alpha+1}(z)p_1(z)}{z}, \quad p_1(z) \in \mathcal{P}(1/2).
\end{aligned}$$

As $f \in \mathcal{S}^*(1/2)$ and $z\sigma_n^{\alpha+1}(z) \in \mathcal{S}^*(1/2)$, by Lemma 2, we have $f(z) * z\sigma_n^{\alpha+1}(z) \in \mathcal{S}^*(1/2)$. From [7, p. 57], we get $\frac{1}{z}\{f(z) * z\sigma_n^{\alpha+1}(z)\} \in \mathcal{P}(1/2)$.

Since $\frac{1}{z}\{f(z) * z\sigma_n^{\alpha+1}(z)\}$ is a polynomial, it is bounded, and hence there exists $\beta_1 < 1$ such that

$$\left|\arg\frac{f(z) * z\sigma_n^{\alpha+1}(z)}{z}\right| < \frac{\beta_1\pi}{2}.$$

Therefore, we have

$$\begin{aligned}
\left|\arg\sigma_n^{\alpha+1}(z, f')\right| &= \left|\arg\frac{f(z) * z\sigma_n^{\alpha+1}(z)}{z} \cdot p_1(z)\right| \\
&\leq \left|\arg\frac{f(z) * z\sigma_n^{\alpha+1}(z)}{z}\right| + |\arg p_1(z)| \\
&\leq \frac{\beta_1\pi}{2} + \frac{\pi}{2} = \beta\pi.
\end{aligned}$$

□

We note that the above result is valid for $f \in \mathcal{S}^*(1/2)$ as well and cannot be improved in general for $f \in \mathcal{S}^*$ using the above method.

In Theorem 2, we verified that zero-free property of f is retained by the approximant $\sigma_n^\alpha(z)$. We are interested in looking for various other source functions and suitable polynomial approximation (Cesáro sums or other polynomial) which makes the approximant zero-free. We are not aware whether the source functions are outer functions (zero-free). The source functions in Theorem 2 and Theorem 3 are derivatives of bounded convex functions and hence zero-free.

Now we consider the general case $f \in \mathcal{R}^*(\alpha)$. To get a result similar to Theorem 2 and Theorem 3 we need to have convolution results related to the class $\mathcal{R}^*(\alpha)$. For this we might prefer a polynomial which is in $\mathcal{K}(\alpha)$. Hence we look at the following polynomial.

Consider the polynomial $q_n^\alpha(z)$ defined by

$$zq_n^\alpha(z) = \sum_{k=0}^{\infty} \frac{(1,n)}{(2-2\alpha,n)} \frac{(2-2\alpha,k)}{(1,k)} \frac{(2-2\alpha,n-k)}{(1,n-k)} \frac{z^{k+1}}{k+1}, \quad z \in \mathbb{D}.$$

Here the Pochhammer symbol (a,r) is defined by $(a,r) = a(a+1)\cdots(a+r-1)$ and $(a,0) = 1$. It is known that $zq_n^\alpha(z) \in \mathcal{K}(\alpha)$ for $\alpha \leq 1/2$ (see [6, p. 1118]). We write $z\,q_n^\alpha(z,f) = z\,q_n^\alpha(z) * f(z)$ where $f \in \mathcal{A}$.

Theorem 4. *Let $f \in \mathcal{R}^*(\alpha), \alpha \leq 1/2$. Then, $q_n^\alpha(z,f')$ are zero-free in $\mathbb{D}$.*

Proof. Proceeding similarly to Theorem 2, we get

$$\begin{aligned} q_n^\alpha(z,f') &= q_n^\alpha(z) * f'(z) \\ &= \frac{1}{z}\left\{zf'(z) * zq_n^\alpha(z)\right\} \\ &= \frac{1}{z}\left\{f(z) * z\left(zq_n^\alpha(z)\right)'\right\}. \end{aligned} \tag{1}$$

$zq_n^\alpha(z) \in \mathcal{K}(\alpha)$ implies that there exists $g(z) \in \mathcal{S}^*(\alpha)$ such that $z(zq_n^\alpha(z))' = g(z)p(z)$ where $p(z) \in \mathcal{P}(0)$. Therefore, using Theorem 3, (1) gives

$$\begin{aligned} q_n^\alpha(z,f') &= \frac{1}{z}\left\{f(z) * g(z)p(z)\right\} \\ &= \frac{1}{z}\left\{f(z) * g(z)\right\} p_1(z), \quad p_1(z) \in \mathcal{P}(0). \end{aligned}$$

Now by Theorem 2 we get $f \in \mathcal{R}^*(\alpha), g \in \mathcal{S}^*(\alpha) \implies f(z) * g(z) \in \mathcal{S}^*(\alpha)$ and $f(z) * g(z) = 0$ if and only if $z = 0$. Since $\operatorname{Re} p_1(z) > 0$ we deduce $q_n^\alpha(z,f') \neq 0$ and the proof is complete. $\square$

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A. Swaminathan
Department of Mathematics
Indian Institute of Technology
IIT-Kharagpur
Kharagpur - 721 302
India

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