ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LIX, 2005 SECTIO A 19–26

JAROSŁAW GÓRNICKI

Another proof of the existence of fixed points of rotative nonexpansive mappings

Dedicated to W. A. Kirk on the occasion of his receiving an Honorary Doctorate from Maria Curie-Skłodowska University

Abstract. We give a new elementary proof that the condition of rotativeness assures in all Banach spaces, the existence of fixed points of nonexpansive mappings, even without weak compactness, or another special geometric structure.

1. Introduction and Preliminaries. Let C be a nonempty closed convex subset of a Banach space E and $T\colon C \to C$ be a *nonexpansive* mapping, i.e.

$$\|Tx - Ty\| \le \|x - y\| \qquad \text{for all} \quad x, y \in C. \tag{1}$$

In general, to assure the existence of fixed points for nonexpansive mappings some assumptions concerning the geometry of the space are added, see [2], [1].

Given integer $n \ge 2$ and real $a \in [0, 1)$, we say that a mapping $T\colon C \to C$ is (a, n)-*rotative* if for any $x \in C$,

$$\|x - T^n x\| \le a\|x - Tx\|. \tag{2}$$

2000 *Mathematics Subject Classification.* Primary 47H09, 47H10.
Key words and phrases. Nonexpansive mappings, rotative mappings, fixed points.

We will simply say that the mapping is n-rotative if it is (a,n)-rotative with some $a < n$, and *rotative* if it is n-rotative for some $n \geq 2$.

In 1981, K. Goebel and M. Koter [3] proved the following theorem:

Theorem 1. *If C is a nonempty closed convex subset of a Banach space E, then any nonexpansive rotative mapping $T\colon C \to C$ has a fixed point.*

Note in particular that this result does not require any compactness assumption on C nor does it require special geometric conditions on the underlying Banach space.

For more accurate studies we refer to [7] where the current state of knowledge concerning such mappings and related topics are presented.

The term "rotative" originates from the fact that all rotations in the Euclidean plane satisfy this condition. Any periodic mapping T with period n (i.e. such that $T^n = I$) is $(0,n)$-rotative, all contractions are rotative for all $n \geq 2$.

We have difficulty in indicating nontrivial examples of rotative mappings. It shows the problem, which has been opened for many years: it is not known whether there exists a $(0,2)$-rotative lipschitzian self-mapping (involution) on a closed convex subset of a Banach space which is fixed point free (see [2], p. 180).

Under such circumstances we illustrate the phenomenon of rotativeness with the following examples.

Example 1. Let $C[0,1]$ be the space of continuous real valued functions on $[0,1]$ with the standard supremum norm. Set

$$K = \{x \in C[0,1] \colon 0 = x(0) \leq x(t) \leq x(1) = 1\}.$$

A mapping $T\colon K \to K$ defined by $Tx(t) = tx(t)$, $x \in K$, $t \in [0,1]$ is nonexpansive (even contractive, i.e. $\|Tx - Ty\| < \|x - y\|$ if $x \neq y$) and fixed point free. This mapping is not rotative, which can be concluded from Theorem 1.

Example 2. Let K be defined as in Example 1 and let K_1 be the set of all functions $x \in K$ which are nondecreasing. For $n \in \mathbb{N}$, $n \geq 2$ and $x \in K_1$ let

$$Tx(t) = \begin{cases} x\left(\frac{n-1}{n} + t\right) - x\left(\frac{n-1}{n}\right) & \text{if } t \in [0, \frac{1}{n}], \\ x\left(t - \frac{1}{n}\right) + \left[1 - x\left(\frac{n-1}{n}\right)\right] & \text{if } t \in [\frac{1}{n}, 1]. \end{cases}$$

It is easy to see that $T : K_1 \to K_1$ is nonexpansive, n-rotative (because, $T^n x(t) = x(t)$) and $x(t) = t$, $t \in [0,1]$, is a fixed point of T.

The rotativeness condition (2) is actually independent of nonexpansiveness of T (and, in fact, this condition may be satisfied by very irregular mappings).

A mapping $T\colon C \to C$ is said to be *k-lipschitzian* if

$$\|Tx - Ty\| \leq k\|x - y\| \qquad \text{for all} \quad x, y \in C.$$

We can also consider k-lipschitzian rotative mappings with $k > 1$. (In this setting, rotativeness assures the existence of fixed points with k slightly greater than 1, see [2], [5], [7].) Recall that even in a Hilbert space one can construct a selfmapping of the unit ball which is fixed point free and $(1+\varepsilon)$-lipschitzian with $0 < \varepsilon < 1$ arbitrarily small (see [1], Example 1). Of course this mapping is not rotative.

Example 3. Let K be defined as in Example 1. For $k > 1$ set

$$Tx(t) = k \max\left\{x(t) - \left(1 - \frac{1}{k}\right), 0\right\}.$$

It is easy to see that $T\colon K \to K$ is k-lipschitzian, $(\sum_{j=1}^{n} \frac{1}{k^{j-1}}, n)$-rotative and fixed point free. Here, $a > 1$. For 2-rotative mappings, aside from some observations recording $a = 0$, no examples are known in the case $0 < a \leq 1$.

The following lemma plays an important role in the paper.

Lemma 1 ([4]). *Let C be a nonempty closed convex subset of a Banach space E and let $T\colon C \to C$ be k-lipschitzian. Assume that $A, B \in \mathbb{R}$ and $0 \leq A < 1$ and $0 < B$. If for an arbitrary $x \in C$ there exists $z \in C$ such that*

$$\|Tz - z\| \leq A\|Tx - x\|$$

and

$$\|z - x\| \leq B\|Tx - x\|,$$

then T has a fixed point in C.

The next simple lemma has technical character.

Lemma 2. *If $x \in \mathbb{R}_+ \setminus \{1\}$ and $n \geq 3$, then*

$$\sum_{j=2}^{n-1} jx^{j-1} = \frac{2x - nx^{n-1} - x^2 + (n-1)x^n}{(1-x)^2}. \tag{3}$$

Proof. It is consequence of a simple calculation:

$$\begin{aligned}
\sum_{j=2}^{n-1} jx^{j-1} &= 2x + 3x^2 + \cdots + (n-1)x^{n-2} \\
&= [x^2]' + [x^3]' + \cdots + [x^{n-1}]' \\
&= \left[\sum_{j=2}^{n-1} x^j\right]' = \left[\frac{x^2 - x^n}{1-x}\right]'
\end{aligned}$$

and the right site of (3) is obvious. □

2. Main result. Now, we give a new elementary and constructive proof of Theorem 1. Our proof is based on finding a sequence which converges to a fixed point of the investigated mapping. In this paper we use Halpern's idea of iterative procedure [6]. Halpern used his procedure to construct an infinite sequence in a Hilbert space which converges to a fixed point of a nonexpansive mapping. We use his idea in a different way and in Banach spaces.

Proof of Theorem 1. CASE $n = 2$. We consider

$$z = \frac{1}{2}\left(T^2x_0 + Tx_0\right),$$

where x_0 is an arbitrary point in C, and the sequence $\{z_p\}$ generated by

$$\begin{aligned} z_1 &= z, \\ z_{p+1} &= \frac{1}{2}\left(T^2z_p + Tz_p\right), \quad p = 1, 2, \dots . \end{aligned}$$

It is not difficult to see (using only the triangle inequality, (1) and (2)) that

$$\|z - Tz\| \le \left(\frac{1}{2} + \frac{a}{4}\right)\|x_0 - Tx_0\| \tag{4}$$

and

$$\|z - x_0\| \le \frac{1}{2}(a+1)\|x_0 - Tx_0\|. \tag{5}$$

Since $\frac{1}{2} + \frac{a}{4} < 1$ for $a < 2$, by inequalities (4) and (5), Lemma 1 implies the existence of fixed points of T in C and guarantees that the sequence $\{z_p\}$ is strongly convergent to a fixed point of T.

CASE $n \ge 3$. We consider a sequence generated by the following iteration:

$$\begin{aligned} x_0 &= x \in C, \\ x_1 &= \alpha T^n x_0 + (1-\alpha)Tx_0, \\ x_2 &= \alpha T^n x_0 + (1-\alpha)Tx_1, \\ &\dots\dots\dots\dots\dots\dots \\ x_{n-1} &= \alpha T^n x_0 + (1-\alpha)Tx_{n-2}, \end{aligned}$$

where $\alpha \in (0,1)$. Put $z = x_{n-1}$, then

$$\begin{aligned} \|z - Tz\| &= \|\alpha T^n x_0 + (1-\alpha)Tx_{n-2} - Tz\| \\ &\le \alpha\|T^{n-1}x_0 - z\| + (1-\alpha)\|x_{n-2} - z\| \\ &= \alpha\|T^{n-1}x_0 - \alpha T^n x_0 - (1-\alpha)Tx_{n-2}\| \\ &\quad + (1-\alpha)\|\alpha T^n x_0 + (1-\alpha)Tx_{n-3} - \alpha T^n x_0 - (1-\alpha)Tx_{n-2}\| \\ &\le \alpha^2\|x_0 - Tx_0\| + \alpha(1-\alpha)\|T^{n-2}x_0 - x_{n-2}\| \\ &\quad + (1-\alpha)^2\|x_{n-3} - x_{n-2}\|. \end{aligned} \tag{6}$$

An estimation for $\alpha(1-\alpha)\|T^{n-2}x_0 - x_{n-2}\|$ is the following

$$\begin{aligned}
\alpha(1-\alpha)\|T^{n-2}x_0 - x_{n-2}\| &= \\
&= \alpha(1-\alpha)\|T^{n-2}x_0 - \alpha T^n x_0 - (1-\alpha)Tx_{n-3}\| \\
&= \alpha(1-\alpha)\|\alpha(T^{n-2}x_0 - T^n x_0) + (1-\alpha)(T^{n-2}x_0 - Tx_{n-3})\| \\
&\le \alpha^2(1-\alpha)\|x_0 - T^2x_0\| + \alpha(1-\alpha)^2\|T^{n-3}x_0 - x_{n-3}\| \\
&= \alpha^2(1-\alpha)\|x_0 - T^2x_0\| \\
&\quad + \alpha(1-\alpha)^2\|T^{n-3}x_0 - \alpha T^n x_0 - (1-\alpha)Tx_{n-4}\| \\
&= \alpha^2(1-\alpha)\|x_0 - T^2x_0\| \\
&\quad + \alpha(1-\alpha)^2\|\alpha(T^{n-3}x_0 - T^n x_0) + (1-\alpha)(T^{n-3}x_0 - Tx_{n-4}\| \\
&\le \alpha^2(1-\alpha)\|x_0 - T^2x_0\| + \alpha^2(1-\alpha)^2\|x_0 - T^3x_0\| \qquad (7) \\
&\quad + \alpha(1-\alpha)^3\|T^{n-4}x_0 - x_{n-4}\| \le \cdots \\
&\le \alpha^2(1-\alpha)\|x_0 - T^2x_0\| + \alpha^2(1-\alpha)^2\|x_0 - T^3x_0\| \\
&\quad + \alpha^2(1-\alpha)^3\|x_0 - T^4x_0\| + \cdots \\
&\qquad + \alpha^2(1-\alpha)^{n-2}\|x_0 - T^{n-1}x_0\|
\end{aligned}$$

(using only the triangle inequality and (1))

$$\le \left[\alpha^2 \sum_{j=2}^{n-1} j(1-\alpha)^{j-1}\right] \|x_0 - Tx_0\|.$$

The evaluation for the next expression in (6) is the following

$$\begin{aligned}
(1-\alpha)^2\|x_{n-3} - x_{n-2}\| &\le (1-\alpha)^3\|x_{n-4} - x_{n-3}\| \le \cdots \\
&\le (1-\alpha)^{n-1}\|x_1 - x_0\| \\
&= (1-\alpha)^{n-1}\|\alpha(T^n x_0 - x_0) + (1-\alpha)(Tx_0 - x_0)\| \qquad (8) \\
&\le [\alpha(1-\alpha)^{n-1}a + (1-\alpha)^n]\|x_0 - Tx_0\|.
\end{aligned}$$

Combining (6) with (7) and (8) yields

$$\begin{aligned}
\|z - Tz\| \le \Bigg\{\alpha^2 + \alpha^2 \sum_{j=2}^{n-1} j(1-\alpha)^{j-1} \qquad (9) \\
+ \alpha(1-\alpha)^{n-1}a + (1-\alpha)^n\Bigg\}\|x_0 - Tx_0\|.
\end{aligned}$$

Putting $\alpha = \frac{1}{n}$ and applying Lemma 2, from (9) we obtain

$$\begin{aligned} \|z - Tz\| \le \Bigg\{ \frac{1}{n^2} + \frac{1}{n^2} \sum_{j=2}^{n-1} j \left(\frac{n-1}{n} \right)^{j-1} \\ + \frac{1}{n} \left(\frac{n-1}{n} \right)^{n-1} a + \left(\frac{n-1}{n} \right)^{n} \Bigg\} \|x_0 - Tx_0\| \\ = \left\{ 1 + \left(\frac{a}{n} - 1 \right) \left(\frac{n-1}{n} \right)^{n-1} \right\} \|x_0 - Tx_0\|. \end{aligned} \tag{10}$$

Moreover, we have

$$\begin{aligned} \|z - x_0\| &= \|\alpha(T^n x_0 - x_0) + (1-\alpha)(Tx_{n-2} - x_0)\| \\ &\le \alpha a \|x_0 - Tx_0\| + (1-\alpha)\|Tx_{n-2} - x_0\| \\ &\le \alpha a \|x_0 - Tx_0\| + (1-\alpha)[\|Tx_{n-2} - T^n x_0\| + \|T^n x_0 - x_0\|] \\ &\le a\|x_0 - Tx_0\| + (1-\alpha)\|x_{n-2} - T^{n-1}x_0\| \\ &= a\|x_0 - Tx_0\| \\ &\quad + (1-\alpha)\|\alpha(T^n x_0 - T^{n-1}x_0) + (1-\alpha)(Tx_{n-3} - T^{n-1}x_0)\| \\ &\le a\|x_0 - Tx_0\| + \alpha(1-\alpha)\|x_0 - Tx_0\| \\ &\quad + (1-\alpha)^2\|x_{n-3} - T^{n-2}x_0\| \\ &\le a\|x_0 - Tx_0\| + \alpha(1-\alpha)\|x_0 - Tx_0\| \\ &\quad + \alpha(1-\alpha)^2\|x_0 - T^2x_0\| + \alpha(1-\alpha)^3\|x_0 - T^3x_0\| + \cdots \\ &\qquad + \alpha(1-\alpha)^{n-2}\|x_0 - T^{n-2}x_0\| + (1-\alpha)^{n-1}\|x_0 - Tx_0\| \end{aligned} \tag{11}$$

(using only the triangle inequality and (1))

$$\begin{aligned} &\le \big\{a + \alpha(1-\alpha) + 2\alpha(1-\alpha)^2 + \cdots \\ &\quad +(n-2)\alpha(1-\alpha)^{n-2} + (1-\alpha)^{n-1}\big\} \|x_0 - Tx_0\| \\ &\le \{n + 1 + 2 + \cdots + (n-2) + 1\} \|x_0 - Tx_0\| \\ &= \frac{1}{2}(n^2 + n + 2)\|x_0 - Tx_0\|. \end{aligned}$$

Since

$$1 + \left(\frac{a}{n} - 1 \right) \left(\frac{n-1}{n} \right)^{n-1} < 1 \qquad \text{for} \quad a < n,$$

by inequalities (10) and (11), Lemma 1 implies the existence of fixed points of T in C. □

Remark 1. From the above proof it follows that the sequence $\{z_p\}$ generated by the following iteration process:

$$z_1(x) = x_{n-1}(x),\ z_2(x) = x_{n-1}\big(z_1(x)\big), \ldots, z_{p+1}(x) = x_{n-1}\big(z_p(x)\big)$$

for $p = 1, 2, 3, \ldots$, converges strongly to a fixed point of T.

Moreover, the mapping $R\colon C \to C$ defined by

$$R(x) = \lim_{p\to\infty} z_p(x) \tag{12}$$

is a nonexpansive retraction of C onto the fixed point set of T.

(Recall, a continuous mapping $r\colon C \to F$ is called a *retraction* if $r(x) = x$ for all $x \in F$.)

Proof. Note that a mapping $T_\alpha\colon C \to C$ defined by

$$T_\alpha x_0 = (1-\alpha)Tx_{n-2} + \alpha T^n x_0, \quad x_0 \in C, \quad 0 < \alpha < 1,$$

is nonexpansive. Indeed, for any $x_0, y_0 \in C$ by nonexpansiveness of T, we have

$$\begin{aligned}
\|T_\alpha x_0 - T_\alpha y_0\| &= \|(1-\alpha)Tx_{n-2} + \alpha T^n x_0 - (1-\alpha)Ty_{n-2} - \alpha T^n y_0\| \\
&\le (1-\alpha)\|Tx_{n-2} - Ty_{n-2}\| + \alpha\|T^n x_0 - T^n y_0\| \\
&\le (1-\alpha)\|x_{n-2} - y_{n-2}\| + \alpha\|x_0 - y_0\| \\
&= (1-\alpha)\|(1-\alpha)Tx_{n-3} + \alpha T^n x_0 - (1-\alpha)Ty_{n-3} - \alpha T^n y_0\| \\
&\quad + \alpha\|x_0 - y_0\| \\
&\le (1-\alpha)\big[(1-\alpha)\|x_{n-3}-y_{n-3}\| + \alpha\|x_0-y_0\|\big] + \alpha\|x_0-y_0\| \\
&\le (1-\alpha)^2\|x_{n-3} - y_{n-3}\| + \big[(1-\alpha)\alpha + \alpha\big]\|x_0 - y_0\| \le \cdots \\
&\le (1-\alpha)^{n-1}\|x_0 - y_0\| \\
&\quad + \big[(1-\alpha)^{n-2} + (1-\alpha)^{n-3} + \cdots + (1-\alpha) + 1\big]\alpha\|x_0 - y_0\| \\
&= \left\{(1-\alpha)^{n-1} + \frac{1}{\alpha}\big[1 - (1-\alpha)^{n-1}\big]\alpha\right\}\|x_0 - y_0\| \\
&= \|x_0 - y_0\|.
\end{aligned}$$

Routine calculation shows that a fixed point of T is a fixed point of T_α.

Now, we can consider two sequences

$$\begin{aligned}
&x_0 \longrightarrow z_1 \longrightarrow z_2 \longrightarrow \ldots \longrightarrow z \\
&y_0 \longrightarrow z_1^* \longrightarrow z_2^* \longrightarrow \ldots \longrightarrow z^*
\end{aligned}$$

generated for $x_0, y_0 \in C$ by $z_{k+1} = T_\alpha z_k$, $z_{k+1}^* = T_\alpha z_k^*$, $k = 1, 2, \ldots$. Since T_α is nonexpansive,

$$\|z_{k+1} - z_{k+1}^*\| = \|T_\alpha z_k - T_\alpha z_k^*\| \le \|z_k - z_k^*\|,$$

and the sequence $\{\|z_k - z_k^*\|\}$, as weakly decreasing and bounded, is convergent. Let

$$\lim_{k\to\infty}\|z_k - z_k^*\| = \|z - z^*\|.$$

Because the norm is continuous and

$$\|z - z^*\| \le \|x_0 - y_0\|$$

we have

$$\|Rx_0 - Ry_0\| = \|z - z^*\| \leq \|x_0 - y_0\|,$$

and the proof is complete. □

Remark 2. Note, that the above iteration procedure and its modification are applicable to the proofs of the existence of fixed points of k-lipschitzian n-rotative mappings with $n \geq 3$ (see [5]). The results obtained in [5] are better than these which are known until nowadays.

References

[1] Goebel, K., *Metric environment of the topological fixed point theorems*, Handbook of Metric Fixed Point Theory (W. A. Kirk, B. Sims, eds.), Kluwer Academic Publishers, Dordrecht–Boston–London, 2001, pp. 577–611.

[2] Goebel, K., W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge, 1990.

[3] Goebel, K., M. Koter, *A remark on nonexpansive mappings*, Canadian Math. Bull. **24** (1981), 113–115.

[4] Górnicki, J., *Fixed points of involutions*, Math. Japonica **43**, no. 1 (1996), 151–155.

[5] Górnicki, J., K. Pupka, *Fixed points of rotative mappings in Banach spaces*, J. Nonlinear and Convex Anal. **6** (2005) (to appear).

[6] Halpern, B., *Fixed points of nonexpansive maps*, Bull. Amer. Math. Soc. **73** (1967), 957–961.

[7] Kaczor, W., M. Koter-Mórgowska, *Rotative mappings and mappings with constant displacement*, Handbook of Metric Fixed Point Theory (W. A. Kirk, B. Sims, eds.), Kluwer Academic Publishers, Dordrecht–Boston–London, 2001, pp. 323–337.

Jarosław Górnicki
Department of Mathematics
Rzeszów University of Technology
P.O. Box 85, 35-595 Rzeszów
Poland
e-mail: gornicki@prz.rzeszow.pl

Received February 24, 2005