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# A note on transversally Finsler foliations

Abstract. In the paper [5] a definition of transversally Finsler foliation was given. In this paper we prove a theorem which gives an alternative description of such foliations similar to the case of Riemannian ones. In our considerations transversal cone plays important role. This is a Finsler counterpart of the subspace orthogonal to the leaves.

**1. The subduced transversal metric.** Let $V$ be a finite dimensional vector space over reals $\mathbb{R}$.

**Definition 1.1.** We say that a function $F : V \to \mathbb{R}$ is a Minkowski norm on $V$ if it has the following properties:

(i) $F(v) \geq 0$ for any $v \in V$ and $F(v) = 0$ if and only if $v = 0$,
(ii) $F(\lambda v) = \lambda F(v)$ for any $v \in V$ and $\lambda > 0$,
(iii) $F$ is $C^\infty$ on $V \setminus \{0\}$,
(iv) for any $0 \neq v \in V$ the bilinear, symmetric form $g_v : V \times V \to \mathbb{R}$

$$g_v(u, w) = \frac{1}{2}\frac{\partial^2 F^2(v + tu + sw)}{\partial t \partial s}|_{t=0,\, s=0}$$

is an inner product.

A pair $(V, F)$ is called a Minkowski space.

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The condition (iv) can be written in the following equivalent form. Let $e_1, \dots, e_n$ be a basis of the vector space $V$ and $(v^1, \dots, v^n)$ be coordinates of a vector $v$. Then we can express $F(v)$ as a function $F(v^1, \dots, v^n)$ and (iv) is equivalent to

(iv)$'$ the matrix $\dfrac{\partial^2 F^2}{\partial v^i \partial v^j}$ is positively definite at any $v \neq 0$.

It can be proved [1] that a Minkowski norm $F$ satisfies the triangle inequality

$$F(v_1 + v_2) \leq F(v_1) + F(v_2).$$

A set $B_F = \{v \in V : F(v) \leq 1\}$ is called a unit ball of the norm $F$. It is known [3] that a unit ball is a strictly convex set.

Let $F_1 : V_1 \to \mathbb{R}$ and $F_2 : V_2 \to \mathbb{R}$ be the Minkowski norms on the finite dimensional vector spaces $V_1$ i $V_2$. Let $B_{F_1}$ and $B_{F_2}$ be the corresponding unit balls.

**Definition 1.2** ([2])**.** A surjective linear map $\pi : V_1 \to V_2$ is called an isometric submersion if $\pi(B_{F_1}) = B_{F_2}$.

Let $W \subset V$ be a subspace of a Minkowski space $(V, F)$. Put $Q = V/W$ and let $\pi : V \to Q$ be a projection. We can define Minkowski norm $F_Q$ in $Q$ in the following way. For $[v] = \{v + w : w \in W\} \in Q$ we put

$$F_Q([v]) = \inf\{F(v + w) : w \in W\} = \inf\{F(u) : u \in [v]\}.$$

Geometrically $F_Q([v])$ equals to the distance from the origin to the affine subspace $\pi^{-1}([v]) \subset V$. Observe that strict convexity of the unit ball implies that there exists exactly one $w_0 \in W$ such that $F_Q([v]) = F(v+w_0)$. Indeed, suppose that $F_Q([v]) = F(v + w_1) = F(v + w_2) = \lambda, w_1 \neq w_2$. Then for any $t \in (0, 1)$

$$F(v + tw_1 + (1 - t)w_2) \leq tF(v + w_1) + (1 - t)F(v + w_2) = F_Q([v]) = \lambda.$$

But $tw_1 + (1 - t)w_2$ is an interior point of a strictly convex set

$$B_F^\lambda = \lambda \cdot B_F = \{u \in V : F(u) \leq \lambda\},$$

so $F(v + tw_1 + (1 - t)w_2) < \lambda$.

**Proposition 1.1.** *$F_Q$ is a Minkowski norm in $Q = V/W$ and $\pi : V \to Q$ is an isometric submersion of Minkowski spaces $(V, F)$ and $(Q, F_Q)$.*

**Proof.** It is clear that $F_Q([v]) \geq 0$ and $F_Q([v]) = 0$ if and only if $[v] = 0$. For any $\lambda \geq 0$ we have

$$\begin{aligned} F_Q(\lambda[v]) &= F_Q([\lambda v]) = \inf\{\lambda v + w :\ w \in W\} = \lambda \inf\left\{v + \frac{1}{\lambda} w :\ w \in W\right\} \\ &= \lambda \inf\{v + w :\ w \in W\} = \lambda F_Q([v]). \end{aligned}$$

Let $u_1 \in [v_1]$, $u_2 \in [v_2]$ and $F_Q([v_1]) = F(u_1)$, $F_Q([v_2]) = F(u_2)$. Then

$$\begin{aligned} F_Q([v_1] + [v_2]) &= F_Q([v_1 + v_2]) = \inf\{F(u) :\ u \in [v_1 + v_2]\} \\ &\le F(u_1 + u_2) \le F(u_1) + F(u_2) = F_Q([v_1]) + F_Q([v_2]). \end{aligned}$$

We shall prove that $F_Q$ has the property (iv)$'$.

Let $G = \frac{1}{2}F^2$ and $G_Q = \frac{1}{2}F_Q^2$. Fix a basis $v_1, \dots, v_p$, $u_1, \dots, u_q$ such that $p + q = \dim V$ and $W = \operatorname{lin}\{v_1, \dots, v_p\}$. For any $[v] \in Q$, $[v] = y^1[u_1] + \dots + y^q[u_q]$ we have

$$F_Q([v]) = \inf\{F(x^1, \dots, x^p, y^1, \dots, y^q) : (x^1, \dots, x^p) \in \mathbb{R}^p\}.$$

Let $x^1(y^1, \dots, y^q), \dots, x^p(y^1, \dots, y^q)$ be the functions such that

$$F_Q([v]) = F(x^1(y^1, \dots, y^q), \dots, x^p(y^1, \dots, y^q), y^1, \dots, y^q).$$

We want to prove that $x^1(y^1, \dots, y^q), \dots, x^p(y^1, \dots, y^q)$ are $C^\infty$ functions on $\mathbb{R}^q \backslash \{0\}$. Observe that

$$F_Q(v) = \inf\{F(u) : u \in [v]\} \Leftrightarrow G_Q(v) = \inf\{G(u) :\ u \in [v]\}.$$

For fixed $[v] = y^1[u_1] + \dots + y^q[u_q]$ we can calculate $x^1(y^1, \dots, y^q), \dots,$ $x^p(y^1, \dots, y^q)$ as a solution of a system of $p$ equations

$$\begin{gathered} \frac{\partial G}{\partial x^1}(x^1, \dots, x^p, y^1, \dots, y^q) = 0 \\ \vdots \\ \frac{\partial G}{\partial x^p}(x^1, \dots, x^p, y^1, \dots, y^q) = 0. \end{gathered}$$

From the condition (iv)$'$ it follows that one can use the implicit function theorem to solve this system with respect to $x^1, \dots, x^p$ and the solutions are the $C^\infty$ functions of $y^1, \dots, y^q$ at any $(y^1, \dots, y^q) \neq (0, \dots, 0)$.

We have proved that

$$F_Q([v]) = F(x^1(y^1, \dots, y^q), \dots, x^p(y^1, \dots, y^q), y^1, \dots, y^q)$$

is $C^\infty$ functions on $Q \backslash \{0\}$. Since

$$G_Q(y^1, \dots, y^q) = G(x^1(y^1, \dots, y^q), \dots, x^p(y^1, \dots, y^q), y^1, \dots, y^q)$$

we have

$$\frac{\partial^2 G_Q}{\partial y^\alpha \partial y^\beta} = \frac{\partial^2 G}{\partial x^k \partial x^i} \frac{\partial x^k}{\partial y^\alpha} \frac{\partial x^i}{\partial y^\beta} + \frac{\partial^2 G}{\partial x^k \partial y^\beta} \frac{\partial x^k}{\partial y^\alpha} + \frac{\partial^2 G}{\partial y^\alpha \partial x^k} \frac{\partial x^k}{\partial y^\beta} + \frac{\partial^2 G}{\partial y^\alpha \partial y^\beta},$$

where $k, i \in \{1, \dots, p\}$, $\alpha, \beta \in \{1, \dots, q\}$. For $v = y^1[u_1] + \dots + y^q[u_q]$ we put

$$z^1 = \frac{\partial x^i}{\partial y^\alpha} y^\alpha, \dots, z^p = \frac{\partial x^p}{\partial y^\alpha} y^\alpha,$$

$w^1 = y^1, \dots, w^q = y^q$. Then

$$\frac{\partial^2 G_Q}{\partial y^\alpha \partial y^\beta} y^\alpha y^\beta = \frac{\partial^2 G}{\partial x^k \partial x^i} z^k z^i + 2 \frac{\partial^2 G}{\partial x^i \partial y^\alpha} z^i w^\alpha + \frac{\partial^2 G}{\partial y^\alpha \partial y^\beta} w^\alpha w^\beta > 0$$

for $(y^1, \dots, y^q) \neq (0, \dots, 0)$. □

The metric $F_Q$ is called a subduced metric on $Q$. Let $S = \{u \in V : F_Q([u]) = F(u)\}$.

**Proposition 1.2.** *$S$ is a cone in $V$ and $S \setminus \{0\}$ is a surface in $V$. The natural projection $\pi$ restricted to $S \setminus \{0\}$ is a diffeomorphism onto $Q \setminus \{0\}$.*

**Proof.** Let $u \in S$. There exists $[v]$ such that

$$F(u) = F_Q([v]) = \inf\{F(v + w) : w \in W\}.$$

We have

$$\begin{aligned} F_Q(\lambda[v]) &= \inf\{F(\lambda v + w) : w \in W\} = \lambda \inf \left\{ F\left(v + \frac{1}{\lambda} w\right) : w \in W \right\} \\ &= \lambda \inf\{F(v + w) : w \in W\} = \lambda F(u) = F(\lambda u), \end{aligned}$$

so $\lambda u \in S$ for $\lambda > 0$. The rest part of the proposition follows from the proof of the Proposition 1.1. □

**Example.** Let $V = \mathbb{R}^3$ and

$$F(v^1, v^2, v^3) = \sqrt{(v^1)^2 + (v^2)^2 + (v^3)^2} + \alpha_1 v^1 + \alpha_2 v^2 + \alpha_3 v^3$$

where $(v^1)^2 + (v^2)^2 + (v^3)^2 < 1$. The function $F$ is a Minkowski norm on $\mathbb{R}^3$. Take $W = \{(0, 0, v^3) : v^3 \in \mathbb{R}\}$. Then $\mathbb{R}^3 / W = Q = \mathbb{R}^2$, $F_{\mathbb{R}^2}(v^1, v^2) = \sqrt{(1 - \alpha_3^2)(v^1)^2 + (v^2)^2} + \alpha_1 v^1 + \alpha_2 v^2$ and

$$S = \left\{ (v^1, v^2, v^3) \in \mathbb{R}^3 : v^3 = \frac{-\alpha_3}{\sqrt{1 - \alpha_3^2}} \sqrt{(v^1)^2 + (v^2)^2} \right\}.$$

Let $M$ be a smooth manifold.

**Definition 1.3** ([1], [3], [2])**.** A smooth positive function $F$ on $TM \setminus \{0\}$ such that for each $x \in M$ the restriction of $F$ to $T_x M$ is a Minkowski norm, is called a Finsler metric on $M$.

**Definition 1.4** ([4])**.** A diffeomorphism $f : M \to M$ is called a Finsler isometry if

$$F(f(x), f_*(v)) = F(x, v)$$

for any $x \in M$, $v \in T_x M$.

**Definition 1.5.** A vector field $v : M \to TM$ is called a Killing vector field if the local 1-parameter transformations of $v$ are local isometries.

**2. Transversally Finsler foliations.** Let $(M, \mathcal{F})$ be a foliated manifold equipped with a Finsler metric $F : TM \to \mathbb{R}$ . We denote by $T_x\mathcal{F}$ the subspace of $T_xM$ tangent to the foliation and put $Q_x = T_xM/T_x\mathcal{F}$. $Q = \bigcup_{x\in M} Q_x$ is called a normal bundle of a foliation. We suppose that $\mathcal{F}$ is a foliation of codimension $q$ and $\dim M = p + q$. If $(x^1, \dots, x^p, y^1, \dots, y^q)$ are foliated coordinates in an open set $U \subset M$, $(a^1, \dots, a^p, b^1, \dots, b^q)$ are vector coordinates with respect to the basis $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^p}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^q}$, then $(y^1, \dots, y^q)$ are coordinates in $\overline{U} = U/\mathcal{F}$ and $(y^1, \dots, y^q, b^1, \dots, b^q)$ are coordinates in $T\overline{U}$. Denote by $\pi$ a natural projection $TM \to Q$ and let $p$ be a local projection $U \to \overline{U}$, $p(x^1, \dots, x^p, y^1, \dots, y^q) = (y^1, \dots, y^q)$. We shall identify the vectors $\frac{\partial}{\partial y^i}$ with the corresponding vectors tangent to $\overline{U}$. For each $x$ we can define the subduced Minkowski norm $F_{Q_x} : Q_x \to \mathbb{R}$.

**Proposition 2.1.** *The function $F_Q : Q \to \mathbb{R}$, $F_Q|_{Q_x} = F_{Q_x}$ has the following properties:*

(I) *for any $v \in Q_x$ $F_Q(v) \geq 0$ and $F_Q(v) = 0 \Leftrightarrow v = 0$,*
(II) *$F_Q(\lambda v) = \lambda F_Q(v)$ for any $\lambda > 0$,*
(III) *$F_Q$ is smooth on $Q \setminus \{0\}$,*
(IV) *for any $x \in M$ and $v, w \in Q_x$ the bilinear form*

$$\frac{\partial^2 F_Q^2(x, tv + sw)}{\partial t \partial s}|_{t=0,\ s=0}$$

*is an inner product in $Q_x$.*

**Proof.** Proof follows from the Propositions 1.1 and 1.2. □

$F_Q$ will be called a subduced metric on a normal bundle $Q$. Let $S_x$ denote a cone at $x$. $S_x$ will be called a transversal cone at $x$. Let $B_x$, $B_{Q_x}$ denote the unit balls of metrics $F$ and $F_Q$ respectively.

We recall a definition of a Finsler foliation.

**Definition 2.1** ([5]). A foliated cocycle $\{U_i, f_i, \gamma_{ij}\}$ on a manifold $M$ is said to be a Finsler foliation $\mathcal{F}$ if

a) $\{U_i\}$ is an open covering of $M$,
b) $f_i : U_i \to W$ is is a submersion, where $(W, F)$ is a Finsler manifold,
c) $\gamma_{ij}$ is a local Finsler isometry of $(W, F)$ such that for each $x \in U_i \cap U_j$

$$f_i(x) = (\gamma_{ij} \circ f_j)(x).$$

The Finsler manifold $(W, F)$ will be called the transversal manifold of foliation $\mathcal{F}$.

**Theorem 2.1.** *The following conditions are equivalent:*

(I) *$(M, \mathcal{F})$ is a Finsler foliation,*
(II) *there exists a Finsler metric $F$ on $M$ such that for an arbitrary foliated coordinate system $(x^1, \dots, x^p, y^1, \dots, y^q)$ on an open set $U \subset M$*

*and for any locally projectable vector field $V$ such that $v_x \in S_x$, $F(v)$ does not depend on $(x^1, \dots, x^p)$,*

(III) *there exists a Finsler metric $F$ on $M$ such that for any foliated coordinate system $(x^1, \dots, x^p, y^1, \dots, y^q)$ on $U$ the image of the unit ball under the local projection $p : U \to \overline{U}$ is constant along the leaves,*

(IV) *there exists a metric $F_Q$ in $Q$ satisfying the conditions* (I)–(IV) *of Proposition 2.1 such that in any foliated coordinate system the image of the unit ball $\{v \in Q : F_Q(v) \leq 1\}$ under the local projection $p : U \to \overline{U}$ is constant along the leaves.*

**Proof.** (I) $\Rightarrow$ (II). Let $g$ be an arbitrary Riemannian metric on $M$. Denote by $\mathcal{F}^\perp$ the bundle orthogonal to the leaves of the foliation. Let $f_i : U_i \to U_i/\mathcal{F} = W_i$ be a local submersion onto an open set $W_i$ of the transversal space $(W, F_W)$. Consider foliated coordinates $(x^1, \dots, x^p, y^1, \dots, y^q)$ in $U_i$ such that $(y^1, \dots, y^q)$ are coordinates in $W_i$. Choose a basis of $\mathcal{F}^\perp$ of the form

$$\frac{\partial}{\partial y^1} + A_1^i \frac{\partial}{\partial x^i}, \dots, \frac{\partial}{\partial y^q} + A_q^i \frac{\partial}{\partial x^i}.$$

For any $v \in T_xM$, $v = v_{\mathcal{F}} + v_{\mathcal{F}^\perp} = v^i \frac{\partial}{\partial x^i} + v^\alpha \left( \frac{\partial}{\partial y^\alpha} + A_\alpha^i \frac{\partial}{\partial x^i} \right)$ put

$$F(x, v) = \sqrt{g(v_{\mathcal{F}}, v_{\mathcal{F}}) + F_W^2(f_j^*(v_{\mathcal{F}^\perp}))} = \sqrt{\|v_{\mathcal{F}}\|^2 + F_W^2(f_j^*(v_{\mathcal{F}^\perp}))}.$$

The function $F$ is globally defined on $TM$ because of condition c) of Definition 2.1.

We have

$$\frac{\partial^2 F^2}{\partial v^i \partial v^j} = g_{ij}, \qquad \frac{\partial^2 F^2}{\partial v^i \partial v^\alpha} = 0, \qquad \frac{\partial^2 F^2}{\partial v^\alpha \partial v^\beta} = \frac{\partial^2 F_M^2}{\partial v^\alpha \partial v^\beta}.$$

It is easy to check that for the metric $F$ a transversal cone at $x$ is equal to $\mathcal{F}_x^\perp$ and the values of $F$ on the locally projectable vector fields with values in the transversal cone are constant along the leaves.

(II)$\Rightarrow$ (III). Let $(U, \phi)$ be a foliated coordinate system, $p : U \to U/\mathcal{F} = \overline{U}$ a local submersion. If $B_x$ is a unit ball at $x \in U$ and $S_x$ is a transversal cone then it is clear that $p^*(B_x) = p^*(B_x \cap S_x)$. Suppose that $\overline{w} \in p^*(B_{x_2} \cap S_{x_2})$ and $\overline{w} \notin p^*(B_{x_1} \cap S_{x_1})$, $\overline{x} = p(x_1) = p(x_2)$, $\overline{w} = p^*(w_{x_2})$, $w_{x_2} \in B_{x_2} \cap S_{x_2}$. We can suppose that $F(x_2, w_2) = 1$. Let $v$ be a projectable vector field along $p^{-1}(\overline{x})$ such that $v_x \in S_x$ and $v_{x_2} = w_{x_2}$. Then $F(x, w) = 1$ at any $x \in p^{-1}(\overline{x})$ . In particular $F(x, w_1) = 1$ and $p^*(v_{x_1}) = p^*(w_{x_2}) = \overline{w} \in p^*(B_{x_1} \cap S_{x_1})$.

(III)$\Rightarrow$ (IV). Suppose that $F$ is a Finsler metric such that for any foliated coordinate system $(x^1, \dots, x^p, y^1, \dots, y^q)$ on $U$ the image of the unit ball under the projection $p : U \to \overline{U}$ is constant along the leaves. Let $F_Q$ be the metric in $Q$ from Proposition 2.1. We know that for any $x \in U$ $p(B_x) = p(B_x \cap S_x)$. Any vector from $Q_x$ has a unique representation of the form $[v]$ where $v \in S_x$ and $B_{Q_x} = \{[v] \in Q_x : v \in S_x,\ F_{Q_x}[v] \leq 1\} = \{[v] \in Q_x : v \in S_x,\ F_x(v) \leq 1\}$ which implies that the image of the unit ball with respect to the metric $F_Q$ is constant along the leaves.

(IV) $\Rightarrow$ (I). Cover $M$ with domains $U_i$ of foliated coordinate systems. Then $p_i^*$ induces an isomorphism $Q_x \to T_{p_i(x)}\overline{U}_i$. Thus we can define a family $F_{\overline{U}_i x}$ of Finsler metrics $F_{\overline{U}_i x} : T\overline{U}_i \to \mathbb{R}$. But if $p_i(x_1) = p_i(x_2)$ then $p_i^*(B_{Q_{x_1}}) = p_i^*(B_{Q_{x_2}})$. It means that $F_{\overline{U}_i x_1} = F_{\overline{U}_i x_2}$. Suppose that $U_i \cap U_j \neq \emptyset$ and $\gamma_{ji} : p_i(U_i \cap U_j) \to p_j(U_i \cap U_j)$. From the definition of the metrics $F_{U_i}$ it follows that if $\overline{x}_j = \gamma_{ji}(\overline{x}_i)$ then $F_{\overline{U}_j}(\overline{x}_j, \gamma_{ji}^*(w)) = F_{\overline{U}_i}(\overline{x}_i, w)$, $w \in T_{\overline{x}_i}\overline{U}_i$. Gluing together the local transversal manifolds $\overline{U}_i$ we get a transversal Finsler structure $(W, F_W)$ and $\mathcal{F}$ is a Finsler foliation. $\square$

**Example 2.1.** Let $v$ be a Killing vector field without singularities for the metric $F$. We shall prove that the integral curves of $v$ form a 1-dimensional Finsler foliation. Take a foliated coordinate system $(y^0, y^1, \ldots, y^q)$ defined on $U$ such that locally $v = \frac{\partial}{\partial y^0}$. The change of this type of coordinates is of the form

$$y^{0'} = y^{0'}(y^0, y^1, \ldots, y^q),\ y^{1'} = y^{1'}(y^0, y^1, \ldots, y^q), \ldots,\ y^{q'} = y^{q'}(y^0, y^1, \ldots, y^q).$$

A one-parameter transformation group of the local diffeomorphisms can be written as follows

$$\phi_t(y^0, y^1, \ldots, y^q) = (y^0 + t, y^1, \ldots, y^q).$$

Let $v_x \in S_x \subset T_xM$ be a vector such that

$$F(x, v_x) = \inf\{F(x, v) : v \in T_xM,\ p^*(v) = w\},$$

$w \in T_{p(x)}\overline{U}$. If $x' \in U$ and $p(x) = p(x')$ then there exists $t$ such that $x' = \phi_t(x)$, $p^*(\phi_t^*(v)) = w$ and

$$F(x', \phi_t^*(v)) = \inf\{F(x', v')) : v' \in T_{x'}M,\ p^*(v') = w\}.$$

We have proved that the norm $F$ is constant along the leaves on the locally projectable vector fields.

**Example 2.2.** Let $g$ be a bundle-like metric of a Riemannian foliation $\mathcal{F}$. Take a basic 1-form $\omega$ such that $\|\omega\| \leq 1$. Then $F(v) = \sqrt{g(v,v)} + \omega(v)$ is a Finsler metric and the foliation $\mathcal{F}$ is transversally Finslerian. The metric $F$ is called a transversal Randers metric.

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