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Curvatures for horizontal lift of a Riemannian metric

ABSTRACT. In this paper we give formulas for coefficients of linear connection and basic curvatures of the bundle of volume forms with the horizontal lift of metric.

1. Introduction. Kurek in [4] defined the horizontal lift of a linear connection on a manifold M to the bundle of linear frames. Mikulski in [7] deduced natural linear operators transforming vector fields from n-dimensional manifold M to the vector bundle $T^{(r),a}M$ and in [6] the author defined the natural vector bundle $T^{(0,0),a}M$ associated to the principal bundle LM of linear frames on the smooth manifold M. On the other hand, Mozgawa and Miernowski in [5] presented a systematic approach of basic types of tensor with respect to a symmetric linear connection to the horizontal lift to the bundle of volume forms. In the present paper we are going to determine coefficients of a linear connection and its basic curvatures of the bundle of volume forms with the horizontal lift of a Riemannian metric.

Let M be an orientable n-dimensional manifold equipped with Riemannian metric $g = (g_{ij})$ and let V be the bundle of volume forms over M.

Throughout the paper we assume that $m, n, \ldots = 1, 2, 3, \ldots$ and indices $\alpha, \beta, \ldots = 0, 1, 2, \ldots$. The Einstein summation convention will be used with

respect to these system of indices. We also denote that

$$\Gamma^r_{r[a|n]} = \frac{1}{2}\left(\Gamma^r_{ra|n} - \Gamma^r_{rn|a}\right),$$

$$\Gamma^r_{r(a|n)} = \frac{1}{2}\left(\Gamma^r_{ra|n} + \Gamma^r_{rn|a}\right),$$

$$\Gamma^n_{[ik}\Gamma^t_{nt|j]} = \frac{1}{2}\left(\Gamma^n_{ik}\Gamma^t_{nt|j} - \Gamma^n_{jk}\Gamma^t_{nt|i}\right)$$

and we will use the following convention $g_{jk|i} = \frac{\partial g_{jk}}{\partial x^i}$ (see [3]).

Let M be equipped with the Levi–Civita connection Γ^k_{ij}. Then Γ^k_{ij} are given by the following expressions (see [2])

$$\Gamma^m_{ij} = \frac{1}{2}g^{km}\left(g_{jk|i} + g_{ki|j} - g_{ij|k}\right),$$

where $\left(g^{ij}\right)$ is the inverse matrix to (g_{ij}).

Moreover, the differentiability conditions for the existence of such a volume forms in local coordinates are given by

Lemma 1.1 ([9], [1]). *The Christoffel symbols for the metric g satisfy the following equality*

$$\Gamma^p_{kp|n} = \Gamma^p_{np|k}.$$

In the sequel of the paper we make use of the following theorem.

Theorem 1.2 ([5]). *Let g be a Riemannian metric on M. Then $\gamma = g^H$ is a Riemannian metric on V and*

$$\gamma = \begin{bmatrix} 1 & \Gamma^k_{ik} \\ \Gamma^k_{ik} & g_{ij} + \Gamma^k_{ik}\Gamma^t_{jt} \end{bmatrix}$$

$$\gamma^{-1} = \left(\gamma^{ij}\right) = \begin{bmatrix} 1 + g^{ij}\Gamma^k_{ik}\Gamma^t_{jt} & -g^{ij}\Gamma^k_{ik} \\ -g^{ij}\Gamma^k_{ik} & g^{ij} \end{bmatrix}.$$

2. Curvatures of the lifted metric. In this chapter we prove some basic facts on the properties of the curvatures of the lifted metric to the bundle of volume forms on M.

Theorem 2.1. *The Levi–Civita connection $\widetilde{\Gamma}^\delta_{\alpha\beta}$ for the horizontal lift metric γ are given by the following formulas:*

(a) $\widetilde{\Gamma}^s_{0n} = 0$,
(b) $\widetilde{\Gamma}^0_{00} = 0$,
(c) $\widetilde{\Gamma}^0_{n0} = 0$,
(d) $\widetilde{\Gamma}^s_{mn} = \Gamma^s_{mn}$,
(e) $\widetilde{\Gamma}^0_{mn} = \Gamma^t_{tm|n} - \Gamma^j_{mn}\Gamma^t_{jt}$.

Proof. Ad. (a). From the formula for a Levi–Civita connection we have

$$
\begin{aligned}
\widetilde{\Gamma}^s_{0n} &= \frac{1}{2}\left(\gamma_{n\alpha|0} + \gamma_{0\alpha|n} - \gamma_{on|\alpha}\right)\gamma^{s\alpha} \\
&= \frac{1}{2}\left(\gamma_{n0|0} + \gamma_{00|n} - \gamma_{0n|0}\right)\gamma^{s0} + \frac{1}{2}\left(\gamma_{na|0} + \gamma_{0a|n} - \gamma_{0n|a}\right)\gamma^{sa} \\
&= \frac{1}{2}\left(\gamma_{0a|n} - \gamma_{0n|a}\right)\gamma^{sa} = \frac{1}{2}\left(\Gamma^p_{kp|n} - \Gamma^p_{np|k}\right)g^{sa}.
\end{aligned}
$$

Moreover, from Lemma 1.1 we have that $\Gamma^p_{kp|n} = \Gamma^p_{pn|k}$. Hence $\widetilde{\Gamma}^s_{0n} = 0$.

Ad. (b), (c). These statements follow from definition of Levi–Civita connection, Theorem 1.2 and simple calculations.

Ad. (d). Similarly as above, we have

$$
\begin{aligned}
\widetilde{\Gamma}^s_{mn} &= \frac{1}{2}\left(\gamma_{m\alpha|n} + \gamma_{n\alpha|m} - \gamma_{mn|\alpha}\right)\gamma^{\alpha s} \\
&= \frac{1}{2}\left(\gamma_{m0|n} + \gamma_{n0|m} - \gamma_{mn|0}\right)\gamma^{0s} + \frac{1}{2}\left(\gamma_{ma|n} + \gamma_{na|m} - \gamma_{mn|a}\right)\gamma^{as} \\
&= -\Gamma^t_{t(m|n)}g^{js}\Gamma^t_{jt} + \Gamma^s_{mn} + g^{as}\Gamma^r_{ar}\Gamma^t_{t(m|n)} + g^{as}\left(\Gamma^t_{mt}\Gamma^r_{r[a|n]} + \Gamma^t_{nt}\Gamma^r_{r[a|m]}\right) \\
&= \Gamma^s_{mn},
\end{aligned}
$$

since from Lemma 1.1 we have $\Gamma^r_{r[a|m]} = 0$ and $\Gamma^r_{r[a|n]} = 0$.

Ad. (e). From the formula for a Levi–Civita connection and Theorem 1.2 we have

$$
\begin{aligned}
\widetilde{\Gamma}^0_{mn} &= \frac{1}{2}\left(\gamma_{m\alpha|n} + \gamma_{n\alpha|m} - \gamma_{mn|\alpha}\right)\gamma^{\alpha 0} \\
&= \frac{1}{2}\left(\gamma_{m0|n} + \gamma_{n0|m} - \gamma_{mn|0}\right)\gamma^{00} + \frac{1}{2}\left(\gamma_{ma|n} + \gamma_{na|m} - \gamma_{mn|a}\right)\gamma^{a0} \\
&= \Gamma^t_{t(m|n)}\left(1 + g^{ij}\Gamma^r_{ir}\Gamma^s_{js}\right) - \Gamma^j_{mn}\Gamma^s_{js} \\
&\quad - g^{aj}\Gamma^s_{js}\left(\Gamma^t_{mt}\Gamma^r_{r[a|n]} + \Gamma^t_{nt}\Gamma^r_{r[a|m]} + \Gamma^r_{ar}\Gamma^t_{t(m|n)}\right) \\
&= \Gamma^t_{t(m|n)}\left(1 + g^{ij}\Gamma^r_{ir}\Gamma^s_{js}\right) - \Gamma^j_{mn}\Gamma^s_{js} - g^{aj}\Gamma^s_{js}\Gamma^r_{ar}\Gamma^t_{t(m|n)} \\
&= \Gamma^t_{tm|n} - \Gamma^j_{mn}\Gamma^s_{js},
\end{aligned}
$$

since $\Gamma^t_{t(m|n)} = \Gamma^t_{tm|n}$. $\square$

Corollary 2.1. *If $n = m$ then Christoffel symbols for lifted metric γ are given by the following formulas:*

$$
\begin{aligned}
\widetilde{\Gamma}^s_{mm} &= \Gamma^s_{mm}, \\
\widetilde{\Gamma}^0_{mm} &= \Gamma^p_{mp|m} - \Gamma^j_{mm}\Gamma^t_{jt}.
\end{aligned}
$$

Let R be the curvature tensor of a Riemannian manifold M and let R^s_{ijk} be coefficients of R. Then R^s_{ijk} is expressed in terms of the coefficients Γ^k_{ij}

of the Riemannian connections by the formula (see [2])

$$R^s_{ikj} = \Gamma^l_{ik}\Gamma^s_{jl} - \Gamma^l_{jk}\Gamma^s_{il} + \Gamma^s_{ik|j} - \Gamma^s_{jk|i}.$$

Theorem 2.2. *Let* $\widetilde{R}^\sigma_{\alpha\beta}$ *be the coefficients of the curvature tensor* $\widetilde{R}$ *of the lifted metric* γ. *Then* $\widetilde{R}^\sigma_{\alpha\beta}$ *are related to the curvature tensor* R^s_{ijk} *of the matric* g *by the following formulas:*

(a) $\widetilde{R}^s_{0j0} = 0,$

(b) $\widetilde{R}^s_{ij0} = 0,$

(c) $\widetilde{R}^s_{0jk} = 0,$

(d) $\widetilde{R}^0_{ijk} = -R^n_{ijk}\Gamma^t_{nt} + 2\Gamma^n_{[jk}\Gamma^t_{nt|i]},$

(e) $\widetilde{R}^s_{ijk} = R^s_{ijk}.$

Proof. Ad. (a). From the definition one get

$$\widetilde{R}^s_{0j0} = \widetilde{\Gamma}^l_{00}\widetilde{\Gamma}^s_{lj} - \widetilde{\Gamma}^l_{j0}\widetilde{\Gamma}^s_{l0} + \widetilde{\Gamma}^s_{00|j} - \widetilde{\Gamma}^s_{j0|0}.$$

From Theorem 2.1 we have $\widetilde{\Gamma}^s_{l0} = 0$. Hence $\widetilde{R}^s_{0j0} = 0$.

Ad. (b), (c), (e). These statements follow from definition, Theorem 2.1 and simple calculations.

Ad. (d). From Lemma 1.1 we have $\Gamma^t_{tn|k} = \Gamma^t_{tk|n}$. Hence $\Gamma^t_{t(l|j)} = \Gamma^t_{tl|j}$. Next, we have

$$\widetilde{R}^0_{ijk} = \widetilde{\Gamma}^l_{ik}\widetilde{\Gamma}^0_{lj} - \widetilde{\Gamma}^l_{jk}\widetilde{\Gamma}^0_{li} + \widetilde{\Gamma}^0_{ik|j} - \widetilde{\Gamma}^0_{jk|i}$$

$$= \Gamma^l_{ik}\left(\Gamma^t_{tl|j} - \Gamma^n_{lj}\Gamma^t_{nt}\right) - \Gamma^l_{jk}\left(\Gamma^t_{tl|i} - \Gamma^n_{li}\Gamma^t_{nt}\right)$$

$$\quad + \left(\Gamma^t_{ti|k} - \Gamma^n_{ik}\Gamma^t_{nt}\right)_{|j} - \left(\Gamma^t_{tj|k} - \Gamma^n_{jk}\Gamma^t_{nt}\right)_{|i}$$

$$= \Gamma^t_{nt}\left(\Gamma^l_{jk}\Gamma^n_{li} - \Gamma^l_{ik}\Gamma^n_{lj} + \Gamma^n_{jk|i} - \Gamma^n_{ik|j}\right) + \Gamma^n_{jk}\Gamma^t_{nt|i} - \Gamma^n_{ik}\Gamma^t_{nt|j}$$

$$= -R^n_{ijk}\Gamma^t_{nt} + 2\Gamma^n_{[jk}\Gamma^t_{nt|i]}.$$

$\square$

Let $R_{ik} = R^j_{ijk}$ be the coefficients of the Ricci tensor ([2], [8]). Then from Theorem 2.2 we get directly

Theorem 2.3. *Let* $\widetilde{R}_{\alpha\beta}$ *are the coefficients of a Ricci tensor of the lifted metric* γ. *Then*

(a) $\widetilde{R}_{00} = 0,$

(b) $\widetilde{R}_{i0} = 0,$

(c) $\widetilde{R}_{ik} = R_{ik},$

where R_{ij} *are the coefficients of Ricci tensor of* g *on the Riemannian manifold* M.

Theorem 2.4. *Let $\widetilde{K}$ be a scalar curvature of the lifted metric γ. Then*

$$\widetilde{K} = \frac{n-1}{n+1}K,$$

where K is a scalar curvature of g on a Riemmanian manifold M.

Proof. From the definition of the scalar curvature, Theorem 2.3 and formula for (γ^{ij}) we have

$$\widetilde{K} = \frac{1}{n(n+1)}\widetilde{R}_{ik}\gamma^{ik} = \frac{1}{n(n+1)}R_{ik}g^{ik} = \frac{n-1}{n+1}K.$$

$\square$

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